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ON THE „POINCARÉ–LYAPUNOV“ – LIKE SYSTEMS OF THREE DIFFERENTIAL EQUATIONS

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The problem of determining a periodic regime in many controllable processes modelled by nonlinear differential systems has been recently the subject of increasing attention. This paper consists of two parts. The first having a survey character gives a brief analysis of the theory developed in this field up to now, while the second establishes a new theorem generelizing some of the results commented.

By the "Poincaré-Lyapunov"-like systems (PLS) we shall as usually mean systems of the type

$$
\begin{aligned}
x' &= a_{11}x + a_{12}y + a_{13}z + p(t,x,y,z) \\
y' &= a_{21}x + a_{22}y + a_{23}z + q(t,x,y,z) \qquad \text{(PL)} \\
z' &= a_{31}x + a_{32}y + a_{33}z + r(t,x,y,z)
\end{aligned}
$$

with reals a_{ij} $(i,j = 1,2,3)$ and the smooth enough functions p, q,r satisfying a Lipschitz condition with a sufficiently small constant.

1. Such systems were probably studied for the first time by H. P o i n c a r é /1/, who did a complete classification of the singular points for the linear autonomous case just in the last century. But only since the fiftieths autonomous PLS in R^3 were subjected to a deeper consideration by the Soviet mathematicians, especially with respect to a study of "the Aizerman's problem"; i.e. practically (see e.g. /2/) the problem of the global asymptotic stability of a trivial solution of some special types of (PL). Namely N.N.K r a s o v s k i i /3/ studied one with three nonlinearities:

$$
\begin{aligned}
x' &= f(x) + a_{12}y + a_{13}z \\
y' &= F(x) + a_{22}y + a_{23}z \qquad (2)\\
z' &= \varphi(x) + a_{32} \ ,
\end{aligned}
$$

while A.P. T u z o v /5/, /6/ resp. V.A. P l i s s /7/ - /11/ (and A.P. T u z o v /4/ again) those with the single nonlinearity:

$$
(T)\quad
\begin{aligned}
x' &= a_{11}x + a_{12}y + a_{13}z \\
y' &= F(x) + a_{22}y + a_{23}z \\
z' &= a_{31}x + a_{32}y + a_{33}z
\end{aligned}
\quad \text{resp.} \quad (P)\quad
\begin{aligned}
x' &= f(x) + a_{12}y + a_{13}z \\
y' &= a_{21}x + a_{22}y + a_{23}z \\
z' &= a_{31}x + a_{32}y + a_{33}z.
\end{aligned}
$$

R. R e i s s i g /12/ - /14/ still completed and improved the proving techniques for some of these results in the case (T), but since the end of the sixtieths either PLS with at least two nonlinearities like (B) or PLS with the one involving two variables like (G) has started to be of main interest. This era is connected with names as M.A. B a l i t i n o v /15/ - /17/ for (B) and I.V. G a i s h u n /18/, /19/ (see also /20/) for (G), where

$$\text{(B)}\quad \begin{aligned} x' &= a_{11}x + g(y) + h(z) \\ y' &= a_{21}x + a_{22}y + a_{23}z \\ z' &= a_{31}x + a_{32}y + a_{33}z \end{aligned} \qquad \text{(G)}\quad \begin{aligned} x' &= a_{11}x + a_{12}y + a_{13}z \\ y' &= a_{21}x + a_{22}y + a_{23}z \\ z' &= \varphi(x,y) + a_{33}z \ . \end{aligned}$$

If the fiftieths may be called as "the one-nonlinearity PLS period", while the sixtieths as "the two-nonlinearities PLS one", the seventieths might be called as "the three-nonlinearities PLS period", because several authors like I.G. E g o r o v /21/, Z. Z a p r i a n o v - P. K a l i t z o v a /22/ this time studied some special types of the system

$$\begin{aligned} x' &= a_{11}x + a_{12}y + a_{13}z \\ y' &= a_{21}x + a_{22}y + a_{23}z \\ z' &= \varphi(x,y,z) \end{aligned}$$

and others, like e.g. V.I. S h i r i a e v /23/ the autonomous system (PL) with the perturbating functions depending on the only variable; namely $p(x)$, $q(y)$, $r(z)$.

Besides /13/, where the Popov's frequency criterium (cf. /2/) was employed, all the previous authors used the Lyapunov's second method, consisting of a construction of so the called Lyapunov functions (see e.g. /20/), sometimes combinated (especially by V.A. P l i s s - cf. e.g. /11/) by the geometrical techniques, for the solvability of the Aizerman's problem.

The autonomous system (PL), considered as the linear perturbed one, has been already examined very precisely by I. B i h a r i /24/ jointly with A. E l b e r t /25/ in order to describe asymptotic properties of its solutions. These authors proceeded, similarly as K.R. S ch n e i d e r /26/, /27/ for the general autonomous system in R^3, by the topological approach.

While K.R. S ch n e i d e r studied the problem of the existence of periodic solutions in the autonomous case, the same question appeared naturally in connection with investigating the periodically perturbed systems (T) resp. (P), too. The answer can be either partly deduced from the boundedness results by E.S. A n i t o v a /28/, /29/ and R. R e i s s i g /2/ ((T)-case) resp. W. M ü l l e r /31/ ((P)-case) obtained by means of Lyapunov-like functions again or derived directly employing e.g. a functionally-analytical approach, similarly as G. V i l l a r i /30/ ((P)-case) resp. the present author in the second section of this paper.

In closing let us supply that we omitted here many results dealing with third order equations corresponding to PLS and that there also exist two survey papers on the third order systems by W. M ü l l e r /32/ and G. S a n s o n e /33/ since the sixtieths.

2. In this part we consider the following system, with respect to the periodic solutions problem:

$$\begin{aligned} x' &= f(x) + p(t,x,y,z) \\ y' &= F(x) + G(y) + Cz + q(t) \qquad (1) \\ z' &= h(x) + g(y) + cz + r(t,x,y,z) \end{aligned}$$

where $F(x)$, $G(y)$, $q(t+\theta) \equiv q(t) \in C^1(R^1)/ \Rightarrow y \in C^2(R^1)/$; $f(x)$, $h(x)$, $g(y) \in C(R^1)$; $p(t+\theta, X) \equiv p(t,X)$, $r(t+\theta, X) \equiv$ $\equiv r(t,X) \in C(R^4)$; $X = (x,y,z)$; $\theta > 0, C, c \in R^1 = (-\infty, \infty)$.

Furthermore, let us assume for in (1) appearing functions, resp. for some of their derivatives, the boundedness by constants in the following way:

$$(2) \qquad \begin{array}{lll} |p(t,X)| \leq P, & |q(t)| \leq Q, & |q'(t)| \leq Q', \\ |r(t,X)| \leq R, & |F'(x)| \leq F', & |G'(y)| < \dfrac{2\pi}{\theta} := G', \end{array}$$

holding for all arguments and

$$(3) \qquad \begin{array}{ll} \displaystyle\int_0^\theta q(t)dt = 0, & \displaystyle\int_0^\theta q'(t)dt = 0 \ ; \\ \displaystyle\int_0^\theta p(t)dt = 0, \text{ when } p(t) \equiv p(t,X) \ . & \end{array}$$

Now let us consider a one-parameter family of systems:

$$(1_\mu) \qquad \begin{array}{ll} x' = (1-\mu)ax + \mu\left[f(x) + p(t,x,y,z)\right] \ \ldots\ldots\ldots & (1_\mu^1) \\ y' = (1-\mu)By + \mu\left[F(x) + G(y) + Cz + q(t)\right] \ \ldots & (1_\mu^2) \\ z' = (1-\mu)dz + \mu\left[h(x) + g(y) + cz + r(t,x,y,z)\right] & (1_\mu^3) \end{array}$$

with suitable nonzero constants a, B, d, which magnitude will be specified later, and the parameter μ being $\mu \in \langle 0,1 \rangle$.

It is clear that for $\mu = 1$ we get the original system (1); i.e. $(1) \sim (1_1)$, while for $\mu = 0$ the following three equations

$$(1_0) \qquad x' = ax, \qquad y' = By, \qquad z' = dz \quad (a \neq 0,\ B \neq 0,\ d \neq 0),$$

from which no one has a nontrivial Θ-periodic solution and hence, according to the classical "Leray-Schauder fixed point technique" (see e.g. /2/), the sufficient condition for the existence of a Θ-periodic solution of (1) will be: an a priori estimate of all the Θ-periodic solutions of (1_μ), independently on $\mu \in (0,1\rangle$.

T h e o r e m .

The system (1) admits a Θ-periodic solution, provided (2), (3), $C \neq 0$ and

1) $\liminf_{|x|\to\infty} f(x)\operatorname{sgn} x > P$ resp. $\limsup_{|x|\to\infty} f(x)\operatorname{sgn} x < -P$

/P:= 0, when $p(t) \equiv p(t,X)$/,

2) $\lim_{|y|\to\infty} |cBy + cG(y) - Cg(y)| = \infty$,

3) $|c + G'(y)| \geq \delta >$ 0...const. for all $y \in R^1$.

P r o o f.

I. Let us start from the equation (1^1_μ) multiplied by $x'(t)$, where $x(t+\Theta) \equiv x(t)$ is the first component of a fixed solution of the system (1_μ). Integrating over one period, we obtain, with respect to (2) that

$$\int_0^\Theta x'^2(t)dt \leq |\mu\int_0^\Theta p(t,X(t))x'(t)| \leq P\sqrt{\Theta}\sqrt{\int_0^\Theta x'^2(t)dt}; \text{ i.e.}$$

(4) $$\int_0^\Theta x'^2(t)dt \leq P^2\Theta \quad ,$$

while integrating (1^1_μ) we get, with respect to (2):

$$\left|\frac{(1-\mu)}{\mu} a \int_0^\Theta x(t)dt + \int_0^\Theta f(x(t))dt\right| = \left|\int_0^\Theta p(t,X(t))dt\right| \leq P\Theta.$$

Since the last relation implies together with 1) that (for more details see e.g. /34/)

$$\min_{t\in\langle 0,\Theta\rangle} |x(t)| := |x(t_1)| \leq D_o$$

holds evidently with a suitable constant D_o, it will be satisfied according to (4) also

$$|x(t)| \leq |x(t_1)| + \left|\int_0^\Theta x'(t)dt\right| \leq D_o + \sqrt{\Theta}\sqrt{\int_0^\Theta x'^2(t)dt} \leq D_o + \Theta P := D_1;$$

i.e.

(5) $$|x(t)| \leq D_1$$

and furthermore with respect to (1^1_μ), (2), (5):

(6) $$|x'(t)| \leq |aD_1| + \max_{|x|\leq D_1} |f(x)| + P := D_1' .$$

II. Let us consider the last two equations from (1_μ); i.e. (1^2_μ) and (1^3_μ), from which we get for $C \neq 0$ that

$$y'' = (1-\mu)By' + \mu\Big\{F'(x)x' + G'(y)y' + \mu C\big[h(x) + g(y) + r(t,X)\big] + \big[(\mu-1)d-c\big]\Big[\frac{(1-\mu)}{\mu} By + F(x) + G(y) + q(t) - \frac{y'}{\mu}\Big] + q'(t)\Big\} ;$$

i.e.

$$y'' - \left[c + (1-\mu)(B + d) + \mu G'(y)\right] y' - \mu^2 Cg(y) + \left[(1-\mu)d + c\right] \cdot$$

$$(7) \qquad \cdot \left[(1-\mu)By + \mu G(y)\right] = \mu\{F'(x(t))x'(t) +$$

$$+ \mu C\left[h(x(t)) + r(t,X(t))\right] + \left[(\mu-1)d-c\right]\left[F(x(t)) +\right.$$

$$\left. + q(t)\right] + q'(t)\} := \mu Q(t).$$

Multiplying (7) by $y'(t)$, where $y(t+\Theta) \equiv y(t)$ is the second component of a fixed solution of (1μ), we get after integration over the period Θ that

$$\int_0^\Theta \left[c + (1-\mu)(B + d) + \mu G'(y(t))\right] y'^2(t)dt = \mu\int_0^\Theta Q(t)y'(t)dt$$

and from here under 3) and with suitably chosen constants B,d ($|B|$, $|d|$ - small enough) firstly

$$(8) \qquad |c + (1-\mu)(B + d) + \mu G'(y(t))| := |H_\mu(y)| \geq \mu\frac{\delta}{2} > 0$$

and then with respect to (2), (5) - (8):

$$\mu\frac{\delta}{2}\int_0^\theta y'^2(t)dt \leq \int_0^\theta |H_\mu(y(t))y'^2(t)|\,dt \leq \mu\int_0^\theta |Q(t)y'(t)|\,dt \leq$$

$$\leq \mu\left[F'D_1' + |c|(H+R) + (|d| + |c|)(F+Q) +\right.$$

$$\left. + Q'\right]\sqrt{\Theta}\cdot\sqrt{\int_0^\theta y'^2(t)dt} := \mu\mathcal{X}\sqrt{\Theta}\sqrt{\int_0^\theta y'^2(t)dt}\ ,$$

where

$$(9)\qquad F := \max_{|x| \leq D_1} |F(x)| \quad , \qquad H := \max_{|x| \leq D_1} |h(x)| \qquad ; \text{ i.e.}$$

$$(10)\qquad \int_0^{\Theta} y'^2(t)dt \qquad \frac{4\chi^2\Theta}{\delta^2} \; := D_2'^2 \quad .$$

Furthermore, after integration (7) over the period Θ , we get for (2), (3), (9) that

$$|- \mu^2 c \int_0^{\Theta} g(y(t))dt + [(1-\mu)d + c]\,[(1-\mu)B \int_0^{\Theta} y(t)dt +$$

$$+ \mu \int_0^{\Theta} G(y(t))dt]| = |\mu \{\mu c \int_0^{\Theta} [h(x(t)) + r(t,X(t))]dt -$$

$$- [(1-\mu)d + c] \int_0^{\Theta} F(x(t))dt\}| \leq \mu\Theta[\mu|c|(H + R) +$$

$$+ (1-\mu)(|d| + |c|)F] \; ,$$

which surely implies under 2) the existence of such constants D_0', D_2 that

$$\min_{t \in \langle 0,\Theta \rangle} |y(t)| \; := |y(t_2)| \leq D_0$$

and consequently (cf. (10))

$$|y(t)| \leq |y(t_2)| + |\int_0^{\Theta} y'(t)dt| \leq D_0' + \sqrt{\Theta}\sqrt{\int_0^{\Theta} y'^2(t)dt} \leq$$

$$\leq D_0' + \Theta D_2' := D_2$$

holds; i.e.

$$(11)\qquad |y(t)| \leq D_2 \quad .$$

III. Now multiplying equation (7) by $y''(t)$, we get after integration over the period Θ that the following is satisfied with respect to (2), (6), (7), (9) - (11):

$$\int_0^\Theta y''^2(t)dt = \Big|\mu\int_0^\Theta G'(y(t))y'(t)y''(t)dt + \mu^2 C\int_0^\Theta g(y(t))y''(t)dt +$$

$$+ \big[(1-\mu)d+c\big]\Big[(1-\mu)B\int_0^\Theta y'^2(t)dt - \mu\int_0^\Theta G(y(t))y''(t)dt\Big] +$$

$$+ \mu\int_0^\Theta Q(t)y''(t)dt\Big| \leq Q_1 + Q^+\int_0^\Theta |y''(t)|dt +$$

$$+ G'\int_0^\Theta |y'(t)y''(t)|\,dt \quad ,$$

where

$$(12) \qquad g := \max_{|y|\leq D_2}|g(y)| \quad , \qquad G := \max_{|y|\leq D_2}|G(y)| \quad , \qquad G' := \max_{|y|\leq D_2}|G'(y)|$$

and

$$Q^* := F'D_1' + |C|(H+R) + (|d| + |c|)(F+Q) + Q'$$

$$Q^+ := |Cg| + (|d| + |c|)G + G^* \quad , \quad Q_1 := B\Theta^2 D_2' \quad ,$$

which gives by means of the Wirtinger-type (cf.e.g. /35/) and Schwarz inequalities:

$$\int_0^\Theta y''^2(t)dt \leq \frac{Q_1 + Q^+\sqrt{\Theta}\sqrt{\int_0^\Theta y''^2(t)dt}}{1 - G'\dfrac{\Theta}{2\pi}}$$

resp. (cf.(2)) implies the existence of such a constant D_2'' that

$$\int_0^{\Theta} y^{\cdot\cdot 2}(t)dt \leq D_2^{\cdot\cdot} \qquad \text{holds.}$$

Since it is, however, always satisfied the relation

$$\max_{t\in\langle 0,\Theta\rangle} y^{\cdot 2}(t) \leq \Theta\int_0^{\Theta} y^{\cdot\cdot 2}(t)dt \ ,$$

then

$$(13) \qquad |y^{\cdot}(t)| \leq \sqrt{\Theta D_2^{\cdot\cdot}}$$

must be true and consequently (cf. (1^2_{jn}), (2), (5), (9), (11) - (13)):

$$(14) \qquad |z(t)| \leq \frac{1}{C}\left[\sqrt{\Theta D_2^{\cdot\cdot}} + BD_2 + F + G + Q\right] \ := D_3 \ .$$

Thus (5), (11), (14) obviously imply the existence of such a constant $D := 3\max(D_1, D_2, D_3)$ with

$$\max_{t\in\langle 0,\Theta\rangle}\left[|x(t)| + |y(t)| + |z(t)|\right] \leq D \ldots /X(t+\Theta) \equiv X(t)/,$$

which completes the proof.

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SOUHRN

O SYSTÉMECH TŘÍ DIFERENCIÁLNÍCH ROVNIC "POINCARÉHO-LJAPUNOVOVA" TYPU

JAN ANDRES

Pomocí Leray-Schauderovy alternativy je nalezeno dosti obecné kriterium existence periodického řešení soustavy (1). Rovněž jsou uvedeny dosud dosažené základní výsledky o zvláštních případech soustavy (PL), z nichž mnohé byly v této práci zobecněny.

РЕЗЮМЕ

О СИСТЕМАХ ТРЕХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ТИПА "ПУАНКАРЕ-ЛЯПУНОВА"

ЯН АНДРЕС

В работе дается достаточно общее условие существования периодического решения системы (1) при помощи альтернативы Лере-Шаудера. Представлены тоже основные результаты, достиженные до сих пор для частых случаев системы (PL), многие из которых были обобщены в этой работе.

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