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SIMULATION OF ANALOG COMPUTER IN SOLVING NON-LINEAR DIFFERENTIAL EQUATIONS BY A DIGITAL COMPUTER

KAREL BENEŠ

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Abstract mathematical models realized on analog or digital computers present a significant part of simulative models. Analog computers are preferred for their speed and descriptiveness while digital computers have the advantage of greater accuracy, universality, reproductivity of solutions and easy documentation. The inaccuracy appearing with analog computers is due to the fact that the actual computing units differ somewhat from the ideal computing units (in our computing design considered) and thus work with a certain error may be influenced by a suitably chosen structure of the computing unit as say in diod functional transformators, where the accuracy may be influenced by choosing a number of linear sections or by the choice of the distributions of breakpoints.

The author is concerned with the influence of the distribution of breakpoints of an approximating line of the accuracy in solving some non-linear differential equations.

One way to generate functional dependencies on an analog computer is the use of diode functional transformators approximating the functional dependence required, by a linear broken line. The accuracy of this approximation primarily depends on the number of broken sections and on the distributions of breakpoints of this broken line.

Assume the function $f(x)$ in $\langle a; b\rangle$ being approximated by a broken line consisting of k linear sections. Let the breakpoints of this broken line have the coordinates $[x_i; y_i]$ $i = 0, 1, 2, \ldots, k$; $x_0 = a$, $x_k = b$. The output value of the functional transformators in the interval $\langle x_{i-1}; x_i\rangle$ is expressible by

$$y = \frac{y_i - y_{i-1}}{x_i - x_{i-1}}(x - x_i) + y_i = \tag{1}$$

$$= \frac{y_i - y_{i-1}}{x_i - x_{i-1}} x + y_i - \frac{y_i - y_{i-1}}{x_i - x_{i-1}} x_i = S_i x + Q_i,$$

where

$$S_i = \frac{y_i - y_{i-1}}{x_i - x_{i-1}}, \qquad (2)$$

$$Q_i = y_i - S_i x_i.$$

We see that the transformator makes an error given by the relation

$$\varepsilon(x) = f(x) - S_i x - Q_i.$$

This error assumes its maximal value at the point $x_{\max i}$ for which

$$\left[\frac{\mathrm{d}\varepsilon}{\mathrm{d}x}\right]_{x=x_{\max i}} = 0, \qquad (3)$$

i.e.

$$\left[\frac{\mathrm{d}\varepsilon}{\mathrm{d}x}\right]_{x=x_{\max i}} = S_i.$$

The simplest distribution of breakpoints in $\langle a; b\rangle$ is a uniform distribution whose breakpoints lie on the graph of the function approximated. In this distribution the i-th point has the coordinates

$$a + i\frac{b-a}{k}, \qquad f\left(a + i\frac{b-a}{k}\right),$$
$$i = 0, 1, 2, \ldots, k.$$

From the total accuracy point of view this distribution is not optimal. The errors in the individual sections are generally different, hence the maximal error of the whole approximation may be reduced.

If we require the maximal total error of the approximation with a given number of sections to be minimal, then there must hold

$$\begin{aligned} \varepsilon_{\max 1} &= \varepsilon_{\max 2} \\ \varepsilon_{\max 2} &= \varepsilon_{\max 3} \\ &\vdots \\ \varepsilon_{\max k-1} &= \varepsilon_{\max k}. \end{aligned} \qquad (4)$$

To solve this system of equations analytically is in most cases impossible. For a numerical computation we may apply the algorithm whose block-diagram is shown in Figure 1, where $p = 10^{-7}$ was chosen.

If $f''(x) = 0$ holds throughout the interval, then the accuracy of the approximation may be increased by shifting the approximating line by a value $\frac{\varepsilon_{\max}}{2}$ in the direction of axis $f(x)$. In this case the function $_{,}{}^{J}x)$ will be approximated with the accuracy of $\pm\frac{\varepsilon_{\max}}{2}$.

From the probability point of view it is important for the mean value of the approximated function in $\langle a; b\rangle$ to be equal to the mean value of the approximating broken line, i.e. there must hold the equality

$$\frac{\int_a^b f(x)\,dx}{b-a} = \frac{\sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} (S_i x + Q_i)\,dx}{b-a}. \tag{5}$$

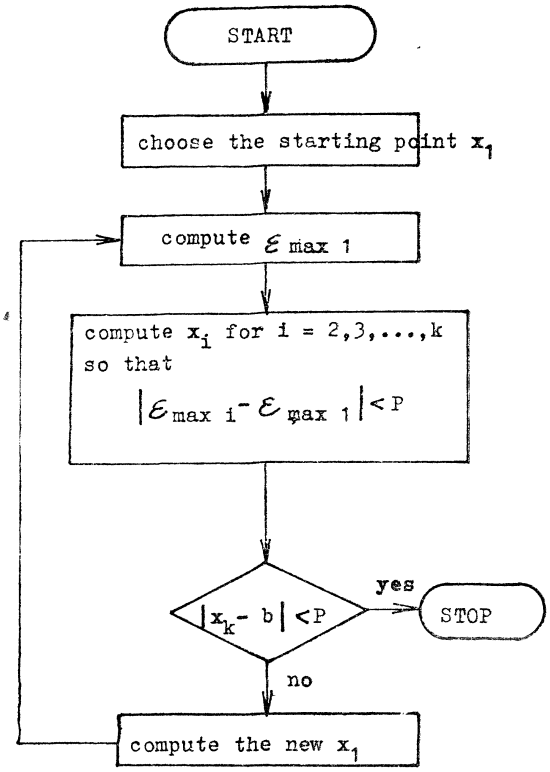

Fig. 1

This requirement may be satisfied by shifting the line whose breakpoints lie on the graph of the approximated function in the direction of the axis $f(x)$. The shifting value P may be determined in the following way.

It follows from the geometrical meaning of a definite integral that the numerator on the right side of (5) equals to

$$\sum_{i=1}^{k} \frac{[(f(x_{i-1}) + P) + (f(x_i) + P)](x_{i-1} - x_i)}{2} =$$

$$\sum_{i=1}^{k} \frac{(f(x_{i-1}) + f(x_i)) \cdot (x_i - x_{i-1}) + 2P \cdot (x_i - x_{i-1})}{2} =$$

$$\sum_{i=1}^{k} \frac{(f(x_{i-1}) + f(x_i))(x_i - x_{i-1})}{2} + P(b - a).$$

Inserting the above expression into (5) gives

$$P = \frac{F(b) - F(a) - \frac{1}{2} \sum_{i=1}^{k} (f(x_{i-1}) + f(x_i)) \cdot (x_i - x_{i-1})}{b - a},$$

where F is a primitive function to the function f.

It appears that the best starting distribution for shifting is the distribution with the same maximal errors in the particular sections. This distribution satisfies both the requirement for the best accuracy and preserves the equality of mean values for the particular sections.

By Figure 1 there were determined the distribution of breakpoints and the errors for the functions $f(x) = x^2$, $f(x) = x^3$ a $f(x) = \sqrt{x}$ and for $k = 5$ and $k = 10$.

x_i	ε_i
0,2	0,010
0,4	0,010
0,6	0,010
0,8	0,010
1,00	0,010

The shift by $\frac{\varepsilon}{2} = -0\,0050$.

The shift for the distribution with the same mean value $P = -0{,}006\,666\,7$.

$f(x) = x^2$, $k = 5$

x_i	ε_i
0,1	0,00250
0,2	0,00250
0,3	.
0,4	.
0,5	.
0,6	
0,7	
0,8	
0,9	
1,0	0,00250

The shift by $\frac{\varepsilon}{2} = -0{,}001250$.

The shift for the distribution with the same mean value $P = -0\,0016667$.

$f(x) = x^2$, $k = 10$

x_i	ε_i
0,3292622	0,0137396
0,5349974	.
0,7067264	.
0,8596470	.
1,0000000	0,0137396

The shift by $\frac{\varepsilon}{2} = -0{,}0068698$.

The shift for the distribution with the same mean value $P = -0{,}0090778$.

$f(x) = x^3$, $k = 5$

x_i	ε_i
0,2063920	0,0033840
0,3353534	.
0,4429986	.
0,5388541	.
0,6268318	.
0,7090175	.
0,7866884	.
0,8606980	.
0,9316520	.
1,0000000	0,0033840

The shift by $\frac{\varepsilon}{2} = -0{,}0016920$.

The shift for the distribution with the same mean value $P = -0{,}0022433$.

$f(x) = x^3$, $k = 10$

x_i	ε_i
0,0044444	0,0166667
0,0400000	.
0,1600000	.
0,4444444	.
1,0000000	0,0166667

The shift by $\frac{\varepsilon}{2} = 0{,}0083333$.

The shift for the distribution with the same mean value $P = 0{,}0111111$.

$f(x) = \sqrt{x}$, $k = 5$

x_i	ε_i
0,0003306	0,0045455
0,0029752	.
0,0119008	.
0,0330579	.
0,0743802	.
0,1457851	.
0,2591736	.
0,4284298	.
0,6694215	.
1,0000000	0,0045455

The shift by $\frac{\varepsilon}{2} = 0{,}0022727$.

The shift for the distribution with the same mean value $P = 0{,}0030303$.

$f(x) = \sqrt{x}$, $k = 10$.

Using functional transformators in solution of nonlinear differential equations may cause that the computing network in fact solves a sequence of linear equations instead of the equations under consideration. The first equation of this system has its initial conditions coincident with the given nonlinear equation. Passing on to a new equation always takes place in that moment when a magnitude entering into the functional transformators crosses the bounds of the section wherein it moved so far. In every new equation the initial conditions are determined by the values of the solution relative to the preceding equation in the moment of crossing the breakpoint. Figure 2 shows an analog computer network set up for solving

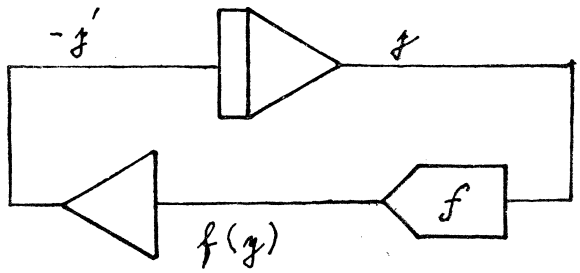

Fig. 2

the nonlinear differential equation $y' = f(y)$. Yet instead of this equation the network solves the sequence of the equation

$$y' = S_i y + Q_i. \tag{6}$$

The question arises what is the influence on the accuracy of solution if functional transformators are applied. To answer this question we created a subprogram ERROR simulating the course of the solution generated by a given analog computing net on a digital computer. This subprogram ensures the fabulation of the error arisen with the step $h = 0{,}03$ and its recording in the printer. Besides the subprogram ensures the graphing of the correct solution as well as the graphing of the error arisen, on the peripheral device DIGIGRAF.

Figure 3 illustrates a simplified program shedule of the algorithm by which the subprogram is working. The subprogram consideres errors arisen in using functional transformators only. Other computing units are assumed to work ideally.

The subprogram ERROR has two formal parameters, namely FD and FN. Both parameters denote the subprograms of the type FUNCTION with the parameters I and T. The subprogram FD computes the value of the (I-1)-st derivative of the analog solution in the time T. If I has a value equal to the value of the order of the derivative increased by one, then the subprogram FD computes the value of the correct solution in the time T. The subprogram FN computes the value of a magnitude entering into the I-st transformator in the time T. To ensure the

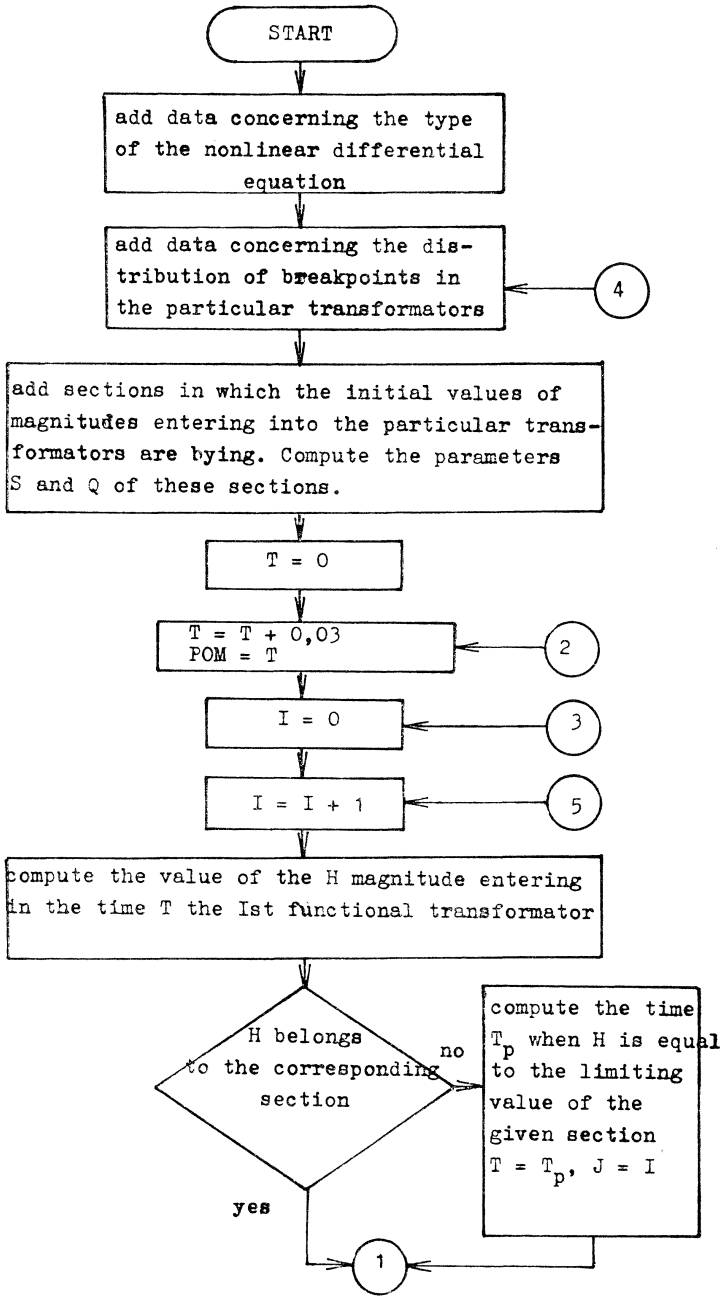
START
add data concerning the type of the nonlinear differential equation
add data concerning the distribution of breakpoints in the particular transformators
4
add sections in which the initial values of magnitudes entering into the particular transformators are bying. Compute the parameters S and Q of these sections.
T = 0
T = T + 0,03
POM = T
2
I = 0
3
I = I + 1
5
compute the value of the H magnitude entering in the time T the Ist functional transformator
H belongs to the corresponding section
no
yes
compute the time T_p when H is equal to the limiting value of the given section
T = T_p, J = I
1

Fig. 3a

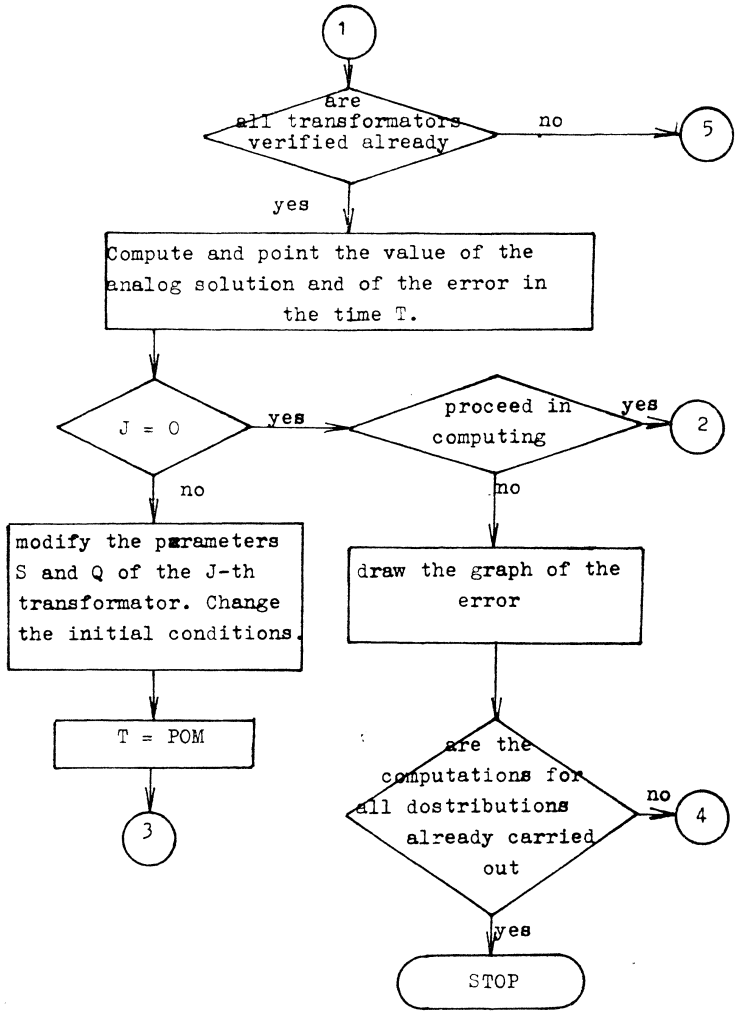

Fig. 3b

right functions of the subprogram, it is necessary for the subprograms FD and FN to contain the declarators DIMENSION S(5), Q(5), PP(5) and COMMON (B3) S, Q, TO, PP. Likewise it is necessary for the parameters of the linear section wherein the magnitude entering into the I-st transformator occors, to be denoted by the variables S(I), Q(I) and the initial conditions of the (I-1)-st derivative of the analog solution by the variables TO and PP(I). The subprogram ERROR uses for the output results the auxiliary subprogrames RAM 1, RAM 2, PAGE.

RAM 1 ensures the drawing of the axial cross for the graph of the correct solution, RAM 2 draws the frame for the graphs of the errors and PAGE ensures paginating of the output code in the printer.

In practical application we directed our attention to equations containing nonlinear dependences of the type y^2, y^3 and $\sqrt{y}$. Computations were performed for the values $k = 5$ and $k = 10$, and for 4 types of the distribution of breakpoints.

Distribution number 1: a uniform distribution. The breakpoints lie on the graph of the function approximated.

Distribution number 2: a distribution with the same maximal errors in the particular sections. The breakpoints lie on the graph of the function approximated.

Distribution number 3: the distribution number 2 shifted by the value $\frac{\varepsilon_{max}}{2}$ in the direction of the axis $f(x)$.

Distribution number 4: the distribution number 2 shifted in the direction of the axis $f(x)$ so that both the approximating and the approximated functions have the same mean values in the interval $\langle 0; 1 \rangle$.

Valuation of the results:

It is natural that all our results depend on the differential equation considered. However, we may say that in applying the distribution with 10 linear sections, the resulting error is about 3–4 times smaller than in the same distribution with 5 sections.

In considering the influence of the distribution of breakpoints on the accuracy of solution we focus our attention on comparing the distribution number 1 with that of number 2. Here it cannot be admittedly determined which is more advantageous. Which distribution affects the resultant analog solution with a smaller error is decided by the course of the magnitude entering into the functional transformator. Concretely the more advantageous distribution is that, whose error of approximation is smaller in the sections which the magnitude entering into the transformator is going through. The fact that in these distributions the error of approximation does not change its sign is to great disadvantage of both. This causes in equations of the first order with a monotone nonconvergent solution that the derivative of the analog solution is continually greater (or smaller) than the derivative of the correct solution. Thus the analog solution increases continually more guickly (or more slowly) than the right solution (see for instance the equations $y' = y^2$, $y(0) = 0{,}2$ $\left(y = \frac{1}{5 - t}\right)$, $y' = 2y^3$, $y(0) = 0{,}25$ $\left(y = \right.$ $\left. = \sqrt{\frac{1}{4(4 - t)}}\right)$, $y' = 0{,}4\sqrt{y}$ $y(0) = 0{,}04$, $(y = 0{,}04 \,.\, (t + 1)^2)$. The above disadvantage may cause in equations of the second order with a convergent solution

that the analog computer generates a parabolic solution (for instance the equation $y'' = 2y^3$, $y(0) = 1$, $y'(0) = -1$, $\left(y = \frac{1}{t+1}\right)$ and equation $y'' = 6y^2$, $y(0) = 1$, $y'(0) = -2$, $\left(y = \frac{1}{(t+1)^2}\right)$.

The distribution number 3 has not this disadvantage which reflects on greater accuiacy of the solution. But also here the error of the analog solution does not mostly change its sign. It would be optimal if the resultant error continually changed its sign and if it were zero in passing to a new equation. This would ensure the error of the analog computer not to be affected by the variety of the initial conditions of the equation considered and that solved by the analog network. These requirements are best approached in the distribution number 4. Here the error of the analog solution far less depends on the course of the magnitude entering into the transformator than in the above distributions.

The accuracy of the analog solution of nonlinear differential equations may be considerably increased by appropriate choosing the distribution of the broken line through which the functional transformator approximates the function given. The values of the errors obtained in simulating the analog solution of the second order equation prove that in using the functional transformators the solution may become completely worthless and this by inappropriate choosing the distribution of the breakpoints (see figure 5a, b). On the contrary if we use a suitable distribution,

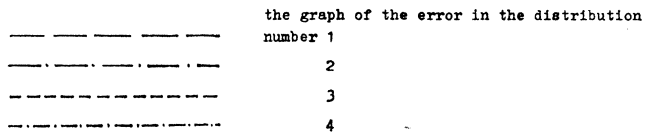

Fig. 4

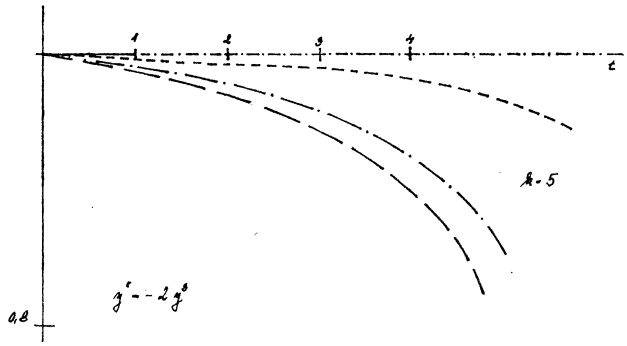

Fig. 5a

the error of the analog solution assumes virtually neglectable values. The most advantageous distribution of the breakpoints appears to be the distribution number 4.

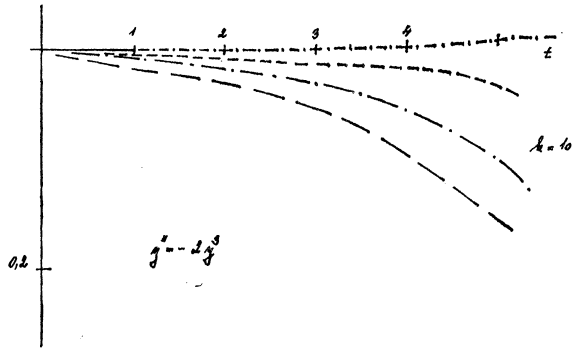

Fig. 5b

Souhrn

SIMULACE ANALOGOVÉHO POČÍTAČE PŘI ŘEŠENÍ NELINEÁRNÍCH DIFERENCIÁLNÍCH ROVNIC ČÍSLICOVÝM POČÍTAČEM

KAREL BENEŠ

V práci je vyšetřována teoretická přesnost analogového řešení nelineárních diferenciálních rovnic s použitím diodových funkčních transformátorů.

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Реэюме

СИМУЛЯЦИЯ АНАЛОГОВОЙ ВЫЧИСЛИТЕЛЬНОЙ МАШИНЫ ПРИ РЕШЕНИЮ НЕЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИИ ЦИФРОВОЙ МАШИНОЙ

КАРЕЛ БЕНЕШ

Работа занимается теоретической точностью аналогового решения нелинейных дифферентиальных уравнений с использованием диодных функциональных преобразователей.