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ON TRANSFORMATIONS OF TWO HOMOGENEOUS LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS OF A GENERAL AND OF STURM FORMS

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Introduction

The theory of transformations of two linear second order differential equations of Jacobian form

$$y'' = q(t)y \quad , \qquad (q)$$

$$Y'' = Q(T)Y \quad , \qquad (Q)$$

with continuous coefficients q, Q has been expounded by O. B o r ů v k a in /1/.

If j, J are open intervals and $t \in j$, $T \in J$, then by a transformation of the differential equation (Q) into the differential equation (q) we mean the ordered pair $[f,h]$ of functions $f(t)$, $h(t)$ defined in an open interval i, $i \subset j$, and having such properties that $h(i) = I$, $I \subset J$, $f \in C^{(2)}(i)$, $h \in C^{(3)}(i)$, $f(t)\,h'(t) \neq 0$ for every $t \in i$, and that for every solution Y of (Q) the function

$$y(t) = f(t)\, Y\left[h(t)\right] \tag{1}$$

is a solution of (q) in i.

The transformation problem for general equations of the n-th order has been studied by F. N e u m a n in /2/.

In this paper we shall be concerned with the transformation problem for linear homogeneous differential equations of the second order in a general and Sturm forms. Let us recall that by a solution of a linear second order differential equation of a general form

$$y'' + a(t)y' + b(t)y = 0\ , \qquad \text{(ab)}$$

where $a, b \in C^{(0)}(j)$, we mean every function y, $y \in C^{(2)}(j)$ satisfying the equation (ab) identically. By a solution of a linear second order differential equation of Sturm form

$$(p(t)y')' + q(t)y = 0\ , \qquad \text{(pq)}$$

where $p, q \in C^{(0)}(j)$, $p(t) \neq 0$ in j, we mean every function y, $y \in C^{(1)}(j)$, $py' \in C^{(1)}(j)$, satisfying the equation (pq) identically.

Consider now the following general linear differential equations of the second order

$$y'' + a(t)y' + b(t)y = 0\ , \qquad \text{(ab)}$$

$$Y'' + A(T)Y' + B(T)Y = 0\ , \qquad \text{(AB)}$$

where $a, b \in C^{(0)}(j)$, $A, B \in C^{(0)}(J)$, and the linear differential equations of the second order of Sturm form

$$(p(t)y')' + q(t)y = 0\ , \qquad \text{(pq)}$$

$$(P(T)Y')' + Q(T)Y = 0\ , \qquad \text{(PQ)}$$

where $p, q \in C^{(0)}(j)$, $p(t) \neq 0$ in j, $P, Q \in C^{(0)}(J)$, $P(T) \neq 0$ in J.

By a direct calculation we derive necessary and sufficient conditions set on the functions $f = f(t)$, $h = h(t)$ for every solution Y of (AB) and (PQ) to be transformed by equation (1) into the solution y of (ab) and (pq), respectively.

a) Linear differential equation of the second order of a general form

Definition 1. By a transformation of the general linear second order differential equation (AB) into the general linear second order differential equation (ab) we mean an ordered pair $[f,h]$ of functions $f(t)$, $h(t)$, defined in an open interval i, $i \subset j$, and having such properties that $h(i) = I$, $I \subset J$, $f \in C^{(2)}(i)$, $h \in C^{(2)}(i)$, $fh' \neq 0$ for $t \in i$, and that for every solution Y of (AB) the function y defined by (1) in i is a solution of (ab).

The function f will be called the multiplier and the function h the parametrization of the transformation $[f,h]$.

Theorem 1. Let (ab), (AB) be the linear second order differential equations of a general form, whereby $a,b \in C^{(0)}(j)$, $A,B \in C^{(0)}(J)$. The differential equation (AB) is transformed into (ab) by the transformation $[f,h]$ if and only if the parametrization h satisfies the nonlinear differential equation

$$\frac{1}{2}\left(A(h)h'(t)-\frac{h''(t)}{h'(t)}-a(t)\right)'+\frac{1}{4}\left(A(h)h'(t)-\frac{h''(t)}{h'(t)}-a(t)\right)^2+ \tag{2}$$

$$+\frac{1}{2}a(t)\left(A(h)h'(t)-\frac{h''(t)}{h'(t)}-a(t)\right)-B(h)h'^2(t)+$$

$$+b(t)=0$$

in i, and the multiplier f is given by the formula

$$f(t)=\frac{k}{\sqrt{|h'(t)|}}\,e^{-\frac{1}{2}\int a(t)dt}\,e^{\frac{1}{2}\int A(h)h'(t)dt},\quad t\in i, \tag{3}$$

where $k \neq 0$ is a multiplicative constant.

Proof. Let the differential equation (AB) be transformed into (ab) by means of $[f,h]$. We seek conditions which the functions f,h must fulfil for the function $y(t) = f(t)Y[h(t)]$ to be a solution of (ab) for every solution Y of (AB). Since $f,h \in C^{(2)}(i)$, we can by differentiating twice the equation

$$y(t) = f(t)Y[h(t)]$$

obtain

$$y'(t) = f(t)h'(t)Y'[h(t)] + f'(t)Y[h(t)] \quad , \tag{4}$$

$$y''(t) = f(t)h'^2(t)Y''[h(t)] + [2f'(t)h'(t) + f(t)h''(t)]\, Y'[h(t)] + f''(t)Y[h(t)] \, . \tag{5}$$

Inserting now y, y', y'' into (ab), we get the identity in i

$$f(t)h'^2(t)Y''[h(t)] + [2f'(t)h'(t) + f(t)h''(t) + a(t)f(t)h'(t)]\, Y'[h(t)] + [f''(t) + a(t)f'(t) + b(t)f(t)]\, Y[h(t)] = 0 \, ,$$

which is true for every solution Y of (AB). Comparing this with equation (AB), then with respect to the assumption

$$f(t)h'(t) \neq 0 \quad \text{for} \quad t \in i, \quad h(i) \subset J \, ,$$

we obtain

$$\frac{2f'(t)}{f(t)h'(t)} + \frac{h''(t)}{h'^2(t)} + \frac{a(t)}{h'(t)} = A[h(t)] \quad , \quad t \in i \, , \tag{6}$$

$$\frac{f''(t)}{f(t)h'^2(t)} + \frac{a(t)f'(t)}{f(t)h'^2(t)} + \frac{b(t)}{h'^2(t)} = B[h(t)] \, , \; t \in i \, . \tag{7}$$

We can very simply deduce from (6) that

$$\frac{f'(t)}{f(t)} = \frac{1}{2}\left[A(h)h'(t) - \frac{h''(t)}{h'(t)} - a(t)\right] . \tag{8}$$

Since $f \in C^{(2)}(i)$, then $\frac{f'}{f} \in C^{(1)}$, which enables us to write

$$A(h)h'(t) - \frac{h''(t)}{h'(t)} - a(t) \in C^{(1)}(i) \ . \tag{9}$$

Since $\frac{f''}{f} = \left(\frac{f'}{f}\right)' + \left(\frac{f'}{f}\right)^2$, we obtain from (7) that

$$\left(\frac{f'(t)}{f(t)}\right)' + \left(\frac{f'(t)}{f(t)}\right)^2 = B(h)h'^2(t) - a(t)\,\frac{f'(t)}{f(t)} - b(t) \ , \tag{10}$$

which on eliminating the expression $\frac{f'}{f}$ by means of (8), reduces to

$$\frac{1}{2}\left(A(h)h'(t) - \frac{h''(t)}{h'(t)} - a(t)\right)' + \frac{1}{4}\left(A(h)h'(t) - \frac{h''(t)}{h'(t)} - a(t)\right)^2 + \frac{1}{2}a(t)\left(A(h)h'(t) - \frac{h''(t)}{h'(t)} - a(t)\right) - B(h)h'^2(t) + b(t) = 0$$

Thus the parametrization h fulfils the differential equation (2). On multiplying (6) by the function $h'(\neq 0)$ we get

$$2\,\frac{f'(t)}{f(t)} + \frac{h''(t)}{h'(t)} + a(t) - A(h)h'(t) = 0 \tag{11}$$

which after integration yields

$$\ln f^2(t) + \ln|h'(t)| + \int a(t)dt - \int A(h)h'(t)dt = \ln|c| ,$$

$c \neq 0$, a constant,

whence

$$f(t) = \frac{k}{\sqrt{|h'(t)|}}\, e^{-\frac{1}{2}\int a(t)dt}\, e^{\frac{1}{2}\int A(h)h'(t)dt} \ ,$$

where $k = \pm\sqrt{|c|}$, $c \neq 0$. Hence the multiplier f is given by formula (3).

Suppose conversely that assumptions (2) and (3) on functions $f \in C^{(2)}(i)$, $h \in C^{(2)}(i)$ are satisfied. Then we have to prove that the transformation $[f,h]$ transforms (AB) into (ab).

Let Y be a solution of the differential equation (AB). We will show that the function y expressed by equation (1): $y(t) = f(t)Y[h(t)]$ is a solution of the differential equation (ab). With respect to (1), (4) and (5) we obtain

$$f(t)Y(h) = y(t)$$

$$f(t)h'(t)Y'(h) = y'(t) - \frac{f'(t)}{f(t)} y(t)$$

$$f(t)h'^2(t)Y''(h) = y''(t) - \frac{1}{f(t)h'(t)} (2f'(t)h'(t) + f(t)h''(t)) \cdot (y'(t) - \frac{f'(t)}{f(t)} y(t)) - \frac{f''(t)}{f(t)} y(t) \quad .$$

Inserting h(t) for the independent variable T in (AB), we obtain with respect to the foregoing equations (10) and (11) following from (2) and (3), successively

$$0 = Y''(h) + A(h)Y'(h) + B(h)Y(h) = \frac{1}{f(t)h'^2(t)} \cdot$$

$$\cdot \left[y''(t) - \frac{1}{f(t)h'(t)} (2f'(t)h'(t)+f(t)h''(t))(y'(t) - \frac{f'(t)}{f(t)} y(t)) - \frac{f''(t)}{f(t)} y(t)\right] +$$

$$+ A(h)\frac{1}{f(t)h'(t)} (y'(t)- \frac{f'(t)}{f(t)} y(t)) + \frac{1}{f(t)} B(h)y(t) =$$

$$= \frac{1}{f(t)h'^2(t)} \Big[y''(t)- (\frac{1}{f(t)h'(t)}(2f'(t)h'(t)+f(t)h''(t)) +$$

$$+ A(h)h'(t))y'(t) + \Big(\frac{f'(t)}{f(t)} \frac{1}{f(t)h'(t)} (2f'(t)h'(t) +$$

$$+ f(t)h''(t) - \frac{f''(t)}{f(t)} - A(h)\frac{f'(t)h'(t)}{f(t)} + B(h)h'^2(t))y(t)\Big] =$$

$$= \frac{1}{f(t)h'^2(t)}\Big[y''(t) + a(t)y'(t) + b(t)y(t)\Big] ,$$

from which we see that y satisfies the differential equation (ab).

Remark 1. In assuming that $a \in C^{(1)}(i)$, $A \in C^{(1)}(I)$, $I = h(i)$, we obtain with respect to (9) that $h \in C^{(3)}(i)$, and the parametrization h with respect to (2) satisfies at i the nonlinear differential equation of the third order

$$- \{h,t\} + \left(\frac{A^2(h)}{4} + \frac{A'(h)}{2} - B(h)\right) h'^2(t) =$$

$$= \frac{a^2(t)}{4} + \frac{a'(t)}{2} - b(t), \tag{12}$$

where the symbol $\{h,t\} = \frac{1}{2}\frac{h'''(t)}{h'(t)} - \frac{3}{4}\frac{h''^2(t)}{h'^2(t)}$ and denotes the Schwarzian derivative of the function h.

Setting in (ab) and (AB) $a \equiv 0$ in j and $A \equiv 0$ in J and writing $-q$ and $-Q$ instead of b and B, respectively, then the assumptions of Remark 1 are fulfilled and the equations (ab) and (AB) go over into linear second order differential equations of the Jacobian form

$$y'' = q(t)y , \tag{q}$$

$$Y'' = Q(T)Y , \tag{Q}$$

where $q \in C^{(0)}(j)$, $Q \in C^{(0)}(J)$, whose transformations were studied under the assumption that $f \in C^{(2)}$, $h \in C^{(3)}$ by O.Borůvka in /1/.

For this special case we get the following statements on functions f and h as a Corollary of Theorem 1:

Let (q), (Q) be linear second order differential equations of the Jacobian form, whereby $q \in C^{(0)}(j)$, $Q \in C^{(0)}(J)$. The differential equation (Q) is transformed into the differential equation (q) by the transformation $[f,h]$ by Definition 1 exactly if $h \in C^{(3)}(i)$ and the parametrization h satisfies the nonlinear differential equation of the third order

$$-\{h,t\} + Q(h)h'^2(t) = q(t)$$

and the multiplier f is given by the formula

$$f(t) = \frac{k}{\sqrt{|h'(t)|}} \quad , \quad k \neq 0 \quad .$$

b) Linear second order differential equations of Sturm form

Definition 2. By a transformation of the Sturm linear second order differential equation (PQ) into the Sturm linear second order differential equation (pq) we mean an ordered pair $[f,h]$ of functions f(t), h(t), defined in an open interval i, $i \subset j$, and having such properties that $h(i) = I$, $I \subset J$, $f \in C^{(1)}(i)$, $h \in C^{(1)}(i)$, $f(t)h'(t) \neq 0$ for $t \in i$, $\frac{P(h)f'(t)}{f^3(t)h'(t)} \in C^{(1)}(i)$, and for every solution Y of the differential equation (PQ) the function y defined by equation (1) in interval i is a solution of the differential equation (pq).

The function f will be called the multiplier and the function h the parametrization of the transformation $[f,h]$.

Theorem 2. Let (pq), (PQ) be linear second order differential equations of Sturm form, whereby $p,q \in C^{(0)}(j)$, $p \neq 0$ in j, $P,Q \in C^{(0)}(J)$, $P \neq 0$ in J. The differential equation (PQ) is transformed into the differential equation (pq) by the transformation $[f,h]$ exactly if the parametrization h satisfies in i the nonlinear differential equation

$$\frac{1}{2}\left(p(t)\,\frac{p(t)h'(t)}{P(h)}\left(\frac{P(h)}{p(t)h'(t)}\right)'\right)' +$$

$$+\frac{1}{4}\,p(t)\left(\frac{p(t)h'^2(t)}{P(h)}\right)^2\left(\frac{P(h)}{p(t)h'(t)}\right)'^2 -$$

$$-\frac{p(t)h'^2(t)}{P(h)}\,Q(h) + q(t) = 0 \qquad (13)$$

and the multiplier f is given by the formula

$$f(t) = k\sqrt{|P(h)/p(t)h'(t)|}\,, \quad t\in i\,, \qquad (14$$

where k is a nonzero multiplicative constant.

Proof. Let the differential equation (PQ) be transformed into the differential equation (pq) by the transformation $[f,h]$. Let us seek conditions which the functions f,h must fulfil for the function $y(t) = f(t)Y[h(t)]$ to be a solution of (pq) for every solution Y of (PQ). By differentiating the transformation equation

$$y(t) = f(t)Y[h(t)]$$

we obtain

$$y'(t) = f(t)h'(t)Y'(h) + f'(t)Y(h) \quad .$$

Multiplying both sides by $P(h)/f^2(t)h'(t)$ and inserting for Y(h) from the foregoing equation gives after rearrangement

$$\frac{P(h)}{f(t)}\,Y'(h) = \frac{P(h)}{f^2(t)h'(t)}\,y'(t) - \frac{P(h)f'(t)}{f^3(t)h'(t)}\,y(t) \quad .$$

Performing the differentiation, we find that

$$-\frac{f'(t)P(h)}{f^2(t)}\,Y'(h) + \frac{1}{f(t)}(P(h)Y'(h))' = \left(\frac{P(h)}{f^2(t)h'(t)}\,y'(t)\right)' -$$

$$-\left(\frac{P(h)f'(t)}{f^3(t)h'(t)}\right)'y(t) - \frac{P(h)f'(t)}{f^3(t)h'(t)}\,y'(t) \quad .$$

Since $(P(h)Y'(h))' = - Q(h)Y(h)h'(t)$ and inserting for Y, Y' yields

$$- \frac{f'(t)}{f(t)} \left[\frac{P(h)}{f^2(t)h'(t)} y'(t) - \frac{P(h)f'(t)}{f^3(t)h'(t)} y(t) \right] +$$

$$+ \frac{1}{f(t)} \left[- Q(h)h'(t) \frac{y(t)}{f(t)} \right] = \left(\frac{P(h)}{f^2(t)h'(t)} y'(t) \right)' -$$

$$- \left(\frac{P(h)f'(t)}{f^3(t)h'(t)} \right)' y(t) - \frac{P(h)f'(t)}{f^3(t)h'(t)} y'(t)$$

or

$$\left(\frac{P(h)}{f^2(t)h'(t)} y'(t) \right)' + \left[- \frac{P(h)f'^2(t)}{f^4(t)h'(t)} - \left(\frac{P(h)f'(t)}{f^3(t)h'(t)} \right)' + \right.$$

$$\left. + \frac{Q(h)h'(t)}{f^2(t)} \right] y(t) = 0 \, .$$

This is a linear second order differential equation of Sturm form for the function y. It must be therefore identical up to the nonzero multiplicative constant with the linear differential equation (pq) which is of the same form. Consequently

$$\frac{P(h)}{f^2(t)h'(t)} = cp(t) \tag{15}$$

$$\frac{f'^2(t)P(h)}{f^4(t)h'(t)} - \left(\frac{f'(t)P(h)}{f^3(t)h'(t)} \right)' + \frac{h'(t)Q(h)}{f^2(t)} = cq(t) \, . \tag{16}$$

From (15) we obtain

$$\frac{P(h)}{p(t)h'(t)} = cf^2(t) \, . \tag{17}$$

Because of $f \in C^{(1)}(i)$ it is obvious that also $\frac{P(h)}{p(t)h'(t)} \in C^{(1)}(i)$.

Hence

$$\left(\frac{P(h)}{p(t)h'(t)} \right)' = 2cf(t)f'(t) \; . \tag{18}$$

From (18) and (18) we obtain

$$\frac{f'(t)}{f(t)} = \frac{1}{2} \left(\frac{P(h)}{f(t)h'(t)} \right)' \frac{p(t)h'(t)}{P(h)} \; . \tag{19}$$

From (16) on substituting for $\frac{f'}{f}$ from (19) and for f^2 from (17) we get

$$\frac{1}{2}\left((p(t) \frac{p(t)h'(t)}{P(h)} \left(\frac{P(h)}{p(t)h'(t)} \right)'\right)' +$$

$$+ \frac{1}{4} p(t) \left(\frac{p(t)h'(t)}{P(h)} \right)^2 \left(\frac{P(h)}{p(t)h'(t)} \right)'^2 -$$

$$- \frac{p(t)h'^2(t)}{P(h)} Q(h) + q(t) = 0 \; .$$

From (17) we get that the multiplier f is determined by (14), where

$$k = \pm \frac{1}{\sqrt{|c|}} \; .$$

Suppose conversely that the assumptions (13) and (14) on functions f and h are satisfied. We have to prove that the transformation $[f,h]$ transforms the equation (PQ) into (pq).

Let Y be a solution of the differential equation (PQ). We show that the function y expressed by equation (1) is a solution of the differential equation (pq). From the transformation equation (1) and from its derivative follows

$$Y(h) = \frac{1}{f(t)} \; y(t)$$

$$Y'(h) = \frac{1}{f(t)h'(t)}\, y'(t) - \frac{f'(t)}{f^2(t)h'(t)}\, y(t)\ .$$

Multiplying both sides by $\frac{P(h)}{f(t)}$ gives

$$\frac{1}{f(t)}\, P(h)Y'(h) = \frac{P(h)}{f^2(t)h'(t)}\, y'(t) - \frac{P(h)f'(t)}{f^3(t)h'(t)}\, y(t)$$

whence by differentiating

$$-\frac{f'(t)}{f^2(t)}\, P(h)Y'(h) + \frac{1}{f(t)}\, (P(h)Y'(h))' =$$

$$= \left(\frac{P(h)}{f^2(t)h'(t)}\, y'(t)\right)' - \left(\frac{P(h)f'(t)}{f^3(t)h'(t)}\right)' y(t) -$$

$$- \frac{P(h)f'(t)}{f^3(t)h'(t)}\, y'(t)\ .$$

On substituting on the left hand side for $Y'(h)$, we obtain

$$- \frac{P(h)f'(t)}{f^3(t)h'(t)}\, y'(t) + \frac{P(h)f'^2(t)}{f^4(t)h'(t)}\, y(t) + \frac{1}{f(t)}\, (P(h)Y'(h))' =$$

$$= \left(\frac{P(h)}{f(t)h'(t)}\right)' - \left(\frac{P(h)f'(t)}{f^3(t)h'(t)}\right)' y(t) - \frac{P(h)f'(t)}{f^3(t)h'(t)}\, y'(t)$$

or

$$\frac{1}{f(t)}\, (P(h)Y'(h))' = \left(\frac{P(h)}{f^2(t)h'(t)}\, y'(t)\right)' -$$

$$- \left(\frac{P(h)f'(t)}{f^3(t)h'(t)}\right)' y(t) - \frac{P(h)f'^2(t)}{f^4(t)h'(t)}\, y(t)\ .$$

Since

$$0 = \frac{1}{f(t)}\left[(P(h)Y'(h))' + Q(h)Y(h)h'(t)\right] =$$

$$=\left(\frac{P(h)}{f^2(t)h'(t)}\,y'(t)\right)' - \left(\frac{P(h)f'(t)}{f^3(t)h'(t)}\right)'y(t) -$$

$$-\frac{P(h)f'^2(t)}{f^4(t)h'(t)}\,y(t) + \frac{1}{f(t)}\,Q(h)\frac{h'(t)}{f(t)}\,y(t) = (cp(t)y'(t))' +$$

$$+\, cq(t)y(t) = c\left[(p(t)y'(t))' + q(t)y(t)\right],$$

i.e. y is a solution of the differential equation (pq).

Remark 2. In assuming that the coefficients p, P are of class $C^{(2)}$, i.e. $p \in C^{(2)}(i)$, $P \in C^{(2)}(I)$ and $p(t) \neq 0$ in i, $P(T) \neq 0$ in I, $I = h(i)$, it follows with respect to the assumption given in Definition 2

$$\frac{P(h)f'(t)}{f^3(t)h'(t)} \in C^{(1)}(i) \tag{20}$$

that $h \in C^{(3)}(i)$ and $f \in C^{(2)}(i)$.

Indeed, it holds with respect to (19) and (17) that

$$\frac{P(h)f'(t)}{f^3(t)h'(t)} = \frac{cp^2(t)h'(t)}{2P(h)}\left(\frac{P(h)}{p(t)h'(t)}\right)'.$$

Consequently

$$\frac{p^2(t)h'(t)}{P(h)}\left(\frac{P(h)}{p(t)h'(t)}\right)' \in C^{(1)}(i). \tag{21}$$

If we set $Z(t) = \frac{P(h)}{p(t)h'(t)}$, then by our assumption $Z \in C^{(1)}(i)$ and $h'(t) = \frac{P(h)}{Z(t)p(t)}$. Since $\frac{P(h)}{Z(t)p(t)} \in C^{(1)}(i)$, then $h \in C^{(2)}(i)$.

Completely analogous can be deduced from (21) that $h \in C^{(3)}(i)$, as well as from (20) that $f \in C^{(2)}(i)$.

With respect to (13) the parametrization h satisfies in i the nonlinear third order differential equation

$$-\{h,t\} + \left[\frac{1}{2}\frac{P''(h)}{P(h)} - \frac{1}{4}\frac{P'^2(h)}{P^2(h)} - \frac{Q(h)}{P(h)}\right] h'^2(t) =$$

$$= \frac{1}{2}\frac{p''(t)}{p(t)} - \frac{1}{4}\frac{p'^2(t)}{p^2(t)} - \frac{q(t)}{p(t)} , \qquad (22)$$

where $\{h,t\} = \frac{1}{2}\frac{h'''(t)}{h'(t)} - \frac{3}{4}\frac{h''^2(t)}{h'^2(t)}$.

Indeed, setting for simplicity $\frac{P(h)}{p(t)} = X(t)$, yields from (17) that

$$cf^2(t) = \frac{1}{h'(t)} X(t) ,$$

whence by differentiating

$$2cf(t)f'(t) = X'(t)\frac{1}{h'(t)} - X(t)\frac{h''(t)}{h'^2(t)} .$$

From the foregoing relations we get

$$2\frac{f'(t)}{f(t)} = \frac{X'(t)}{X(t)} - \frac{h''(t)}{h'(t)} .$$

Substituting for $\frac{f'}{f}$ and f^2 into (16) gives

$$-\frac{1}{4}\left(\frac{X'(t)}{X(t)} - \frac{h''(t)}{h'(t)}\right)^2 \frac{ch'(t)}{X(t)}\frac{P(h)}{h'(t)} - \left(\frac{1}{2}\left(\frac{X'(t)}{X(t)} - \right.\right.$$

$$\left.\left.- \frac{h''(t)}{h'(t)}\right)\frac{ch'(t)}{X'(t)}\frac{P(h)}{h'(t)}\right)' + \frac{ch'^2(t)Q(h)}{X(t)} = cq(t).$$

Replacing X(t)p(t) for P(h) gives by rearrangement

$$- \frac{p(h)}{4} \left(\frac{X'(t)}{X(t)} - \frac{h''(t)}{h'(t)} \right)^2 - \left(\frac{p(t)}{2} \left(\frac{X'(t)}{X(t)} - \frac{h''(t)}{h'(t)} \right) \right)' +$$

$$+ \frac{Q(h)}{X(t)} h'^2(t) = q(t)$$

and on performing the operations indicated

$$- \frac{p(t)}{4} \left(\frac{X'^2(t)}{X^2(t)} - 2 \frac{X'(t)}{X(t)} \frac{h''(t)}{h'(t)} + \frac{h''^2(t)}{h'^2(t)} \right) -$$

$$- \frac{p'(t)}{2} \left(\frac{X'(t)}{X(t)} - \frac{h''(t)}{h'(t)} \right) - \frac{p(t)}{2} \left(\frac{X''(t)}{X(t)} - \frac{X'^2(t)}{X^2(t)} - \right.$$

$$\left. - \frac{h'''(t)}{h(t)} + \frac{h''^2(t)}{h'^2(t)} \right) + \frac{Q(h)}{X(t)} h'^2(t) = q(t)$$

or

$$\frac{1}{2} \frac{h''(t)}{h'(t)} - \frac{3}{4} \frac{h''(t)}{h'^2(t)} + \frac{Q(h)}{p(t)X(t)} h'^2(t) +$$

$$+ \frac{1}{2} \left(\frac{X'(t)}{X(t)} + \frac{p'(t)}{p(t)} \right) \frac{h''(t)}{h'(t)} - \frac{1}{2} \left(\frac{X'(t)}{X(t)} \right)' -$$

$$- \frac{1}{2} \frac{p'(t)}{p(t)} \frac{X'(t)}{X(t)} - \frac{1}{4} \frac{X'^2(t)}{X^2(t)} = \frac{q(t)}{p(t)} , \qquad (23)$$

where $X = \frac{P(h)}{p(t)}$. Inserting for X into (23) yields

$$- \{h,t\} - \frac{Q(h)}{P(h)} h'^2(t) - \frac{1}{2} \frac{P'(h)}{P(h)} h''(t) + \frac{1}{2} \Big[\frac{P''(h)}{P^2(h)} h'^2(t) -$$

$$- \frac{P'^2(h)}{P^2(h)} h'^2(t) + \frac{P'(h)}{P(h)} h''(t) - \frac{p''(t)}{p(t)} + \frac{p'^2(t)}{p^2(t)} \Big] +$$

$$+ \frac{1}{2} \frac{p'(t)}{p(t)} \Big[\frac{P'(h)}{P(h)} h'(t) - \frac{p'(t)}{p(t)} \Big] + \frac{1}{4} \Big[\frac{P'(h)}{P(h)} h'(t) -$$

$$- \frac{p'(t)}{p(t)} \Big]^2 = - \frac{q(t)}{p(t)} \; . \tag{24}$$

Equation (24) may be arranged into the form of (22).

Putting in the linear differential equations (pq), (PQ) respectively $p \equiv -1$ in j, $P \equiv -1$ in J, then these equations go over into linear second order differential equations of Jacobian form

$$y'' = q(t)y \; , \tag{q}$$
$$Y'' = Q(T)Y \; , \tag{Q}$$

where $q \in C^{(0)}(j)$, $Q \in C^{(0)}(J)$ and the differential equation (22) into the equation

$$-\{h,t\} + Q(h)h'^2(t) = q(t)$$

and the multiplier f is given by the formula

$$f(t) = \frac{k}{\sqrt{|h'(t)|}} \; , \quad k \neq 0 \; .$$

This repeatedly proves the Corollary of Theorem 1.

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SOUHRN

TRANSFORMACE DVOU LINEÁRNÍCH HOMOGENNÍCH DIFERENCIÁLNÍCH ROVNIC 2.ŘÁDU OBECNÉHO A STURMOVA TVARU

JITKA LAITOCHOVÁ

Je definována transformace $[f,h]$ diferenciální rovnice (AB) do diferenciální rovnice (ab) a diferenciální rovnice (PQ) do diferenciální rovnice (pg) pomocí rovnice

$$y(t) = f(t)\,Y[h(t)] \quad ,$$

kde Y značí řešení rovnice (AB), resp. (PQ), y řešení rovnice (ab), resp. (pq). Jsou nalezeny nutné a postačující podmínky pro funkce f,h za nichž transformace existuje. Dále se nacházejí podmínky, za nichž lze úvahy o transformacích rovnice obecného a Sturmova tvaru provést pro lineární diferenciální rovnice Jacobiho tvaru.

РЕЗЮМЕ

ТРАНСФОРМАЦИЯ ДВУХ ЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ 2-ГО ПОРЯДКА ОБЩЕЙ И ШТУРМОВОЙ ФОРМЫ

ЙИТКА ЛАЙТОХОВА

Определяется трансформация $[f, h]$ дифференциального уравнения (AB) в дифференциальное уравнение (ab) и дифференциальное уравнение (PQ) в уравнение (pq) при помощи уравнения

$$y(t) = f(t)\, Y\, [h(t)] ,$$

где Y обозначает решение уравнения (AB) или (PQ), y решение уравнения (ab) или (pq). Найдены необходимые и достаточные условия по отношению к функциям f, h, при которых это преобразование существует. Далее определены условия при которых возможно рассуждения о преобразованиях уравнений общей и Штурмовой формы перенести на линейные уравнения формы Якоби.

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