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Katedra matematické analýzy a numerické matematiky přírodovědecké fakulty University Palackého v Olomouci
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ON PROPERTIES OF DERIVATIVES OF THE BASIC CENTRAL DISPERSION IN AN OSCILLATORY EQUATION $y'' = q(t)y$ WITH AN ALMOST PERIODIC COEFFICIENT q

SVATOSLAV STANĚK

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1. Introduction

The distribution of zeros in solutions of a differential equation

$$y'' = q(t)\,y, \qquad q \in C^0(\mathbf{R}), \tag{q}$$

may be described through the basic central dispersion φ of (q). O. Borůvka proved in [3] the function $\varphi(t) - t$, $\varphi'(t)$, $\varphi''(t)$ and $\varphi'''(t)$ to be π-periodic provided the coefficient q of (q) is a π-periodic function. In [6] the function $\varphi(t) - t$ was proved to be almost periodic if the coefficient q of (q) is an almost periodic function. The present pader demontrates

Theorem 1. *Let φ be the basic central dispersion of an oscillatory equation* (q) *with an almost periodic coefficient q. Then also*

$$\varphi^{(i)}(t), \qquad i = 1, 2, 3,$$

are almost periodic functions.

2. Basic concepts and lemmas

A equation (q) is called oscillatory if $\pm\infty$ are the cluster points of the roots relative to every nontrivial solution of this equation. All equation of the type (q) considered below are assumed to be oscillatory. The trivial solution of (q) will not be considered.

A function $\alpha \in C^0(\mathbf{R})$ is called (first) phase of (q) if there exist its independent solutions u, v such that

$$\operatorname{tg} \alpha(t) = u(t)/v(t) \qquad \text{for } t \in \mathbf{R} - \{t;\, v(t) = 0\}.$$

Every phase α of (q) possesses the following properties:

$$\alpha \in C^3(\mathbf{R}); \qquad \alpha(\mathbf{R}) = \mathbf{R}; \qquad \alpha'(t) \neq 0 \qquad \text{for } t \in \mathbf{R}.$$

Let α be a phase of (q) and put $\varphi(t) := \alpha^{-1}[\alpha(t) + \pi \operatorname{sign} \alpha']$, $t \in \mathbf{R}$. The function φ is called the basic (first kind) central dispersion of (q) and we have

$$\varphi \in C^3(\mathbf{R}); \qquad \varphi(t) > t, \qquad \varphi'(t) > 0 \qquad \text{for } t \in \mathbf{R},$$

(see [2], [3]).

Let us recall at this point that a function $f \in C^0(\mathbf{R})$ is called almost periodic (see e.g. [5]), if there exists to every $\varepsilon > 0$ a positive number $L(= L(\varepsilon))$, such that there exists at least one number τ, on every interval $\langle x, x + L)$ $(x \in \mathbf{R})$, for which

$$| f(t + \tau) - f(t) | < \varepsilon \qquad \text{for } t \in \mathbf{R}.$$

Lemma 1. *Let $q_n \in C^0(\mathbf{R})$ and $\lim_{n\to\infty} q_n(t) = q(t)$ uniformly on* **R**. *Then there exist phases α_n and α of* ($\mathrm{q_n}$) *and* (q), *respectively, such that* $\operatorname{sign} \alpha'_n = \operatorname{sign} \alpha' = 1$ *and*

$$\lim_{n\to\infty} \alpha_n^{(i)}(t) = \alpha^{(i)}(t), \qquad i = 0, 1, 2, 3,$$

uniformly on every compact interval.

Proof. Let u_n, v_n be solutions of ($\mathrm{q_n}$) and u, v be solutions of (q): $u_n(0) = u(0) = v'_n(0) = v'(0) = 0$, $u'_n(0) = u'(0) = v_n(0) = u(0) = 1$. Let us put

$$\beta_n(t) := 1/(u_n^2(t) + v_n^2(t)), \qquad \beta(t) := 1/(u^2(t) + v^2(t)), \qquad t \in \mathbf{R}.$$

Since $\lim_{n\to\infty} u_n^{(i)}(t) = u^{(i)}(t)$, $\lim_{n\to\infty} v_n^{(i)}(t) = v^{(i)}(t)$, $(i = 0, 1, 2)$, uniformly on every compact interval (see [4], Theorem 2.4.) then

$$\lim_{n\to\infty} \beta_n^{(i)}(t) = \beta^{(i)}(t), \qquad i = 0, 1, 2,$$

uniformly on every compact interval. Let us put

$$\alpha_n(t) := \int_0^t \beta_n(s)\,\mathrm{d}s, \qquad \alpha(t) := \int_0^t \beta(s)\,\mathrm{d}s, \qquad t \in \mathbf{R}.$$

Then α_n is a phase of ($\mathrm{q_n}$) and α is a phase of (q) (see [2]) possessing the properties presented in Lemma 1.

Remark 1. Lemma 1 has been proved in [6] in a special case with $i = 0, 1$.

Lemma 2. *Let $q_n \in C^0(\mathbf{R})$ and $\lim_{n\to\infty} q_n(t) = q(t)$ uniformly on* **R**. *Let $\varphi_{(n)}$ and φ be the basic central dispersions of* ($\mathrm{q_n}$) *and* (q), *respectively. Then*

$$\lim_{n\to\infty} \varphi_{(n)}^{(i)}(t) = \varphi^{(i)}(t), \qquad i = 0, 1, 2, 3,$$

uniformly on every compact interval.

Proof. The case with $i = 0$ has been proved in [6]. Let α_n be a phase of (q_n) and α be a phase of (q) possessing the properties stated in Lemma 1. By differentiating the equalitues

$$\alpha_n[\varphi_{(n)}(t)] = \alpha_n(t) + \pi, \qquad \alpha[\varphi(t)] = \alpha(t) + \pi,$$

we obtain

$$\alpha_n'[\varphi_{(n)}(t)]\, \varphi_{(n)}'(t) = \alpha_n'(t), \qquad \alpha'[\varphi(t)]\, \varphi'(t) = \alpha'(t),$$

whence

$$\varphi_{(n)}'(t) - \varphi'(t) = \frac{\alpha_n'(t)}{\alpha_n'[\varphi_{(n)}(t)]} - \frac{\alpha'(t)}{\alpha'[\varphi(t)]}, \qquad t \in \mathbf{R}. \tag{1}$$

From (1) immediately follows the assertion of the Lemma for $i = 1$. With the properties of α_n and α, and by differentiating (1), we become the assertion of the Lemma for $i = 2, 3$.

Lemma 3. *Let φ be the basic central dispersion of* (q) *with an almost periodic coefficient q. Then the composite function $q[\varphi(t)]$ is also almost periodic.*

Proof. Let $\{t_n\}$ be an arbitrary sequence of numbers. According to our assumption, the function q is almost periodic and by [6] such also is the function $d(t) := \varphi(t) - t$, $t \in \mathbf{R}$. Thus, we may choose from $\{t_n\}$ a subsequence – denoted again by $\{t_n\}$ – such that $\lim_{n\to\infty} q(t + t_n) = p(t)$, $\lim_{n\to\infty} d(t + t_n) = s(t)$ uniformly on $\mathbf{R}$. We will show that

$$\lim_{n\to\infty} q[\varphi(t + t_n)] = p[t + s(t)] \qquad \text{uniformly on } \mathbf{R}. \tag{2}$$

Let $\varepsilon > 0$ be an arbitrary number. Since q is uniformly continuous on $\mathbf{R}$, there exists a $\delta(= \delta(\varepsilon)) > 0$, such that $| q(x + \Delta x) - q(x) | < \frac{\varepsilon}{2}$, $x \in \mathbf{R}$, for every Δx, $| \Delta x | < \delta$. Let N be such a positive integer whereby for every $n \geqq N$: $| q(t + t_n) - p(t) | < \frac{\varepsilon}{2}$, $| d(t + t_n) - s(t) | < \delta$, $t \in \mathbf{R}$. Then for $n \geqq N$ and $t \in \mathbf{R}$

$$| q[\varphi(t + t_n)] - p[t + s(t)] | = | q[t + t_n + d(t + t_n)] - p[t + s(t)] | \leqq$$
$$\leqq | q(t + t_n + d(t + t_n)] - q(t + t_n + s(t)] | + | q(t + t_n + s(t)] - p[t + s(t)] | <$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We see that (2) holds. It follows from the continuity of q and from Bohr – Bochner's theorem (cf. [5]) that the function $q[\varphi(t)]$ is almost periodic.

Lemma 4. *Let φ be the basic central dispersion of* (q) *with an almost periodic coefficient q. Then there exist positive numbers k, K, such that*

$$k \leqq \varphi(t) - t \leqq K, \qquad t \in \mathbf{R}.$$

Proof. With respect to Lemma 3 ([6]) it suffices to prove only the inequality

$$k \leqq \varphi(t) - t \qquad \text{for } t \in \mathbf{R},$$

where $k > 0$ is a constant. Assume, there exists a $\{t_n\} : \lim\limits_{n\to\infty} [\varphi(t_n) - t_n] = 0$. For definiteness let, for example, $\lim\limits_{n\to\infty} t_n = \infty$. Let u_n be solutions of (q): $u_n(t_n) = 0$, $u_n'(t_n) = 1$. According to our assumption, q is almost periodic function, which enables us to choose from $\{q(t + t_n)\}$ a subsequence $\{q(t + t_{n_k})\}$ such that $\lim\limits_{k\to\infty} q(t + t_{n_k}) = p(t)$ uniformly on $\mathbf{R}$. Put $v_{n_k}(t) := u_{n_k}(t + t_{n_k})$ for $t \in \mathbf{R}$. Then v_{n_k} is a solution if the equation $y'' = q(t + t_{n_k})\, y$, $v_{n_k}(0) = 0$, $v_{n_k}'(0) = 1$ and $\lim\limits_{n\to\infty} v_{n_k}^{(i)}(t) = v^{(i)}(t)$ uniformly on every compact interval, $(i = 0, 1)$, v being a solution of (p), $v(0) = 0$, $v'(0) = 1$. Since $v_{n_k}[\varphi(t_{n_k}) - t_{n_k}] = 0$, there exists a number $\tau_k \in (0, \varphi(t_{n_k}) - t_{n_k}) : v_{n_k}'(\tau_k) = 0$. Because of $\lim\limits_{\kappa\to\infty} \tau_k = 0$ necessarily $v'(0) = \lim\limits_{k\to\infty} v_{n_k}'(\tau_k) = 0$, which is a contradiction.

Lemma 5. *Let φ be the basic central dispersion of* (q) *with an almost periodic coefficient q. Then there exist positive numbers a, b, c, such that*

$$a \leqq \varphi'(t) \leqq b, \qquad t \in \mathbf{R}, \tag{3}$$

$$|\varphi^{(i)}(t)| \leqq c, \qquad t \in \mathbf{R}; \qquad i = 2, 3. \tag{4}$$

Proof. Let $t_0 \in \mathbf{R}$ and u be a solution of (q): $u(t_0) = 0$, $u'(t_0) = 1$. Then

$$\varphi'(t_0) = 1/u'^2[\varphi(t_0)], \tag{5}$$

(cf. [2]). Assume (3) not valid. As an example, let $\limsup\limits_{t\to\infty} \varphi'(t) = \infty$. Then there exists $\{t_n\}$, $\lim\limits_{n\to\infty} t_n = \infty$ such that $\lim\limits_{n\to\infty} \varphi'(t_n) = \infty$. From (5) we obtain the existence $\{u_n\}$ solutions of (q), $u_n(t_n) = 0$, $u_n'(t_n) = 1$, such that

$$\lim_{n\to\infty} u_n'[\varphi(t_n)] = 0. \tag{6}$$

Put $x_n := \varphi(t_n)$, $n = 1, 2, \ldots$ The fact that we may choose from $\{q(t + x_n)\}$ a subsequence uniformly convergent on $\mathbf{R}$ enables us to assume without any loss of generality that

$$\lim_{n\to\infty} q(t + x_n) = p(t) \qquad \text{uniformly on } \mathbf{R}. \tag{7}$$

If we put $v_n(t) := u_n(t + x_n)$, $t \in \mathbf{R}$, $(n = 1, 2, \ldots)$, then v_n is a solution of the equation $y'' = q(t + x_n)\, y$, $v_n(t_n - x_n) = 0$, $v_n'(t_n - x_n) = 1$, $\lim\limits_{n\to\infty} v_n'(0) = 0$. It follows from Lemma 4 that we may choose from $\{x_n - t_n\} = \{\varphi(t_n) - t_n\}$ a convergent subsequence $\{x_{n_k} - t_{n_k}\} : \lim\limits_{k\to\infty} (x_{n_k} - t_{n_k}) = \alpha$, where $\alpha > 0$. From (7) we obtain $\lim\limits_{k\to\infty} v_{n_k}^{(i)}(t) = v^{(i)}(t)$ uniformly on every compact interval, $(i = 0, 1)$, where v is a solution of (p), $v(\alpha) = 0$, $v'(\alpha) = 1$, $v'(0) = 0$. On account of $u_{n_k}[\varphi(t_{n_k})] = 0$, we have $v(0) = 0$. Then, naturally, $v = 0$, which is a contradiction.

We proceed similarly for $\limsup_{t\to-\infty} \varphi'(t) = \infty$, $\liminf_{t\to\infty} \varphi'(t) = 0$, $\liminf_{t\to-\infty} \varphi'(t) = 0$.

Let us pass to the proof of inequality (4) for $i = 2$. Let this inequality invalid. Formula (6) in [1] yields for every $t_0 \in \mathbf{R}$: $\varphi''(t_0) = 2v^3[\varphi(t_0)]\, v'[\varphi(t_0)]$, v being a solution of (q), $v(t_0) = 1$, $v'(t_0) = 0$. Consequently, there exists a $\{t_n\}$, $\lim_{n\to\infty} |t_n| = \infty$ such that $\lim_{n\to\infty} |\varphi''(t_n)| = \infty$. For definiteness we assume $\lim_{n\to\infty} t_n = \infty$. Let v_n be solutions of (q), $v_n(t_n) = 1$, $v_n'(t_n) = 0$. Then necessarily

$$\lim_{n\to\infty} |v_n^3[\varphi(t_n)]\, v_n'[\varphi(t_n)]| = \infty. \tag{8}$$

Now we prove that $\{v_n[\varphi(t_n)]\}$ and $\{v_n'[\varphi(t_n)]\}$ are bounded, contrary to (8). Let $\{v_n[\varphi(t_n)]\}$ be unbounded. Without any loss of generality it may be assumed that $\lim_{n\to\infty} |v_n[\varphi(t_n)]| = \infty$. If we put $u_n(t) := v_n(t + t_n)$ for $t \in \mathbf{R}$ and $n = 1, 2, 3, \ldots$, then u_n is a solution of the equation $y'' = q(t + t_n)\, y$, $u_n(0) = 1$, $u_n'(0) = 0$. Let $\{t_{n_k}\}$ be such a subsequence $\{t_n\}$ that $\lim_{n\to\infty} q(t + t_{n_k}) = p(t)$ uniformly on $\mathbf{R}$ and let $\varphi(t) - t \leqq K$ for $t \in \mathbf{R}$ (the existence of the positive constant K is guaranteed by Lemma 4). Letting v be a solution of (p), $v(0) = 1$, $v'(0) = 0$ yields

$$\lim_{k\to\infty} u_{n_k}^{(i)}(t) = v^{(i)}(t), \qquad i = 0, 1, \tag{9}$$

uniformly on $\langle 0, K\rangle$, contradicting the fact that $\lim_{k\to\infty} |v_{n_k}[\varphi(t_{n_k})]| = \lim_{k\to\infty} |u_{n_k}[\varphi(t_{n_k}) - t_{n_k}]| = \infty$.

Let $\{v_n'[\varphi(t_n)]\}$ be unbounded and assume again $\lim_{n\to\infty} \{v_n'[\varphi(t_n)]\} = \infty$. We come to the contradiction in a manner analogous to that used above, but this time – uhlike to the foregoin – we will utilize $i = 2$ in (9).

It remains to prove (4) for $i = 3$. This result however immediately follows from the boundedness of the functions q, φ'', from the inequalities (3) and from the equality

$$-\frac{1}{2}\frac{\varphi'''(t)}{\varphi'(t)} + \frac{3}{4}\left(\frac{\varphi''(t)}{\varphi'(t)}\right)^2 + \varphi'^2(t)\, q[\varphi(t)] = q(t),$$

introduced and proved in [2].

3. Proof of Theorem 1

To show the function $\varphi'(t)$ to be almost periodic it suffices to prove (by Bohr–Bochner's theorem – see [5]) that for every sequence of numbers $\{h_n\}$ a subsequence uniformly convergent on $\mathbf{R}$ may be chosen from the sequence of functions

$\{\varphi'(t + h_n)\}$. According to the assumption, q is an almost periodic function so that we may assume without loss of generality that $\lim_{n\to\infty} q(t + h_n) = p(t)$ uniformly on **R**. In analogy with the proof of Theorem 1 ([6]) we may prove the function $\varphi(t + h_n) - h_n$ to be the basic central dispersion of

$$y'' = q(t + h_n)\, y. \tag{10}$$

Thus, by Lemma 2, $\{\varphi'(t + h_n)\}$ is uniformly convergent on every compact interval. Assume $\{\varphi'(t + h_n)\}$ not to be uniformly convergent on **R**. Then there exist a number $a > 0$, $\{t_n\}$ ($\lim_{n\to\infty} | t_n | = \infty$) and increasing sequence of natural number $\{k_n\}$, $\{r_n\}$ such that

$$| \varphi'(t_n + h_{k_n}) - \varphi'(t_n + h_{r_n}) | \geqq a, \qquad n = 1, 2, 3, \dots . \tag{11}$$

By Lemma 4 $\{\varphi'(t_n + h_{k_n})\}$, $\{\varphi'(t_n + h_{r_n})\}$ are bounded. Thus, in passing to appropriate subsequences – to simplify the writing we use the same notation – we may obtain: $\lim_{n\to\infty} \varphi'(t_n + h_{k_n}) = b$, $\lim_{n\to\infty} \varphi'(t_n + h_{r_n}) = c$ and

$$\lim_{n\to\infty} q(t + t_n + h_{k_n}) = p_1(t), \qquad \lim_{n\to\infty} q(t + t_n + h_{r_n}) = p_2(t) \tag{12}$$

uniformly on **R**. With respect to (11) we have

$$| b - c | \geqq a. \tag{13}$$

Next we have $p_1 = p_2$ (see the proof of Theorem 1 ([6])). The function $\varphi(t + t_n + h_{k_n}) - t_n - h_{k_n}$ is the basic central dispersion of the equation $y'' = q(t + t_n + h_{k_n})\, y$ and $\varphi(t + t_n + h_{r_n}) - t_n - h_{r_n}$ is the basic central dispersion of the equation $y'' = q(t + t_n + h_{r_n})\, y$. Consequently, it follows from (12), from the equality $p_1 = p_2$ and from Lemma 2 that $\lim_{n\to\infty} \varphi'(t_n + h_{k_n}) = \lim_{n\to\infty} \varphi'(t_n + h_{r_n})$, contradicting (13).

By an analogous method we can prove that $\varphi''(t)$ is an almost periodic function. Then, we obtain from the equality $-\frac{1}{2}\frac{\varphi'''(t)}{\varphi'(t)} + \frac{3}{4}\left(\frac{\varphi''(t)}{\varphi'(t)}\right)^2 + \varphi'^2(t)\, q[\varphi(t)] = q(t)$, from Lemmas 3 and 5, and from the known properties of almost periodic functions (see e.g. 5, pages 9–11 and 19–21) that $\varphi'''(t)$ is also almost periodic function.

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Souhrn

O VLASTNOSTECH DERIVACÍ ZÁKLADNÍ CENTRÁLNÍ DISPERSE OSCILATORICKÉ ROVNICE $y'' = q(t)\,y$ SE SKOROPERIODICKÝM KOEFICIENTEM q

SVATOSLAV STANĚK

Rozložení nulových bodů řešení rovnice

$$y'' = q(t)\,y, \qquad q \in C^0(\mathbf{R}), \tag{q}$$

lze popsat základní centrální dispersí (1. druhu) φ rovnice (q). Hlavní výsledek je uveden v následující větě: Nechť φ je základní centrální disperse oscilatorické rovnice (q) se skoroperiodickým koeficientem q. Pak pro $i = 1, 2, 3$ jsou $\varphi^{(i)}(t)$ skoroperiodické funkce.

Резюме

О СВОЙСТВАХ ПРОИЗВОДНЫХ ОСНОВНОЙ ЦЕНТРАЛЬНОЙ ДИСПЕРСИИ КОЛЕБЮЩЕГОСЯ УРАВНЕНИЯ $y'' = q(t)\,y$ С ПОЧТИ-ПЕРИОДИЧЕСКИМ КОЭФФИЦИЕНТОМ q

СВАТОСЛАВ СТАНЕК

Разложение корней интервалов уравнения

$$y'' = q(t)y,\ q \in C^0(\mathbf{R}), \tag{q}$$

возможно описать основной центральной дисперсией (1-ого рода) φ уравнения (q). Основной результат работы: Пусть φ-основная центральная дисперсия колеблющегося уравнения (q) с почти-периодическим коэффициентом q. Тогда $\varphi^{(i)}(t)$ почти-периодические функции ($i = 1, 2, 3$).