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ON INFLECTION POINTS OF THE PLANE QUARTIC WITH TWO NODAL AND ONE FLEXNODAL POINTS

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Let a complex projective plane $\mathbf{S}_2$ be given and let us consider an irreducible quartic $(\mathbf{K})$ in $\mathbf{S}_2$ containing three nodal-points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ one of them, say $\mathbf{A}$, be the flexnode, i.e. $\mathbf{A}$ be the intersection point of multiplicity 4 for the curve $(\mathbf{K})$ and for any tangent of $(\mathbf{K})$ at $\mathbf{A}$. We may also say: $\mathbf{A}$ is the centre of two different linear places of $(\mathbf{K})$ each of them is of class 2. Using the second Plücker's formula, we establish, the inflex-point divisor $\mathbf{I}$ on $(\mathbf{K})$ to be of order 4. There is a natural question, under what conditions the divisor $\mathbf{I}$ is determined by a linear form on $(\mathbf{K})$. This article is devoted to the solution of this problem. Let us note, that if the flexnodal point will be replaced by a cuspidal point, the divisor $\mathbf{I}$ will also be of order 4 and will not be determined by a linear form as proved in [1].

Now, let us choose the coordinate frame $(\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2, \mathbf{E})$ in S_2 so that

$$\mathbf{A}_0 = \mathbf{A}, \qquad \mathbf{A}_1 = \mathbf{B}, \qquad \mathbf{A}_2 = \mathbf{C}.$$

Then the quartic $(\mathbf{K})$ will be expressed by the equation:

$$a_0x_1^2x_2^2 + a_1x_0^2x_2^2 + a_2x_0^2x_1^2 + 2b_0x_0^2x_1x_2 + 2b_1x_0x_1^2x_2 + 2b_2x_0x_1x_2^2 = 0, \quad (1)$$

with $a_0 \neq 0$, $a_1 \neq 0$, $a_2 \neq 0$ (cf. [1] pg. 229). Let us rewrite (1) in the form:

$$x_0^2(a_2x_1^2 + 2b_0x_1x_2 + a_1x_2^2) + x_1x_2(2b_1x_0x_1 + a_0x_1x_2 + 2b_2x_0x_2) = 0. \quad (2)$$

Then (2) shows that any common solution of

$$a_2x_1^2 + 2b_0x_1x_2 + a_1x_2^2 = 0, \quad (3)$$

(the equation of tangents at $\mathbf{A}$) and of

$$2b_1x_0x_1 + a_0x_1x_2 + 2b_2x_0x_2 = 0 \quad (4)$$

(the equation of certain conic containing **A** and as well as **B** and **C**) is also the common solution of (3) and (1). However, as **A** is a flexnode, then (1) and (3) have just one common solution namely the point $\mathbf{A} = (1, 0, 0)$; consequently, it must be $b_1 = b_2 = 0$ and (**K**) has with respect to the reper $(\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2, \mathbf{E})$ the equation:

$$(\mathbf{K}): \qquad a_0x_1^2x_2^2 + a_1x_0^2x_2^2 + a_2x_0^2x_1^2 + 2b_0x_0^2x_1x_2 = 0, \tag{5}$$

with $a_0 \neq 0$, $a_1 \neq 0$, $a_2 \neq 0$ and moreover $b_0^2 - a_1a_2 \neq 0$ (in the opposite case the tangents at **A** would not be distinct).*

By a simple calculating we establish the Hessian **H** of the form (**K**) (the left side of (5)):

$$\begin{aligned} \mathbf{H} = 24[(b_0^2 - a_1a_2)\, x_0^4(a_2x_1^2 + 2b_0x_1x_2 + a_1x_2^2) - \\ - a_0a_2x_1^4(a_2x_0^2 + a_0x_2^2) - a_1a_0x_2^4(a_1x_0^2 + a_0x_1^2) - \\ - 2a_0^2b_0x_1^3x_2^3 + 2a_0a_2b_0x_0^2x_1^3x_2 + 2a_0a_1b_0x_0^2x_1x_2 + 6a_0a_1a_2x_0^2x_1^2x_2^2]. \end{aligned} \tag{6}$$

(For the calculation see also [1] pg. 229.)

Let us investigate the common points of the divisor (**H**) and the coordinate axises. For the axis $x_0 = 0$, we get the equation

$$-a_0^2a_2x_1^4x_2^2 - 2a_0^2b_0x_1^3x_2^3 - a_0^2a_1x_1^2x_2^4 = 0.$$

The solution $x_1^2 = 0$ and $x_2^2 = 0$ lead to the points $\mathbf{A}_2 = \mathbf{C}$, $\mathbf{A}_1 = \mathbf{B}$. The coordinates x_1, x_2 of the remaining common points of (**H**), say $\mathbf{X}_0, \mathbf{Y}_0$ under the condition $x_0 = 0$, are determined by

$$a_2x_1^2 + 2b_0x_1x_2 + a_1x_2^2 = 0. \tag{7_0}$$

Let $\mathbf{X}_1, \mathbf{Y}_1$ and $\mathbf{X}_2, \mathbf{Y}_2$ respectively have a similar meaning for the divisor (**H**) and the coordinate axis $x_1 = 0$ resp. $x_2 = 0$. Then the coordinates x_0, x_2 of $\mathbf{X}_1, \mathbf{Y}_1$ resp. the coordinates x_0, x_1 of $\mathbf{X}_2, \mathbf{Y}_2$ are determined as a solution of equation

$$(b_0^2 - a_1a_2)\, x_0^2 - a_0a_1x_2^2 = 0, \tag{7_1}$$

resp.

$$(b_0^2 - a_1a_2)\, x_0^2 - a_0a_2x_1^2 = 0. \tag{7_2}$$

It follows from (7_0), (7_1), (7_2) that all pairs $\mathbf{X}_0, \mathbf{Y}_0, \mathbf{X}_1, \mathbf{Y}_1, \mathbf{X}_2, \mathbf{Y}_2$ are parwice different and distinct from **A**, **B**, **C**. Moreover, (7_1) and (7_2) imply that the cross-ratios $(\mathbf{ACX}_1\mathbf{Y}_1)$ and $(\mathbf{ABX}_2\mathbf{Y}_2)$ equal to -1.

The six points $\mathbf{X}_0, \mathbf{Y}_0, \mathbf{X}_1, \mathbf{Y}_1, \mathbf{X}_2, \mathbf{Y}_2$ are contained on a certain conic (**Q**) namely

$$(\mathbf{Q}): \qquad (b_0^2 - a_1a_2)\, x_0^2 - a_0a_2x_1^2 - a_0a_1x_2^2 - 2a_0b_0x_1x_2 = 0. \tag{8}$$

Using (6), (5) and (8) we find that

$$\mathbf{H} - 24\mathbf{QK} = 72a_0x_0^2x_1x_2[2a_2b_0x_1^2 + (3a_1a_2 + b_0^2)\, x_1x_2 + 2a_1b_0x_2^2]. \tag{9}$$

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*) Remark: We get thus a further geometrical meaning of the condition $b_1 = b_2 = 0$: The tangents at the nodal points **B**, **C** separate the lines ***BA***, ***BC***, resp. ***CA***, ***CB*** harmonically.

Relation (9) shows that the inflection points of (K) lie on the singular conic

$$(\mathbb{Q}_1): \qquad 2a_2b_0x_1^2 + (3a_1a_2 + b_0^2)\,x_1x_2 + 2a_1b_0x_2^2 = 0, \tag{10}$$

more exactly: Denoting by P, P' both (linear) places of (K) with the centre **A**, then the form (Q) on the left side of (10) determines the divisor

$$2\mathsf{P} + 2\mathsf{P}' + \mathsf{I} \tag{11}$$

on (K). Any of the components l_1, l_2 of $(\mathbb{Q}_1)$ determines the divisor

$$\mathsf{P} + \mathsf{P}' + \mathsf{D} \quad \text{resp.} \quad \mathsf{P} + \mathsf{P}' + \mathsf{D}', \tag{12}$$

with

$$\mathsf{D} + \mathsf{D}' = \mathsf{I} \tag{13}$$

on (K). It follows from this that I may not be determined by any linear form.

We conclude by

Theorem: *Let* (**K**) *be a plane quartic with two nodal-points and by a single flexnodal-point. Let* P *and* P' *be its places having the flexnode as the centre and let* I *be the inflex-point divisor. Then* I *is of order 4 and there exists a conic* $(\mathbb{Q}_1)$ *whose singular point is the flexnode of* (**K**) *determining the divisor* 2P + 2P' + I *on* (**K**). *There does not exist any line determining the divisor* I *on* (**K**); *the divisor* I *is of the form* (*13*), *where* (*12*) *are the divisors intersected by components of* $(\mathbb{Q}_1)$ *on* (**K**).

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INFLEXNÍ BODY ROVINNÉ KVARTIKY SE DVĚMA UZLOVÝMI A JEDNÍM FLEKTNODÁLNÍM BODEM

Souhrn

V článku je vyšetřován divizor inflexních bodů trinodální rovinné kvartiky, jejíž právě jeden uzlový bod je navíc flektonodální, tj. je středem dvou lineárních větví druhé třídy.

ТОЧКИ ПЕРЕГИБА АЛГЕБРАИЧЕСКОЙ КРИВОЙ 4-ОГО ПОРЯДКА ОБЛЯДАЮЩЕЙ ДВУМЯ ТОЧКАМИ САМОПЕРЕСЕЧЕНИЯ И ОДНОЙ ТОЧКОЙ КОТОРАЯ ЦЕНТРОМ ДВУХ ЛИНЕЙНЫХ ВЕТВЕЙ ВТОРОГО КЛАССА

Резюме

В статье рассматривается дивизор точек перегиба плоской кривой порядка 4 обладающей тремя особыми точками самопересечения, одна из них является центром двух линейных ветвей второго класса.