Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 23 (1984), No. 1, 153--159

Persistent URL: http://dml.cz/dmlcz/120145

ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM
1984 VOLUMEN 79 PAG. 153—160

Katedra kybernetiky a matematické informatiky přírodovědecké fakulty University Palackého v Olomouci
Vedoucí katedry: Karel Beneš, Doc., Ing., CSc.

SIMULATION OF DYNAMIC SYSTEMS WITH A TRANSFER FUNCTION OF THE TYPE $n = m$ EXCITED BY THE DIRAC FUNCTION

KAREL BENEŠ

(*Received April 30th, 1983*)

Investigations of dynamic systems excited by the Dirac function are of great significance both from the mathematical and the technical point of view. The Dirac function is not directly generable and therefore another equivalent description of the investigated system is sought wherein the Dirac function does not occur, or with smalter requirements on accurasy, the Dirac function aray be approximated by a reslangle or an exponential impulse.

The transfer function is defined as a ratio of the Laplase images of the output and input magnitudes with zero initial conditions, i. e.

$$H(s) = \frac{Y(s)}{Z(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}, \tag{1}$$

where m, n are non-negative integers numbers. Assume $m = n$, a_k, b_k = constant, $a_n = 1$.

The transfer function (1) may be put into an image form of the differential equation

$$s^n Y(s) + a_{n-1} s^{n-1} Y(s) + \dots + a_0 Y(s) = \tag{2}$$
$$= b_m s^m Z(s) + b_{m-1} s^{m-1} Z(s) + \dots + b_0 Z(s),$$

which is also the Laplase image of the differential equation

$$y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_0 y = b_m z^{(m)} + b_{m-1} z^{(m-1)} + \dots + b_0 z. \tag{3}$$

Equation (3) is generally programmed in the form of a system of the differential equations

$$
\begin{aligned}
y_1' &= b_0 z - a_0 y, \\
y_2' &= b_1 z - a_1 y + y_1, \\
y_3' &= b_2 z - a_2 y + y_2, \\
&\vdots \\
y_n' &= b_{n-1} z - a_{n-1} y + y_{n-1}, \\
y &= b_n z + y_n.
\end{aligned} \tag{4}
$$

Certain difficulties arise if $z = \delta(t)$ is the Dirac impulse defined by the relations

$$\delta(t) = \lim_{\varepsilon \to 0} (t, \varepsilon) \tag{5}$$

$$
\begin{aligned}
\delta(t, \varepsilon) &= 0 \qquad \text{for } t < 0, \\
\delta(t, \varepsilon) &= \frac{1}{\varepsilon} \qquad \text{for } 0 \leqq t \leqq \varepsilon \\
\delta(t, \varepsilon) &= 0 \qquad \text{for } t > \varepsilon,
\end{aligned}
$$

$$\int_{-\infty}^{\infty} \delta(t)\, \mathrm{d}t = \int_{0}^{t} \delta(t) = \vartheta(t), \tag{5a}$$

where

$$
\begin{aligned}
\vartheta(t) &= 0 \qquad \text{for } t < 0, \\
\vartheta(t) &= 1 \qquad \text{for } t \geqq 0.
\end{aligned} \tag{5b}
$$

If we bring to the input integrator the Dirac function $\delta(t)$ (see figure 1), then we get a step function $\vartheta(t)$ on its output (under the assumption that the integrator changes its sign), i.e.

$$v(t) = -\int_{0}^{t} (u(t) + \delta(t))\, \mathrm{d}t = -\int_{0}^{t} u(t)\, \mathrm{d}t - \vartheta(t). \tag{6}$$

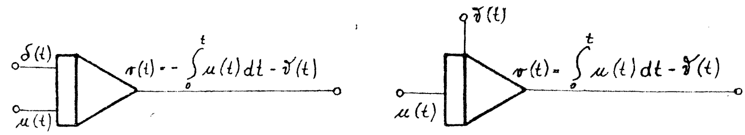

Fig. 1

Integration of the function $\delta(t)$ is thus equivalent to placing the initial condition $-\vartheta(t) = -1$.

Figure 2 illustrates a program block for solutions of system (4) for $m = n - 1$ if the function $z = \delta(t)$ is realizable. If the Dirac impulse is not realizable, then the block in figure 2 may be re-plotted on the ground of relation (6) – see figure 3. Thus, no difficulties arise for $n > m$ in modeling dynamical systems exited by the Dirac function. Another situation is in case of the function $n = m$, where the response y is obtained by the last equation of system (4) on the output of the adder. Figure 4 shows a not realizable program block ($z = \delta(t)$) for such a case. Let us assume the Dirac impulse in time $t = 0$, multiplied by the coefficient b_n to pass from the output of the adder over the coefficients a_j to the inputs of the relative integrators, where – according to (6) – it proves as a further equivalent initial condition $-(-1)^{n-j} a_j b_n$. The program block by figure (4) may then be

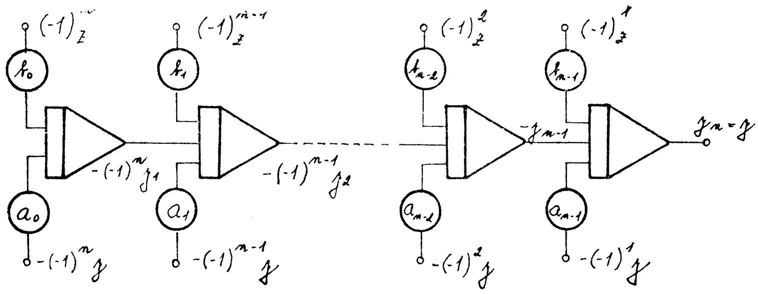

Fig. 2

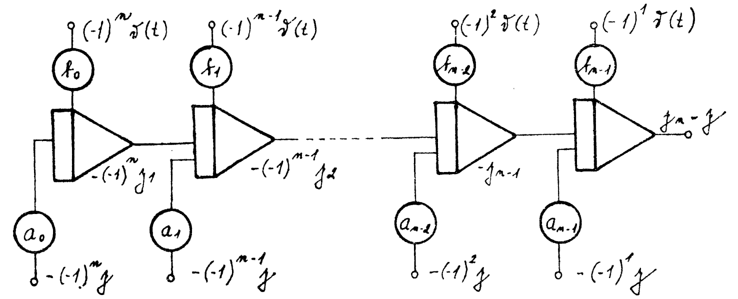

Fig. 3

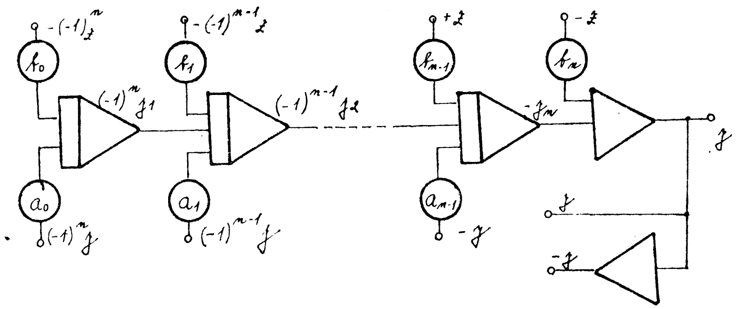

Fig 4

re-plotted to the form of figure 5. The program block in figure 4 is described by the system of differential equations (4). If we put $z = \delta(t)$, then the program block in figure 4 is described by the system of equations

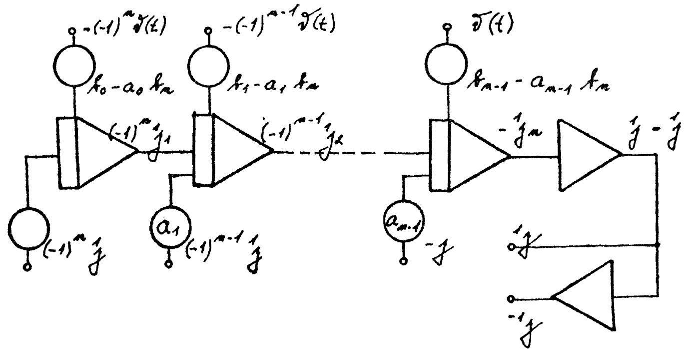

Fig. 5

$$(-1)^n y_1 = -\int_0^t (-1)^n a_0 y \, dt + (-1)^n b_0 \vartheta(t) \tag{7}$$

$$(-1)^{n-1} y_2 = -\int_0^t [(-1)^{n-1} a_1 y + (-1)^n y_1] \, dt + (-1)^{n-1} b_1 \vartheta(t)$$

$$\vdots$$

$$(-1)^{n-j} y_{j+1} = -\int_0^t [(-1)^{n-j} a_j y + (-1)^{n-j+1} y_j] \, dt +$$
$$+ (-1)^{n-j} b_j \vartheta(t)$$

$$\vdots$$

$$-y_n = -\int_0^t [-a_{n-1} y + (-1)^1 y_{n-1}] \, dt - b_{n-1} \vartheta(t)$$

$$y = y_n + b_n \delta(t). \tag{7a}$$

Inserting the value of (7a) for y in (7) we get on the ground of (6) that

$$(-1)^n y_1 = -\int_0^t (-1)^n a_0 y_n \, dt - (-1)^n a_0 b_n \vartheta(t) + (-1)^n b_0 \vartheta(t) \tag{8}$$

$$\vdots$$

$$(-1)^{n-1} y_2 = -\int_0^t [(-1)^{n-1} a_1 y_n + (-1)^n y_1] \, dt - (-1)^{n-1} a_1 b_n \vartheta(t) + (-1)^{n-1} b_1 \vartheta(t)$$

$$\vdots$$

$$(-1)^{n-j} y_{j+1} = -\int_0^t [(-1)^{n-j} a_j y_n + (-1)^{n-j+1} y_j] \, dt - (-1)^{n-j} a_j b_n \vartheta(t) +$$
$$+ (-1)^{n-j} b_j \vartheta(t)$$

$$\vdots$$

$$-y_n = -\int_0^t [-a_{n-1} y_n + (-1)^1 y_{n-1}] \, dt + a_{n-1} b_n \vartheta(t) - b_{n-1} \vartheta(t)$$

The last two expressions on the right sides of system (8) may be regarded as the initial values of the relative functions $(-1)^{n-y}y_{j+1}$. The program block in figure 5 is described by the system of equations (upon substituting ${}^1y = {}^1y_n$).

$$(-1)^n{}^1y_1 = -\int_0^t (-1)^n a_0{}^1y_n\,dt + (-1)^n(b_0 - a_0b_n)\,\vartheta(t), \tag{9}$$

$$(-1)^{n-1}{}^1y_2 = -\int_0^t [(-1)^{n-1}a_1{}^1y_n + (-1)^n{}^1y_1]\,dt + (-1)^{n-1}(b_1 - a_1b_n)\,\vartheta(t),$$

$$\vdots$$

$$(-1)^{n-1}{}^1y_{j+1} = -\int_0^t [(-1)^{n-j}a_j{}^1y_n + (-1)^{n-j+1}{}^1y_j]\,dt + (-1)^{n-j}(b_j - a_jb_n)\,\vartheta(t),$$

$$\vdots$$

$$-{}^1y_n = -\int_0^t [-a_{n-1}{}^1y_n + (-1)^1{}^1y_{n-1}\,dt - (b_{n-1} - a_{n-1}b_n)\,\vartheta(t).$$

On account of the fact that systems (8) and (9) are (up to the notation of variables) identical, it holds $y_n = {}^1y_n$, then also the program block for $t > 0$ in figures 4 and 5 are equivalent. The input value y for $t = 0$ by figure 4 and (7a) is given by the relation $y = y_n + b_n\delta(t)$. The output value ${}^1y$ by figure 5 is given by the relation ${}^1y = {}^1y_n = y_n$. Figures 6 and 6a show the course of the values y and ${}^1y$ by the

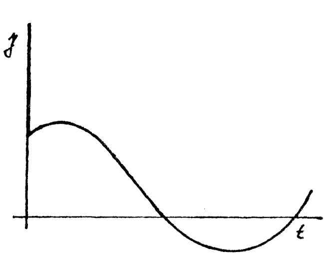

Fig. 6

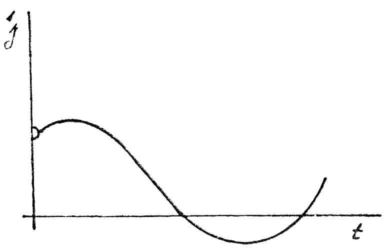

Fig. 6a

program blocks in figures 4 and 5, respectively. If $z = \delta(t)$, then, by (1), the image of the output magnitude y is given by the relation ($n = m$, $a_n = 1$).

$$Y(s) = b_n + \frac{(b_{n-1} - b_na_{n-1})\,s^{n-1} + (b_{n-2} - b_na_{n-2})\,s^{n-2} + \ldots + b_0 - a_0b_n}{s^n + a_{n-1}s^{n-1} + \ldots + a_0} =$$

$$= b_n + \frac{\sum_{i=1}^{n}(b_{n-i} - a_{n-i}b_n)\,s^{n-i}}{s^n + a_{n-1}s^{n-1} + \ldots + a_0}\,. \tag{10}$$

The first term on the right side after the inverse transformation gives $b_n\delta(t)$, the second term (a lineaı system with constant coefficients is concerned) is an image of functions of the type $At^k e^{\alpha t} \cos(\omega t + \varphi)$. If the numerator is a multiple of the denominator, i.e. $b_j = ka_j$, $b_n = k$, then

$$Y(s) = \frac{ks^n + ka_{n-1}s^{n-1} + \ldots + ka_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0} = k \tag{11}$$

and $y_{(t)} = k\delta(t)$ as can be seen in the program block in figure 4. If for instance the system with the transfer function

$$H(s) = \frac{Y(s)}{Z(s)} = \frac{b_2 s^2 + b_1 s + b_0}{s^2 + a_1 s + a_0}, \tag{12}$$

is investigated, then, with $z = \delta(t)$, the image of the response y has the form

$$Y(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^2 + a_1 s + a_0} = b_2 + \frac{(b_1 - a_1 b_2) s + b_0 - a_0 b_2}{s^2 + a_1 s + a_0}. \tag{13}$$

Solving the task by means of the progıam block on the ground of relations (9) and by figure 5, then the program block for $m = n = 2$ has the form by figure 7 and is described by the system of equations (for $t \geqq 0$, $\vartheta(t) = 1$)

(14)
$$-y_2 = -\int_0^t (-a_1 y_2 + y_1)\,dt - b_1 + a_1 b_2,$$
$$y_1 = -\int_0^t a_0 y_2\,dt + b_0 - a_0 b_2,$$

Performing the diffeıentiation we find that

$$-y_2' = a_1 y_2 - y_1, \qquad y_1' = -a_0 y_2, \tag{14a}$$

where for the initial conditions by (14) and figure 7 $y_{2(0)} = b_1 - a_1 b_2$, $y_{1(0)} = b_0 - a_0 b_2$ hold.

Putting the system of (14a) to a second order differential equation for y_2 gives

$$y_2'' + a_1 y_2' + a_0 y_2 = 0 \tag{14b}$$

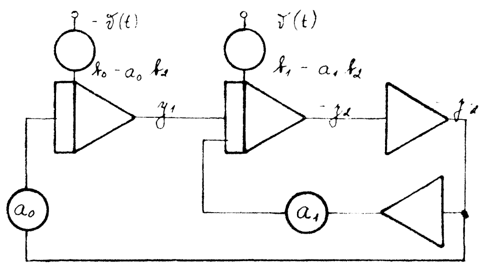

Fig. 7

with the initial conditions $y_{2(0)} = b_1 - a_1 b_2$, $y'_{2(0)} = -a_1(b_1 - a_1 b_2) + b_0 - a_0 b_2$. The Laplace image of (14b) has then the form

$$s^2 Y_2(s) - s(b_1 - a_1 b_2) + a_1(b_1 - a_1 b_2) - b_0 + a_0 b_2 + a_1 s Y_2(s) - \\ - a_1(b_1 - a_1 b_2) + a_0 Y_2(s) = 0,$$

i.e.

$$s^2 Y_2(s) - s(b_1 - a_1 b_2) - b_0 + a_0 b_2 + a_1 s Y_2(s) + a_0 Y_2(s) = 0,$$

whence

$$Y_2(s) = \frac{(b_1 - a_1 b_2) s + b_0 - a_0 b_2}{s^2 + a_1 s \quad a_0}. \tag{15}$$

The image of the response is the same as the second part of the expression on the right side of equation (13). For $t > 0$ is thus $y_2 = y$. And the program block by figure 4 may by replaced by that of figure 5.

Souhrn

SIMULACE DYNAMICKÝCH SYSTÉMŮ S PŘENOSOVOU FUNKCÍ TYPU $m = n$ BUZENÝCH DIRACOVOU FUNKCÍ

KAREL BENEŠ

V práci je popsána možnost modelování přenosových funkcí typu $m = n$ systémů buzených Diracovou funkcí. Na základě odvozených vztahů je ukázáno, že odezvu je možno sledovat pro $t > 0$.

Doc. Ing. Karel Beneš, CSc.
katedra kybernetiky a matematické informatiky
přírodovědecké fakulty University Palackého
Leninova 26
771 46 Olomouc ČSSR

Резюме

СИМУЛЯЦИЯ ДИНАМИЧЕСКИХ СИСТЕМ С ПЕРЕДОТОЧНОЙ ФУНКЦИЕЙ ТИПА $m = n$ ВОЗБУЖДЕННЫХ ФУНКЦИЕИ ДИРАКА

КАРЕЛ БЕНЕШ

В работе описана возможность моделирования передаточных функций типа м=н систем возбуждённых функцией Дирака. На основе показанных отношений показано что выходный сигнал можно повторить для т > 0.