Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 23 (1984), No. 1, 39--44

Persistent URL: `http://dml.cz/dmlcz/120149`

ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM
1984 VOLUMEN 79 PAG. 39–44

Katedra matematické analýzy a numerické matematiky přírodovědecké fakulty University Palackého v Olomouci
Vedoucí katedry: Miroslav Laitoch, Prof., RNDr., CSc.

ON THE BASIC SECOND KIND CENTRAL DISPERSION OF $y'' = q(t)y$ WITH AN ALMOST PERIODIC COEFFICIENT q

SVATOSLAV STANĚK

(*Received September 26., 1982*)

1. Introduction

Let

$$y'' = q(t)\,y, \qquad q \in C^0(\mathbf{R}), \qquad q(t) < 0 \qquad \text{for } t \in \mathbf{R}, \tag{q}$$

be an oscillatory equation with an almost periodic coefficient q. Equations of this type present a natural generalization of differential equations with a periodic coefficient. From the viewpoint of Borůvka's theory of the basic central dispersions (see [1], [2]), these equations were first studied in [5]. The basic first kind central dispersion φ of (q) describes distributions of zeros in solutions of (q). It was proved in [5] that the function $\varphi(t) - t$ is almost periodic. The distributions of zeros of the derivative in solutions of (q) is described by the basic second kind central dispersion of this equation. The present paper proves

Theorem 1. *Let* (q) *be an equation with an almost periodic coefficient* q, $\sup_{t \in \mathbf{R}} q(t) < 0$ *and* ψ *be the basic second kind central dispersion of* (q). *Then the function* $\psi(t) - t$ *is almost periodic.*

2. Basic concepts and lemmas

Let us say that (q) is an oscillatory equation (on $\mathbf{R}$) if $\pm\infty$ are the cluster points of the roots of every (nontrivial) solution of (q).

A function $\alpha \in C^0(\mathbf{R})$ is called a first phase of (q) if there exist independent solutions u, v of this equation such that

$$\operatorname{tg} \alpha(t) = u(t)/v(t) \qquad \text{for } t \in \mathbf{R} - \{t;\, v(t) = 0\}.$$

Let α be a first phase of the oscillatory equation (q). Let us put $\varphi(t) := \alpha^{-1}[\alpha(t) + \pi \,.\, \text{sign}\, \alpha']$, $t \in \mathbf{R}$. The function φ is called the basic first kind central dispersion of (q).

A function $\beta \in C^0(\mathbf{R})$ is called a second phase of (q) if there exist independent solutions u_1, v_1 of (q) such that

$$\text{tg}\, \beta(t) = u_1'(t)/v_1'(t) \qquad \text{for } t \in \mathbf{R} - \{t; v_1'(t) = 0\}.$$

Let β be a second phase of the oscillatory equation (q). Put $\psi(t) := \beta^{-1}[\beta(t) + \pi \,.\, \text{sign}\, \beta']$, $t \in \mathbf{R}$. The function ψ is called the basic second kind central dispersion of (q).

Let α be a first phase of (q). Then there always exists a second phase β of (q) such that

$$n\pi < \beta(t) - \alpha(t) < (n + 1)\,\pi, \qquad t \in \mathbf{R},$$

where n is an integer.

The definitions and properties given below may be found in [1] and [2].

Definition 1. ([3]). *A function $f \in C^0(\mathbf{R})$ is called almost periodic if to every $\varepsilon > 0$ there exists a number L (> 0) such that for every $x \in \mathbf{R}$ there exists at least one number τ in the interval $\langle x, x + L)$, such that*

$$|f(t + \tau) - f(t)| < \varepsilon \qquad \textit{for } t \in \mathbf{R}.$$

Lemma 1. ([3]). *A function $f \in C^0(\mathbf{R})$ is almost periodic exactly if from every sequence of functions $\{f(t + h_n)\}$, $h_n \in \mathbf{R}$, a subsequence of functions may be chosen, uniformly converging on* $\mathbf{R}$.

It was proved in [4] that every equation (q) with an almost periodic coefficient q is either oscillatory or disconjugate.

Lemma 2. *Let $q_n \in C^0(\mathbf{R})$, $\lim_{n\to\infty} q_n(t) = q(t)$ uniformly on every compact interval and let $q_n(t) < 0$, $q(t) < 0$ for $t \in \mathbf{R}$. Then there exist a second phase β_n* of (q_n) *and a second phase β of* (q), *such that*

$$\lim_{n\to\infty} \beta_n(t) = \beta(t), \qquad \lim_{n\to\infty} \beta_n'(t) = \beta'(t),$$

uniformly on every compact interval.

Proof. Let u_n, v_n be solutions of (q_n) and u, v be solutions of (q) satisfying the initial conditions: $u_n(0) = u(0) = v_n'(0) = v'(0) = 0$, $u_n'(0) = u'(0) = v_n(0) = v(0) = 1$. Let us put

$$\gamma_n(t) := \frac{q_n(t)}{u_n'^2(t) + v_n'^2(t)}, \qquad \gamma(t) := \frac{q(t)}{u'^2(t) + v'^2(t)}; \qquad t \in \mathbf{R}.$$

Since $\lim_{n\to\infty} (u_n'^2(t) + v_n'^2(t)) = u'^2(t) + v'^2(t)$ and $\lim_{n\to\infty} q_n(t) = q(t)$ uniformly on every compact interval, also $\lim_{n\to\infty} \gamma_n(t) = \gamma(t)$ is uniformly there. Put

$$\beta_n(t) := \int_0^t \gamma_n(x)\,dx, \qquad \beta(t) := \int_0^t \gamma(x)\,dx, \qquad t \in \mathbf{R}.$$

Then β_n is a second phase of (q_n) and β is a second phase of (q), both having the properties given in Lemma 2.

Lemma 3. *Let the assumptions of Lemma 2 be satisfied. Let next* (q_n) and (q) *oscillatory equations,* $\psi_{(n)}$ *be the basic second kind central dispersion of* (q_n) *and* ψ *be the basic second kind central dispersion of* (q). *Then*

$$\lim_{n \to \infty} \psi_{(n)}(t) = \psi(t)$$

uniformly on every compact interval.

Proof. Let β_n be a second phase of (q_n) and β be a second phase of (q) possessing the properties given in Lemma 2 and defined in its proof. Then $\beta_n(\mathbf{R}) = \mathbf{R}$, $\beta(\mathbf{R}) = \mathbf{R}$, sign $\beta_n' =$ sign $\beta' = -1$. By Lemma 2 $\lim_{n \to \infty} \beta_n^{(i)}(t) = \beta^{(i)}(t)$ uniformly on every compact interval, $(i = 0, 1)$, whence it follows that $\lim_{n \to \infty} \beta_n^{-1}(t) = \beta^{-1}(t)$ uniformly on every compact interval. From this and from the equalities $\psi_{(n)}(t) = \beta_n^{-1}[\beta_n(t) - \pi]$, $\psi(t) = \beta^{-1}[\beta(t) - \pi]$ then follows the assertion of the Lemma.

Lemma 4. *Let* (q) *be an equation with an almost periodic coefficient* q, $\sup_{t \in \mathbf{R}} q(t) < 0$. *Let* ψ *be the basic second kind central dispersion of* (q). *Then there exists a number* $K > 0$ *such that*

$$\psi(t) - t \leqq K, \qquad t \in \mathbf{R}. \tag{1}$$

Proof. Let φ be the basic first kind central dispersion of (q). According to Lemma 3 ([5]), there exists a number $L > 0$: $\varphi(t) - t \leqq L$ for $t \in \mathbf{R}$. Let α be an increasing first phase of (q). Then there exists a second phase β of (q) and an integer n such that

$$n\pi < \beta(t) - \alpha(t) < (n + 1)\,\pi, \qquad t \in \mathbf{R}. \tag{2}$$

Evidently sign $\alpha' =$ sign $\beta' = 1$ and we get from (1)

$$\alpha^{-1}(t - n\pi) > \beta^{-1}(t) > \alpha^{-1}(t - (n + 1)\,\pi), \qquad t \in \mathbf{R}, \tag{3}$$

and from (2) and (3) we obtain

$$\beta^{-1}[\beta(t) + \pi] < \alpha^{-1}[\beta(t) + \pi - n\pi] < \alpha^{-1}[\alpha(t) + 2\pi], \qquad t \in \mathbf{R}. \tag{4}$$

Since $\psi(t) = \beta^{-1}[\beta(t) + \pi]$, $\varphi(t) = \alpha^{-1}[\alpha(t) + \pi]$, we find from (4) that $\psi(t) < \varphi[\varphi(t)]$ for $t \in \mathbf{R}$, and $\varphi[\varphi(t)] - t \leqq 2L$. Then $\psi(t) - t < \varphi[\varphi(t)] - t \leqq 2L$. Obviously, it suffices to put $K := 2L$.

Before passing to the proof of Theorem 1 let us demonstrate the validity of

Theorem 2. *Let* $q \in C^2(\mathbf{R})$, $q(t) < 0$ *for* $t \in \mathbf{R}$ *and put* $\hat{q}(t) := q(t) + \sqrt{-q(t)}\left(\frac{1}{\sqrt{-q(t)}}\right)''$, $t \in \mathbf{R}$. *Let* (q) *be an oscillatory equation and* $\hat{q}$ *an almost function. Let* ψ *be the basic second kind central dispersion of* (q). *Then the function* $\psi(t) - t$ *is almost periodic.*

Proof. It follows from [1] and [2] that the basic second kind central dispersion of (q) is the basic first kind central dispersion of ($\hat{q}$). Then immediately follows from Theorem 1 ([5]) that the function $\psi(t) - t$ is almost periodic.

3. Proof of Theorem 1

To show that $\psi(t) - t$ is an almost periodic function, it suffices—with respect to Lemma 1—to prove that for every number sequence $\{h_n\}$ we may choose from the sequence of function $\{\psi(t + h_n) - t - h_n\}$ a subsequence uniformly convergent on $\mathbf{R}$. According to our assumption, q is an almost periodic function. Consequently, we may choose from the sequence $\{q(t + h_n)\}$ a subsequence uniformly convergent on $\mathbf{R}$. It may be assumed without any loss of generality that $\{q(t + h_n)\}$ is uniformly convergent on $\mathbf{R}$. Let $\lim_{n\to\infty} q(t + h_n) = p(t)$, $t \in \mathbf{R}$. It follows from the assumptions of the Theorem that $p(t) < 0$ for $t \in \mathbf{R}$ and (p) is an oscillatory equation. Let β be a second phase of (q). Then $\beta(t + h_n)$ is a second phase of

$$y'' = q(t + h_n)\, y. \tag{5}$$

Let $\psi_{(n)}$ be the basic second kind central dispersion of (5). It then follows from the equalities $\beta[\psi_{(n)}(t) + h_n] = \beta(t + h_n) + \pi \operatorname{sign} \beta'$, $\beta[\psi(t)] = \beta(t) + \pi \operatorname{sign} \beta'$ that $\psi_{(n)}(t) = \psi(t + h_n) - h_n$. Thus, by Lemma 3, the sequence $\{\psi(t + h_n) - t - h_n\}$ is uniformly convergent on every compact interval. Assume that the sequence $\{\psi(t + h_n) - t - h_n\}$ is not uniformly convergent on $\mathbf{R}$. Then there exist a number $a > 0$ and increasing sequences of positive integers $\{k_n\}$, $\{r_n\}$ such that

$$|\,\psi(t_n + h_{k_n}) - h_{k_n} - \psi(t_n + h_{r_n}) + h_{r_n}\,| \geqq a, \qquad n = 1, 2, \ldots \tag{6}$$

By Lemma 4, the sequences $\{\psi(t_n + h_{k_n}) - t_n - h_{k_n}\}$ and $\{\psi(t_n + h_{r_n}) - t_n - h_{r_n}\}$ are bounded. Therefore, in passing to appropriate subsequences it is obtainable—for simplification of writting apply the same notation—that

$$\lim_{n\to\infty} (\psi(t_n + h_{k_n}) - t_n - h_{k_n}) = b, \qquad \lim_{n\to\infty} (\psi(t_n + h_{r_n}) - t_n - h_{r_n}) = c,$$

$$\lim_{n\to\infty} q(t + t_n + h_{k_n}) = p_1(t), \qquad \lim_{n\to\infty} q(t + t_n + h_{r_n}) = p_2(t)$$

uniformly on $\mathbf{R}$. On account of (6) we see that

$$|\,b - c\,| \geqq a. \tag{8}$$

Analogous to the proof of Theorem 1 ([5]) we can prove: $p_1 = p_2$. Since $\psi(t + t_n + h_{k_n}) - t_n - h_{k_n}$ is the basic second kind central dispersion of $y'' = q(t + t_n + h_{k_n})\, y$ and $\psi(t + t_n + h_{r_n}) - t_n - h_{r_n}$ is the basic second kind central dispersion of $y'' = q(t + t_n + h_{r_n})\, y$, we get from the equality $p_1 = p_2$, from (7) and from Lemma 3

$$\lim_{n\to\infty} (\psi(t_n + h_{k_n}) - t_n - h_{k_n}) = \lim_{n\to\infty} (\psi(t_n + h_{r_n}) - t_n - h_{r_n}),$$

and so $b = c$, contradicting (8).

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Souhrn

ZÁKLADNÍ CENTRÁLNÍ DISPERSE 2. DRUHU ROVNICE $y'' = q(t)\, y$ SE SKOROPERIODICKÝM KOEFICIENTEM q

SVATOSLAV STANĚK

Nechť

$$y'' = q(t)\, y, \qquad q \in C^{\circ}(\mathbf{R}), \qquad q(t) < 0 \qquad \text{pro } t \in \mathbf{R}, \tag{q}$$

je oscilatorická rovnice. Základní centrální disperse 2. druhu ψ rovnice (q) popisuje rozložení nulových bodů derivace řešení této rovnice.

Hlavní výsledek práce je uveden v následující větě: Nechť (q) je rovnice se skoroperiodickým koeficientem q, $\sup_{t\in\mathbf{R}} q(t) < 0$ a nechť ψ je její základní centrální disperse 2. druhu. Pak funkce $\psi(t) - t$ je skoroperiodická.

Резюме

ОСНОВНАЯ ЦЕНТРАЛЬНАЯ ДИСПЕРСИЯ 2-ОГО РОДА УРАВНЕНИЯ $y'' = q(t)y$ С ПОЧТИ-ПЕРИОДИЧЕСКИМ КОЭФФИЦИЕНТОМ q

СВАТОСЛАВ СТАНЕК

Пусть $y'' = q(t)y$, $q \in C^0(\mathbf{R})$, $q(t) < 0$ для $t \in \mathbf{R}$, (q) колеблющиеся уравнение. Основная центральная дисперсия 2-ого рода ψ уравнения (q) описывает разложение корней производной интегралов уравнения (q). Основной результат работы: Пусть (q) уравнение с почти-периодическим коэффициеном q, $\sup_{t \in \mathbf{R}} q(t) < 0$ и ψ-основная центральная дисперсия 2-ого рода уравнения (q). Тогда функция $\psi(t) - t$ тоже почти-периодическая.