Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

Svatoslav Staněk

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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 25 (1986), No. 1, 57--75

Persistent URL: http://dml.cz/dmlcz/120173

ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM

1986 Mathematica XXV Vol. 85

Katedra matematické analýzy a numerické matematiky
přírodovědecké fakulty univerzity Palackého v Olomouci
Vedoucí katedry: Prof.RNDr. Miroslav Laitoch, CSc.

A PHASE OF THE DIFFERENTIAL EQUATION $y'' = Q(t)y$ WITH A COMPLEX COEFFICIENT Q OF THE REAL VARIABLE

SVATOSLAV STANĚK

(Received January 15th, 1985)

1. O.Borůvka /1/ has introduced the concept of a (first) phase of the equation

$$y'' = q(t)y, \quad q \in C^0(j), \tag{q}$$

where $j:=(a,b)$ $(-\infty \leqq a < b \leqq \infty)$, with $C^n(j)$ and $\tilde{C}^n(j)$ denoting the set of real and complex functions, respectively, having continuous derivatives up to and including the order n ($n = 0,1,2, \dots$) on j. There were thus given a real form for a general solution the above equation together with a neat description of the structure of phases of (q) in applying a certain decomposition in the set of functions of class $C^3(j)$ with the derivative different from zero on j. The phases of (q) appeared to be exceedingly suitable to studying global properties of homogenous linear second order differential equation, e.g. global transformations, limit circle classification, stability of solutions, decomposition of zeros of solutions, Floquet theory etc.

This idea inspired the author to introduce a (first) phase of the equation

$$y'' = Q(t)y, \quad Q \in \tilde{C}^0(j), \quad \mathrm{Im}\, Q(t) \neq 0, \tag{Q}$$

in a certain analogy with the above real case. Thus there were given a form of the general solution of (Q) (Theorem 5 and to it related Theorems 6, 7, 8) and a description of phases of (Q) in Theorem 4. Next a proof is given for the fact that the phases of (Q) have properties analogous to those of the real case (Lemma 3, Theorem 3), yet it is also shown that the properties of solutions of (Q) have no analogies with equations having a real coefficient (Lemma 2, Theorems 1, 2, 9).

2. Let $M \subset R \times R$ be a subset of the Cartesian product $R \times R$ and let $m(M)$ be the Lebesgue measure of the set M. Then the validity of the following Lemma may be verified without difficulty.

Lemma 1. *Let* $a < \dots < t_{-n} < \dots < t_{-1} < t_0 < t_1 \dots < t_n \dots < b$, $\lim_{n\to\infty} t_{-n} = a$, $\lim_{n\to\infty} t_n = b$. *Let* $x = x_n(t)$, $y = y_n(t)$ *be real functions continuous first derivatives on the interval* (t_{n-1}, t_n), $n = 0, \pm 1, \pm 2, \dots$. *Setting*

$$M_n := \{(x,y);\ x=x_n(t),\ y=y_n(t),\ t \in (t_{n-1},t_n)\} \subset R \times R\ ,$$

$$M := \bigcup_{n=-\infty}^{\infty} M_n\ ,$$

yields

$m(M_n)$ *for each* n

and

$m(M) = 0.$

3. Let us look now at some properties of solutions of (Q). The trivial solution of (Q) will be excluded throughout this text. It is obvious that to any two complex numbers y_o, $y_o' \in C$ non-vanishing at the same time there exists a unique solution $y=y(t)$ of (Q), defined on j and satisfying the initial conditions $y(t_o)=y_o$, $y'(t_o)=y_o'$ at a point $t_o \in j$. It is next obvious:

(i) The zeros of any solution of (Q) (so far they exist) have no cluster point in j;

(ii) Solutions u, v of (Q) are linearly dependent exactly if $w := uv' - u'v = 0$ on j;

(iii) Let u be a solution of (Q) with $u(t) \neq 0$ for $t \in (a_1,b_1) \subset j$. Let $t_o \in (a_1,b_1)$ and let us set

$$v(t) := u(t) \int_{t_o}^{t} \frac{ds}{u^2(s)}, \quad t \in (a_1,b_1) \quad . \tag{1}$$

Then v is a solution of (Q) on the interval (a_1,b_1) and $uv' - u'v = 1$;

(iv) Let $Q_1(t) := \operatorname{Re} Q(t)$, $Q_2(t) := \operatorname{Im} Q(t)$, $t \in j$. Then the solution $y(t) = y_1(t) + iy_2(t)$ of (Q) is equivalent to the solution $(y_1(t), y_2(t))$ of the system of differential equations

$$y_1'' = Q_1(t)y_1 - Q_2(t)y_2 \; ,$$
$$y_2'' = Q_2(t)y_1 + Q_1(t)y_2 \; .$$

Theorem 1. Equation (Q) has at least one solution with no zero on j.

Proof. Suppose to the contrary that every solution of (Q) has at least one zero on j and suppose u, v are independent solutions of this equation. Then for any two complex numbers c, d, non-vanishing at the same time, the equation $cu(t) + dv(t) = 0$ has at least one root $t_0(=t_0(c,d)) \in j$. Thus, to every $A \in C$, $A \neq 0$, there exists $t_1 \in j$ such that $Au(t_1) - v(t_1) = 0$. If $u(t_1) = 0$, then $v(t_1) = 0$, which is in contradiction to the linear independence of the solutions u, v of (Q). Therefore $u(t_1) \neq 0$ and $\frac{v(t_1)}{u(t_1)} = A$. By the assumption there exists a $t_2 \in j$: $v(t_2) = 0$, hence $\frac{v(t_2)}{u(t_2)} = 0$. The solution u of (Q) has at most countably many zeros on j and let $u(t_n) = 0$ with $a < \dots < t_{-n} < \dots < t_0 < \dots < t_n < \dots < b$. The function $\frac{v(t)}{u(t)}$ maps the set $j - \{\dots, t_{-n}, \dots, t_0, \dots, t_n, \dots\}$ on the set C. Let $u = u_1(t) + iu_2(t)$, $v(t) = v_1(t) + iv_2(t)$ and

$$M_i := \Big\{ (x,y);\ x = \frac{v_1(t)u_1(t) + v_2(t)u_2(t)}{|u(t)|^2},$$

$$y = \frac{v_2(t)u_1(t) - v_1(t)u_2(t)}{|u(t)|^2},\quad t \in (t_{i-1}, t_i) \Big\} \subset R \times R .$$

If the number of terms of the sequence $t_0, t_1, t_2, \dots$ is finite with t_m the greatest of them, then $M_m := \Big\{ (x,y);$

$$x = \frac{v_1(t)u_1(t) + v_2(t)u_2(t)}{|u(t)|^2},\quad y = \frac{v_2(t)u_1(t) - v_1(t)u_2(t)}{|u(t)|^2}$$

$t \in (t_m, b) \Big\} \subset R \times R$. If the number of terms of the sequence $t_0, t_{-1}, t_{-2}, \dots$ is finite with t_{-n} the smallest of them, then

$$M_{-n} := \Big\{ (x,y);\ x = \frac{v_1(t)u_1(t) + v_2(t)u_2(t)}{|u(t)|^2},$$

$y = \dfrac{v_2(t)u_1(t) - v_1(t)u_2(t)}{|u(t)|^2}$, $t \in (a, t_{-n})\} \subset R \times R$. Let us set $M := \bigcup_i M_i$. By Lemma 1 $m(M) = 0$, contradicting thus the fact that $m(M) = m(R \times R) = \infty$. Hence, there exists a solution of (Q) having no zero on j.

Corollary 1. The exist two independent solutions of (Q) having no zero on j.

Proof. By Theorem 1 there exists a solution u of (Q), $u(t) \neq 0$ for $t \in j$. Let $t_0 \in j$. Setting $v(t) := u(t) \int_{t_0}^{t} \frac{ds}{u^2(s)}$, $t \in j$, yields that u, v are independent solutions of (Q). Let us assume that every solution of (Q) independent of u has a zero on j. Then the equation

$$Au(t) - u(t) \int_{t_0}^{t} \frac{ds}{u^2(s)} = 0$$

has for every $A \in C$ at least one root on j. With respect to the assumption $u(t) \neq 0$ for $t \in j$, the function $\int_{t_0}^{t} \frac{ds}{u^2(s)}$ maps the interval j on the set C in contradiction to Lemma 1, by which this function maps the interval j on the set of measure zero.

Lemma 2. The functions u, v are independent solutions of (Q) and $u^2(t) + v^2(t) \neq 0$ for $t \in j$ exactly if u+iv, u-iv are independent solutions of (Q) having no zero on j.

Proof. ($\Longrightarrow$) Let u, v be independent solutions of (Q) and $u^2(t) + v^2(t) \neq 0$ on j. Then u+iv, u-iv are solutions of (Q), $w := (u+iv)(u-iv)' - (u+iv)'(u-iv) = 2i(vu' - v'u) \neq 0$, hence $u+iv$, $u-iv$ are independent solutions of (Q) and $(u+iv)(u-iv) = u^2 + v^2 \neq 0$ on j.

($\Longleftarrow$) Let $u+iv$, $u-iv$ be independent solutions of (Q) having no zero on j. Then u, v are independent solutions of (Q) and $u^2 + v^2 = (u+iv)(u-iv) \neq 0$ on j.

Theorem 2. *There exist independent solutions* u, v *of* (Q), *such that* $u^2(t) + v^2(t) \neq 0$ *for* $t \in j$.

Proof. By Corollary 1 there exist independent solutions y_1, y_2 of (Q) having no zero on j. Set $u(t) := \frac{1}{2}(y_1(t)+y_2(t))$, $v(t) := \frac{i}{2}(y_2(t)-y_1(t))$, $t \in j$. Then u, v are independent solutions of (Q) and since $y_1=u+iv$, $y_2=u-iv$ it follows from Lemma 2 that $u^2 + v^2 \neq 0$ on j.

4. In this part we introduce the notion of a (first) phase of (Q).

Definition 1. *Let* u, v *be independent solutions of* (Q), $u^2(t) + v^2(t) \neq 0$ *for* $t \in j$ *(the existence of such solutions is guarenteed by Theorem 2) and let* $w := uv' - u'v$. *We say that a function* $\alpha \in \tilde{C}^3(j)$ *is a (first) phase of the basis* (u,v) *of* (Q) *if*

$$\alpha'(t) = - \frac{w}{u^2(t)+v^2(t)} , \quad t \in j$$

and $\operatorname{tg} \alpha(t_0) = \frac{u(t_0)}{v(t_0)}$ *at a point* $t_0 \in j$, *where* $v(t_0) \neq 0$. *We say that a function* α *is a (first) phase of* (Q) *if there exists a basis* (u,v) *of* (Q) *such that* α *is a (first) phase of the basis* (u,v).

Convention. Let α be a phase of (Q). Then $\alpha'(t) \neq 0$ for $t \in j$ and from the theory of functions of the complex variable then it follows the existence of a continuous unique branch $\sqrt{\alpha'(t)}$. Hereafter $\sqrt{\alpha'(t)}$ is used to indicate a continuous unique branch of the square root of the function $\alpha'(t)$.

In analogy with the real case (see e.g./1/) we can prove

Lemma 3. *Let α be a phase of a basis (u,v) of (Q). Then*

(i) $\operatorname{tg}\alpha(t) = \dfrac{u(t)}{v(t)}$ *for* $t \in j - \{t;\ v(t)=0,\ t \in j\}$;

(ii) $u(t) = k\dfrac{\sin\alpha(t)}{\sqrt{\alpha'(t)}}$, $v(t) = k\dfrac{\cos\alpha(t)}{\sqrt{\alpha'(t)}}$, *where* $t \in j$ *and* $k \in C$ *is an appropriate number;*

(iii) $\alpha(t) + n\pi$, *where* $n = 0, \pm 1, \pm 2, \ldots$ *exactly all phases of the basis* (u,v) *of* (Q).

Theorem 3. *A function α is a phase of (Q) exactly if it is a solution of the nonlinear differential equation*

$$- \{\alpha, t\} - \alpha'^2(t) = Q(t) \tag{3}$$

on j, *where* $\{\alpha, t\} = \dfrac{1}{2}\dfrac{\alpha'''(t)}{\alpha'(t)} - \dfrac{3}{4}\left(\dfrac{\alpha''(t)}{\alpha'(t)}\right)^2$ *is the Schwarzian derivative of* α *at the point* t.

Proof. ($\Longrightarrow$) Let α be a phase of (Q). Then there exists a basis (u,v) of (Q) such that $u^2 + v^2 \neq 0$ on j and $\alpha'(t) =$

$= - \dfrac{w}{u^2(t)+v^2(t)}$ for $t \in j$ and $w := uv' - u'v$. It may be verified after a computation of the functions α'', α''' from the formula $\alpha' = - \dfrac{w}{u^2 + v^2}$ that $\alpha \in \hat{C}^3(j)$ and α is a solution of (3) on j.

($\Leftarrow$) Let α be a solution of (3) on j. Then $\alpha \in \tilde{C}^3(j)$, $\alpha'(t) \neq 0$ for $t \in j$. Let us set $u(t) := \frac{\sin\alpha(t)}{\sqrt{\alpha'(t)}}$, $v(t) := \frac{\cos\alpha(t)}{\sqrt{\alpha'(t)}}$ for $t \in j$. A direct calculation shows that u, v are independent solutions of (Q) for $w := uv' - u'v = -1$. Next $u^2 + v^2 = \frac{1}{\alpha'}$,

hence $\alpha' = -\frac{w}{u^2 + v^2}$ with $\mathrm{tg}\,\alpha(t_0) = \frac{u(t_0)}{v(t_0)}$ at a point

$t_0 \in j$, where $v(t_0) \neq 0$. Consequently, α is a phase of the basis (u,v) of (Q) and therefore also a phase of (Q).

Corollary 3. Let α be a phase of (Q). Then also the functions $\pm\alpha + c$ are phases of (Q) for every $c \in C$.

Proof. The functions $\pm\alpha + c$, $c \in C$, are solutions of (3) on j and thus by Theorem 3 they are phases of (Q).

Theorem 4. Let α be a phase of (Q), $c_1, c_2, c_3, c_4 \in C$, $c_1c_4 - c_2c_3 \neq 0$ and

$$(c_1\cos\alpha(t)+c_2\sin\alpha(t))^2+(c_3\cos\alpha(t)+c_4\sin\alpha(t))^2 \neq 0 \text{ for } t \in j. \quad (4)$$

Let $t_0 \in j$ and $d \in C$. Then the function β defined as

$$\beta(t) = \int_{\alpha(t_0)}^{\alpha(t)} \frac{c_2c_3 - c_1c_4}{(c_1\cos z+c_2\sin z)^2+(c_3\cos z+c_4\sin z)^2}\,dz ,$$

$$t \in j , \quad (5)$$

is a phase of (Q). Here the integral on the right-hand side of (5) is taken along the curve $z = \alpha(t)$, $t \in j$.

The converse is also true: Let β be a phase of (Q). Then there exist numbers c_1, c_2, c_3, c_4, $d \in C$, $c_1c_4-c_2c_3 \neq 0$ such that (4) and (5) hold, where integral on the right-hand side of (5) is taken along the curve $z = \alpha(t)$, $t \in j$.

Proof. Let α be a phase of (Q), β be defined by (5), where $c_1, c_2, c_3, c_4, d \in C$, $c_1c_4 - c_2c_3 \neq 0$ and (4) be true. Setting $y_1(z) := c_1 \cos z + c_2 \sin z$, $y_2(z) := c_3 \cos z + c_4 \sin z$, $z \in C$, yields that y_1, y_2 are independent solutions of $y'' = -y$ (on C) and $w := y_1'y_2 - y_1y_2' = c_2c_3 - c_1c_4$. The following formulas

$$\rho' = -2w \frac{y_1y_1' + y_2y_2'}{(y_1^2 + y_2^2)^2} ,$$

$$\rho'' = 8w \frac{(y_1y_1' + y_2y_2')^2}{(y_1^2 + y_2^2)^3} - 2w \frac{y_1'^2 + y_2'^2 - y_1^2 - y_2^2}{(y_1^2 + y_2^2)^2} ,$$

hold for the derivatives of the function

$$\rho(z) := \frac{w}{y_1^2(z) + y_2^2(z)} , \quad z \in C_1 ,$$

where $C_1 = \{ z ;\ z \in C,\ y_1^2(z) + y_2^2(z) \neq 0 \}$. Thence

$$-\frac{1}{2} \frac{\rho''}{\rho} + \frac{3}{4} \left(\frac{\rho'}{\rho} \right)^2 - \rho^2 = -1. \qquad (6)$$

From (5) and with reference to the definition of the function ρ we obtain for $t \in j$:

$$\beta'(t) = \alpha'(t)\, \rho[\alpha(t)] ,$$
$$\beta''(t) = \alpha'^2(t)\, \rho'[\alpha(t)] + \alpha''(t)\, \rho[\alpha(t)] ,$$
$$\beta'''(t) = \alpha'^3(t)\, \rho''[\alpha(t)] + 3\alpha'(t)\alpha''(t)\, \rho'[\alpha(t)] + \alpha'''(t)\, \rho[\alpha(t)] ,$$

This yields $\beta'(t) \neq 0$ for $t \in j$ and $\beta \in \mathcal{C}^3(j)$ which on using (6) gives

$$-\frac{1}{2} \frac{\beta'''(t)}{\beta'(t)} + \frac{3}{4} \left(\frac{\beta''(t)}{\beta'(t)} \right)^2 - \beta'^2(t) = \frac{1}{2} \left[\frac{\rho''[\alpha(t)]}{\rho[\alpha(t)]} \alpha'^2(t) + 3 \frac{\rho'[\alpha(t)]}{\rho[\alpha(t)]} \alpha''(t) + \frac{\alpha'''(t)}{\alpha'(t)} \right] +$$

$$+\frac{3}{4}\left[\frac{\rho'[\alpha(t)]}{\rho[\alpha(t)]}\alpha'(t)+\frac{\alpha''(t)}{\alpha'(t)}\right]^2-\alpha'^2(t)\rho[\alpha(t)]=\left[-\frac{1}{2}\frac{\rho''[\alpha(t)]}{\rho[\alpha(t)]}+\right.$$

$$\left.+\frac{3}{4}\left(\frac{\rho'[\alpha(t)]}{\rho[\alpha(t)]}\right)^2-\rho^2[\alpha(t)]\right]\alpha'^2(t)-\frac{1}{2}\frac{\alpha'''(t)}{\alpha'(t)}+\frac{3}{4}\left(\frac{\alpha''(t)}{\alpha'(t)}\right)^2=$$

$$=-\frac{1}{2}\frac{\alpha'''(t)}{\alpha'(t)}+\frac{3}{4}\left(\frac{\alpha''(t)}{\alpha'(t)}\right)^2-\alpha'^2(t)=Q(t),$$

thus

$$-\{\beta,t\}-\beta'^2(t)=Q(t),\ t\in j,$$

and following Theorem 3 we see that β is a phase of (Q).

Suppose β is a phase of (Q). By Lemma 3 the functions $\frac{\sin\alpha(t)}{\sqrt{\alpha'(t)}}$, $\frac{\cos\alpha(t)}{\sqrt{\alpha'(t)}}$ as well as the functions $\frac{\sin\beta(t)}{\sqrt{\beta'(t)}}$, $\frac{\cos\beta(t)}{\sqrt{\beta'(t)}}$ are independent solutions of (Q). Consequently, there exist numbers $c_1,c_2,c_3,c_4\in C$, $c_1c_4-c_2c_3=1$ such that

$$\frac{\sin\beta(t)}{\sqrt{\beta'(t)}}=c_1\cdot\frac{\cos\alpha(t)}{\sqrt{\alpha'(t)}}+c_2\frac{\sin\alpha(t)}{\sqrt{\alpha'(t)}},$$

$$\frac{\cos\beta(t)}{\sqrt{\beta'(t)}}=c_3\frac{\cos\alpha(t)}{\sqrt{\alpha'(t)}}+c_4\frac{\sin\alpha(t)}{\sqrt{\alpha'(t)}}.$$

Thence it follows that

$$\frac{1}{\beta'(t)}=\frac{1}{\alpha'(t)}\left[(c_1\cos\alpha(t)+c_2\sin\alpha(t))^2+\right.$$
$$\left.+(c_3\cos\alpha(t)+c_4\sin\alpha(t))^2\right],$$

hence (4) is valid, which after integration yields (5) where $t_o\in j$, $d:=\beta(t_o)$.

Theorem 5. Let α be a phase of (Q). Then every solution of (Q) may be written either as

$$c_1 \frac{\sin(\alpha(t) + c_2)}{\sqrt{\alpha'(t)}} \tag{7}$$

or as

$$c_3 \frac{e^{i\gamma\alpha(t)}}{\sqrt{\alpha'(t)}}, \tag{8}$$

where $\gamma^2 = 1$, c_1, c_2, $c_3 \in C$, $c_1 \neq 0 \neq c_3$. Also conversely: The functions defined by (7) and (8) are solutions of (Q) for arbitraly complex numbers $c_1 \neq 0$, c_2, $c_3 \neq 0$ and for any number γ, $\gamma^2 = 1$.

Proof. By Lemma 3 the functions $\frac{\sin\alpha(t)}{\sqrt{\alpha'(t)}}$, $\frac{\cos\alpha(t)}{\sqrt{\alpha'(t)}}$ are independent solutions of (Q). Hence, all solutions of (Q) may be written as $\frac{k_1 \sin\alpha(t) + k_2 \cos\alpha(t)}{\sqrt{\alpha'(t)}}$, where k_1, $k_2 \in C$ do not vanish at the same time. The next part of the proof will be devided into two parts:

(i) $k_1^2 + k_2^2 \neq 0$ and let $\cos c_2 = \frac{k_1}{\sqrt{k_1^2 + k_2^2}}$, $\sin c_2 =$ $= \frac{k_2}{\sqrt{k_1^2 + k_1^2}}$, $c_1 := \sqrt{k_1^2 + k_2^2}$. Then

$$\frac{k_1 \sin\alpha(t) + k_2 \cos\alpha(t)}{\alpha(t)} = \frac{c_1}{\sqrt{\alpha'(t)}} \left[\cos c_2 \sin\alpha(t) + \sin c_2 \alpha(t)\right] =$$

$$= c_1 \frac{\sin(\alpha(t) + c_2)}{\sqrt{\alpha'(t)}}$$

(ii) $k_1^2 + k_2^2 = 0$. Then $k_1 = i\gamma k_2$, where $\gamma^2 = 1$. Setting $c_3 := k_2$ yields

$$\frac{k_1 \sin\alpha(t) + k_2 \cos\alpha(t)}{\sqrt{\alpha'(t)}} = c_3 \frac{\cos\alpha(t) + i\gamma \sin\alpha(t)}{\sqrt{\alpha'(t)}} =$$

$$= c_3 \frac{e^{i\gamma\alpha(t)}}{\sqrt{\alpha'(t)}}$$

Inserting this into (Q) readily verifies that the functions defined on j (7) and (8) are solutions of (Q) and this for every complex numbers $c_1 \neq 0$, c_2, $c_3 \neq 0$ and for a number γ, $\gamma^2 = 1$.

Corollary 4. Let α be a phase of (Q). Then

$$y(t) = \frac{1}{\sqrt{\alpha'(t)}} (c_1 e^{i\alpha(t)} + c_2 e^{-i\alpha(t)}), \quad t \in j, \ c_1, c_2 \in C ,$$

is the general solution of (Q).

Proof. By Theorem 5 the functions $y_1(t) := \dfrac{e^{i\alpha(t)}}{\sqrt{\alpha'(t)}}$, $y_2(t) := \dfrac{e^{-i\alpha(t)}}{\sqrt{\alpha'(t)}}$, $t \in j$, are solutions of (Q) and from $y_1' y_2 - y_1 y_2' = 2i$ then follows that y_1, y_2 are independent solutions of (Q).

Theorem 6. Let α be a phase of (Q), $t_0 \in j$. Then all solutions y of (Q) which may be written as in (7) are determined by the initial conditions either

$$y(t_0) = 0$$

or $y(t_0) = c \ (\neq 0)$, $y'(t_0) = c\left(\alpha'(t_0) \operatorname{cotg}(\alpha(t_0) + c_2) - \frac{1}{2} \frac{\alpha''(t_0)}{\alpha'(t_0)}\right)$

where $c_2 \in C$ *is such a number that* $\sin(\alpha(t_0) + c_2) \neq 0$ *and all solutions* y *of (Q) which may be written in the form of* (8) are *determined by the initial conditions* $y(t_0) = c (\neq 0)$, $y'(t_0) =$

$$= c\left(i\gamma\alpha(t_0) - \frac{1}{2} \frac{\alpha''(t_0)}{\alpha'(t_0)}\right) .$$

Proof. Suppose y is a solution of (Q). If $y(t_0)=0$, then y may be written in the form of (7), only. Let $y(t_0)=c(\neq 0)$ and y may be written in the form $y(t)=c_1 \frac{\sin(\alpha(t)+c_2)}{\sqrt{\alpha'(t)}}$,

where $c_1\neq 0$, c_2 are suitable numbers. Then $c=c_1 \frac{\sin(\alpha(t_0)+c_2)}{\sqrt{\alpha'(t_0)}}$

and $y'(t_0)=c_1\left(\alpha'(t_0)\operatorname{cotg}(\alpha(t_0)+c_2)-\frac{1}{2}\frac{\alpha''(t_0)}{\alpha'(t_0)}\right)$.

If y may be written in the form $y(t) = c_3 \frac{e^{i\gamma\alpha(t)}}{\sqrt{\alpha'(t)}}$ where $c_3\neq 0$ is a suitable number and $\gamma^2 = 1$, then

$$c = c_3 \frac{e^{i\gamma\alpha(t_0)}}{\sqrt{\alpha'(t_0)}} \text{ and}$$

$$y'(t_0) = c_3\left(i\gamma\alpha(t_0)-\frac{1}{2}\frac{\alpha''(t_0)}{\alpha'(t_0)}\right).$$

<u>Theorem 7</u>. <u>Let</u> y <u>be a solution of</u> (Q), $y(t)\neq 0$ <u>for</u> $t\in j$. <u>Then there exists a phase</u> α <u>of</u> (Q) <u>and</u> $(0\neq)\ k\in C$ <u>such that</u>

$$y(t) = k\,\frac{e^{i\alpha(t)}}{\sqrt{\alpha'(t)}}, \qquad t\in j. \tag{9}$$

<u>The converse is valid, too: Let</u> α <u>be a phase of</u> (Q), $(0\neq)k\in C$ <u>and</u> y <u>be defined by</u> (9). <u>Then</u> y <u>is a solution of</u> (Q) <u>and</u> $y(t)\neq 0$ <u>for</u> $t\in j$.

Proof. Suppose y is a solution of (Q), $y(t)\neq 0$ for $t\in j$. Then the existence of a solution u of (Q) follows from the proof of Corollary 1 saying that the solutions y, u are independent and $u(t)\neq 0$ for $t\in j$. Set $U:=\frac{1}{2i}(y+u)$, $V:=\frac{1}{2}(y-u)$.

Then U,V are independent solutions of (Q), $U^2(t)+V^2(t)\neq 0$ for $t\in j$. Let α be a phase of the basis (U,V) of (Q). Then

$U(t) = k\dfrac{\sin\alpha(t)}{\sqrt{\alpha'(t)}}$, $V(t) = k\dfrac{\cos\alpha(t)}{\sqrt{\alpha'(t)}}$, where $k\neq 0$ is an appropriate constant. It then follows

$$y(t) = V(t) + iU(t) = k\frac{\cos\alpha(t) + i\sin\alpha(t)}{\sqrt{\alpha'(t)}} = k\frac{e^{i\alpha(t)}}{\sqrt{\alpha'(t)}} .$$

The second part of the Theorem follows from Theorem (5).

Theorem 8. *Let y be a solution of (Q). Then there exist a phase α of (Q) and a number $c\neq 0$ such that*

$$y(t) = c\frac{\sin\alpha(t)}{\sqrt{\alpha'(t)}} , \quad t\in j .$$

Proof. Suppose there exists a number $t_0\in j$ such that $y(t_0)=0$. Let α_1 be a phase of (Q). Then the existence of numbers $c_1\neq 0$, c_2 such that $y(t) = c_1\dfrac{\sin(\alpha_1(t) + c_2)}{\sqrt{\alpha_1'(t)}}$, $t\in j$, follows from Theorems 5 and 6. To get the statement of the Theorem we set $\alpha := \alpha_1 + c_2$.

Let $y(t)\neq 0$ for $t\in j$. Let $t_0\in j$ and A be chosen such that $\left(\int_{t_0}^{t}\frac{ds}{y^2(s)} + A\right)^2 \neq -1$ for $t\in j$. Such an A always exists as it follows from Lemma 1. If we set $z(t):=y(t)\left(\int_{t_0}^{t}\frac{ds}{y^2(s)} + A\right)$, $t\in j$, then z is a solution of (Q) and $y^2(t) + z^2(t) = y^2(t)\left(1+\left(A+\int_{t_0}^{t}\frac{ds}{y^2(s)}\right)^2\right)\neq 0$. Suppose α is a phase of the basis (y,z) of (Q). Then there exists a $c\neq 0$ such that

$$y(t) = c\,\frac{\sin \alpha(t)}{\sqrt{\alpha'(t)}} \quad \text{for} \quad t \in j.$$

Theorem 9. There exist independent solutions u, v *of* (Q) *such that* $u(t)v(t) \neq 0$ *and* $u(t) \neq v(t)$ *for* $t \in j$.

Proof. Suppose α is a phase of (Q) and set $M := \{(x,y);\ x = \operatorname{Re}\alpha(t),\ y = \operatorname{Im}\alpha(t),\ t \in j\} \subset R \times R$. Following Lemma 1 $m(M)=0$, hence there exists a number d such that $\alpha(t) \neq d + k\pi$, $t \in j$, $k = 0, \pm 1, \pm 2, \dots$. If we set $\beta := \alpha - d$, then, by Theorem 5, the functions $\frac{e^{i\beta(t)}}{\sqrt{\beta'(t)}}$, $\frac{e^{-i\beta(t)}}{\sqrt{\beta'(t)}}$ are independent solutions of (Q). Evidently, $u(t)v(t) \neq 0$ for $t \in j$ and the equality $u(t)=v(t)$ holds for a $t=t_0$ ($\in j$) exactly if $e^{2i\beta(t_0)} = 1$, i.e. exactly if for an integer n, $\beta(t_0) = n\pi$, which is a contradiction. Consequently $u(t) \neq v(t)$ for $t \in j$.

5. Applying the theory of phases enables us to find concrete examples of equations of type (Q) whose solutions have some pregiven properties. Thus, there exist equations having exactly one solution (up to a multiplicative multiple) with an infinite number of zeros and every further solution has a finite number of zeros, only. This becomes readily apparent from the following example.

Example. Setting $\alpha(t) := t + \frac{i}{1+t^2}\sin t$, $t \in R$, it yields

$$\alpha'(t) = 1 + i\left(\frac{\cos t}{1+t^2} - \frac{2t \sin t}{(1+t^2)^2}\right) \neq 0 \text{ for } t \in R \text{ and } \alpha \in \tilde{C}^3(R).$$

Suppose $Q(t) := -\{\alpha, t\} - \alpha'^2(t)$, $t \in R$. It then follows from Theorem 3 that α is a phase of (Q) and we get from Theorems 5 and 6 that every solution of (Q) having a zero is to be sought in the form $c_1 \frac{\sin(\alpha(t) + c_2)}{\sqrt{\alpha'(t)}}$, where $c_1 (\neq 0)$, $c_2 \in C$. Investigating the zeros of solutions of (Q) leads therefore to investigating the roots of equation

$$\sin(\alpha(t) + a) = 0,$$

where $a \in C$. Then $\sin\alpha(k\pi) = \sin\left(k\pi + \frac{1}{1 + k^2\pi^2}\sin k\pi\right) = \sin k\pi = 0$ for every integer k, hence the solution $u(t) = \frac{\sin\alpha(t)}{\sqrt{\alpha'(t)}}$ of (Q) has an infinite number of zeros. Suppose $a = a_1 + ia_2 \neq 0$ and the equation $\sin(\alpha(t)+a_1+ia_2) = 0$ has an infinite number of solutions on R. Then there exists a sequence $\{t_n\}$, $t_n \in R$, $\lim_{n\to\infty} |t_n| = \infty$ and a sequence $\{s_n\}$ of integers s_n such that

$$\alpha(t_n) + a_1 + ia_2 = s_n\pi ,$$

so that

$$t_n + a_1 = s_n\pi ,$$

$$\frac{\sin t_n}{1+t^2} + a_2 = 0, \; n=1,2,3,\dots .$$

This yields $t_n = s_n\pi - a_1$, $\sin t_n = -a_2(1+t_n^2)$ whence it follows (as far as $a_2 \neq 0$) $\lim_{n\to\infty} | a_2(1+t_n^2)| = \infty$, which, however, contradicts the boundedness of the function sin t. If $a_2 = 0$, then $\sin t_n = \sin(s_n\pi - a_1) = -(-1)^{s_n}\sin a_1 = 0$, whence $a = a_1 = p\pi$, where p is an integer. Then, naturally, $\frac{\sin(\alpha(t)+a)}{\sqrt{\alpha'(t)}} = \frac{\sin(\alpha(t)+p\pi)}{\sqrt{\alpha'(t)}} = (-1)^p u(t)$. Consequently each solution of (Q) not being of the form $c\,\frac{\sin\alpha(t)}{\sqrt{\alpha'(t)}}$, where $(0 \neq)\; c \in C$ have an finite number of zeros on R.

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SOUHRN

FÁZE DIFERENCIÁLNÍ ROVNICE $y'' = Q(t)y$ S KOMPLEXNÍM KOEFICIENTEM Q REÁLNÉ PROMĚNNÉ

SVATOSLAV STANĚK

V jisté analogii s reálným případem je v práci zaveden pojem fáze rovnice

$$y'' = Q(t)y, \qquad (Q)$$

kde Q je spojitá komplexní funkce reálné proměnné t definovaná na intervalu $j:=(a,b)$ $(-\infty \leqq a < b \leqq \infty)$.

Je dokázáno, že každá rovnice (Q) má nezávislá řešení u, v splňující $u^2(t)+v^2(t)\neq 0$ pro $t \in j$ (věta 2). Na základě tohoto výsledku je zaveden pojem fáze rovnice (Q). Funkce α se nazývá fáze rovnice (Q) jestliže existují její nezávislo řešení u, v taková, že $u^2(t)+v^2(t)\neq 0$ a $\alpha'(t) = -\frac{w}{u^2(t)+v^2(t)}$ $(w := uv' - u'v)$ pro $t \in j$ a v bodě $t_0 \in j$, kde $v(t_0)\neq 0$ je $\operatorname{tg}\alpha(t_0) = \frac{u(t_0)}{v(t_0)}$.

V práci jsou nalezeny všechny fáze rovnice (Q) a užitím fáze rovnice (Q) je uveden tvar jejího obecného řešení. Dále je fáze využito při vyšetřování rozložení nulových bodů řešení rovnice (Q).

РЕЗЮМЕ

ФАЗА ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ $y'' = Q(t)\,y$ С КОМПЛЕКСНЫМ КОЭФФИЦИЕНТОМ Q ВЕЩЕСТВЕННОЙ ПЕРЕМЕННОЙ

СВАТОСЛАВ СТАНЕК

В некоторой аналогии с вещественным случаем вводится в работе понятие фазы уравнения

$$y'' = Q\,(t)\,y, \qquad (Q)$$

где Q - непрерывная комплексная функция вещественной переменной на интервале $j := (a,b)$ $(-\infty \leqq a < b \leqq \infty)$. Доказано, что каждое уравнение (Q) имеет независимые решения u, v такие, что $u^2(t)+v^2(t) \neq 0$ для $t \in j$ (теорема 2). На этом результате основано понятие фазы уравнения (Q). Функция α называется фазой уравнения (Q) если существуют её независимые решения u, v такие, что $u^2(t)+v^2(t) \neq 0$ и $\alpha'(t) = -\dfrac{w}{u^2(t)+v^2(t)}$ $(w := uv' - u'v)$ для $t \in j$ и в точке $t_0 \in j$ где $v(t_0) \neq 0$ имеет место $\operatorname{tg} \alpha\,(t_0) = \dfrac{u(t_0)}{v(t_0)}$.

В работе показаны все фазы уравнения (Q) и с помощью фазы приводится форма общего решения уравнения (Q). Далее использована фаза при исследовании розложения корней решений уравнения (Q).

RNDr. Svatoslav Staněk
přírodovědecká fakulta UP
Leninova 26
Olomouc
771 46

AUPO, Fac.r.nat.85, Mathematica XXV, (1986)