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ON A STRUCTURE OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS HAVING THE SAME DISCRIMINANT

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1. Introduction

Let $\Delta = \Delta(\lambda)$ be the discriminant of a differential equation

$$y'' = (q(t) + \lambda)\, y, \; q \in C^\circ(\mathbf{R}),\; q(t + \pi) = q(t) \;\text{ for }\; t \in \mathbf{R}, \qquad (q+2)$$

$\lambda \in \mathbf{R}$. This paper presents all differential equations of the type

$$y'' = s(t, \lambda)\, y, \quad s \in C^0(\mathbf{R}\times\mathbf{R}), s(t + \pi, \lambda) = s(t, \lambda) \quad \textit{for } (t, \lambda) \in \mathbf{R}\times\mathbf{R}, \qquad (1)$$

whose discriminant is equal to $\Delta(\lambda)$.

2. Basic concepts and auxiliary results

Let us consider the differential equation of the type

$$y'' = p(t)\, y, \qquad p \in C^\circ(\mathbf{R}), \qquad p(t + \pi) = p(t) \qquad \text{for } t \in \mathbf{R}. \qquad (p)$$

The trivial solution of (p) is excluded from our considerations.

As is well-known (see [11]), the equation (p) is either oscillatory (i.e. ∞ and $-\infty$ are cluster points of zeros of any solution of (p)), or disconjugate (i.e. any solution of (p) has at most one zero on $\mathbf{R}$). If (p) is disconjugate, then it may be either pure disconjugate (i.e. there exist two linearly independent solutions of (p) not possessing any zero on $\mathbf{R}$) or special disconjugate (i.e. there exists one and only one solution of (p), up to a multiplicative constant, not possessing any zero on **R**) (see [14]).

Say that a function $\alpha \in C^0(\mathbf{R})$ is (the first elliptic) phase of (p) (see [2], [3]) if there exist linearly independent solutions u, v of (p) such that

$$\operatorname{tg} \alpha(t) = u(t)/v(t) \qquad \text{for } t \in \mathbf{R} - \{t; t \in \mathbf{R}, v(t) = 0\}.$$

Every phase α of (p) possesses the following properties:

(i) $\alpha \in C^3(\mathbf{R})$,

(ii) $\alpha'(t) \neq 0$ for $t \in \mathbf{R}$,

(iii) $-\{\alpha, t\} - \alpha'^2(t) = p(t)$ for $t \in \mathbf{R}$,

where $\{\alpha, t\} := \alpha'''(t)/(2\alpha'(t)) - (3/4)(\alpha''(t)/\alpha'(t))^2$ denotes the Schwarz derivative of α at the point t.

Let (p) be an oscillatory equation, n an integer and α a phase of (p). Let us set $\varphi_n(t) := \alpha^{-1}[\alpha(t) + n\pi \operatorname{sign} \alpha']$, $t \in \mathbf{R}$, where α^{-1} denotes the inverse function to the function α. The values of the function φ_n are independent of the choice of the phase α. The function φ_n is called the (first kind) central dispersion of (p) with the index n. The function φ_1, or more briefly φ, is called the (first kind) basic central dispersion of (p). This function possesses the following properties:

(i) $\varphi \in C^3(\mathbf{R})$,

(ii) $\varphi(t) > t$ for $t \in \mathbf{R}$,

(iii) $\varphi'(t) > 0$ for $t \in \mathbf{R}$,

(iv) $\varphi(t + \pi) = \varphi(t) + \pi$ for $t \in \mathbf{R}$,

(v) $\underbrace{\varphi\varphi \ldots \varphi(t)}_{n} = \varphi_n(t)$, $\varphi_{-n}(t) = \varphi_n^{-1}(t)$ for $t \in \mathbf{R}$,

(see [2], [3]).

Let (p) be a pure disconjugate equation. Say that a function $\beta \in C^0(\mathbf{R})$ is a hyperbolic phase of (p) if there exist linearly independent solutions u, v of (p) satisfying: $|u(t)| < |v(t)|$ and $\operatorname{tgh} \beta(t) = u(t)/v(t)$ for $t \in \mathbf{R}$. Then $\beta \in C^3(\mathbf{R})$, $\beta'(t) \neq 0$ and $p(t) = -\{\beta, t\} + \beta'^2(t)$ for $t \in \mathbf{R}$ (see [7], [9]).

Let (p) be a special disconjugate equation. Say that a function $\gamma \in C^0(\mathbf{R})$ is a parabolic phase of (p) if there exist linearly independent solutions u, v of (p), $v(t) \neq 0$ for $t \in \mathbf{R}$ such that $\gamma(t) = u(t)/v(t)$, $t \in \mathbf{R}$. Then $\gamma \in C^3(\mathbf{R})$, $\gamma'(t) \neq 0$ and $p(t) = -\{\gamma, t\}$ for $t \in \mathbf{R}$ (see [8], [9]).

Let $c \in C^3(\mathbf{R})$, $c'(t) \neq 0$ for $t \in \mathbf{R}$. Say that c is an elementary phase if $c(t + \pi) = c(t) + \pi \operatorname{sign} c'$, $t \in \mathbf{R}$ (see [2], [3]).

Let (p) be an oscillatory equation. The equation (p) is of category $(1, n)$, where n is a positive integer, if there exists an $x \in \mathbf{R}$: $\varphi_n(x) = x + \pi$. The equation (p) is of category $(2, m)$, where m is an integer, if there exists a number $a \in (0, 1)$ and a phase α of (p) such that $\alpha(t + \pi) = \alpha(t) + (2m + a)\pi$ (see [3]). All solutions of (p) are π-periodic or π-halfperiodic iff $\varphi_n(t) = t + \pi$ for $t \in \mathbf{R}$, where n an is even or an odd number. All solutions of (p) are bounded and are not π-periodic or π-halfperiodic iff (p) is of category $(2, m)$.

Convention. Let $u = u(t, \lambda)$ be a function defined on $\mathbf{D} \subset \mathbf{R} \times \mathbf{R}$, depending on the parameter λ. From now on (if there is no risk of confusion) we shall simplify matters by writing $u^{(i)}(t, \lambda)$ instead of $\frac{\partial^i u}{\partial t^i}(t, \lambda)$.

Following Floquet's theory every equation (1) may be associated with a quadratic equation

$$\varrho^2 - \Delta(\lambda)\,\varrho + 1 = 0,$$

whose roots are called the characteristic multipliers of (1) and $\Delta(\lambda)$ is called the discriminant of (1). Let $u = u(t, \lambda)$, $v = v(t, \lambda)$ be solutions of (1) satisfying the initial conditions: $u(0, \lambda) = v'(0, \lambda) = 0$, $u'(0, \lambda) = v(0, \lambda) = 1$. Then $\Delta(\lambda) = v(\pi, \lambda) + u'(\pi, \lambda)$ (see [1], [3], [6], [10]).

Let now $\Delta(\lambda)$ be the discriminant of (q + λ). We know from [1], [6] and [10] that the function $\Delta(\lambda)$ possesses derivatives of all orders on **R** and that there exists consequences $\{\lambda_i\}_{i=0}^{\infty}$, $\{\lambda'_i\}_{i=1}^{\infty}$,

$$\ldots < \lambda'_4 \leqq \lambda'_3 < \lambda_2 \leqq \lambda_1 < \lambda'_2 \leqq \lambda'_1 < \lambda_0, \tag{2}$$

such that $\Delta(\lambda) = 2$ iff $\lambda = \lambda_i$ $(i = 0, 1, 2, \ldots)$ and $\Delta(\lambda) = -2$ iff $\lambda = \lambda'_i$ $(i = 1, 2, 3, \ldots)$. The intervals $[\lambda_{2n}, \lambda_{2n-1}]$, $[\lambda'_{2n}, \lambda'_{2n-1}]$ $(n = 1, 2, 3, \ldots)$ are called the intervals of instability of (q + λ). For λ lying within these intervals, all solutions of (q + λ) are unbounded and the equation (q + λ) possesses two different real characteristic multipliers. The intervals $(\lambda'_{2n+1}, \lambda_{2n})$, $(\lambda_{2n+1}, \lambda'_{2n+2})$ $(n = 0, 1, 2, \ldots)$ are called the intervals of stability of (q + λ). For λ lying within these intervals, all solutions of (q + λ) are bounded and the equation (q + λ) possesses complex characteristic multipliers. If $\lambda'_{2n-1} = \lambda'_{2n}(\lambda_{2n-1} = \lambda_{2n})$ for a positive integer n, then all solutions of (q + λ'_{2n}) ((q + λ_{2n})) are π-halfperiodic (π-periodic). If $\lambda'_{2n-1} > \lambda'_{2n}(\lambda_{2n} < \lambda_{2n-1})$, then the equations (q + λ'_{2n-1}) and (q + λ'_{2n}) ((q + λ_{2n}) and (q + λ_{2n-1})) possess bounded (π-halfperiodic or π-periodic) solutions as well as unbounded solutions. The equation (q + λ_0) is special disconjugate and (q + λ) is for $\lambda > \lambda_0$ a pure disconjugate one.

Lemma 1. *There exists a phase* $\alpha = \alpha(t, \lambda)$ *of* (q + λ) *with the following properties:*

(i) $\dfrac{\partial^{i+j}\alpha(t, \lambda)}{\partial t^i \partial \lambda^j}$ *are continuous functions on* $\mathbf{R} \times \mathbf{R}$ *for* $i = 0, 1, 2, 3$ *and* $j = 0, 1, 2, \ldots$,

(ii) $\alpha(0, \lambda) = 0$ *for* $\lambda \in \mathbf{R}$,

(iii) $\alpha'(t, \lambda) \neq 0$ *on* $\mathbf{R} \times \mathbf{R}$.

Proof. Let $u = u(t, \lambda)$, $v = v(t, \lambda)$ be solutions of (q + λ) satisfying the initial conditions: $u(0, \lambda) = v'(0, \lambda) = 0$, $u'(0, \lambda) = v(0, \lambda) = 1$. Then it follows from the Theorem on continuous dependence of solutions on parameters ([5]) that $\dfrac{\partial^{i+j}u(t, \lambda)}{\partial t^i \partial \lambda^j}$ and $\dfrac{\partial^{i+j}v(t, \lambda)}{\partial t^i \partial \lambda^j}$ are continuous on $\mathbf{R} \times \mathbf{R}$ for $i = 0, 1, 2$ and $j =$

$= 0, 1, 2 \ldots$ Let us put

$$\alpha(t, \lambda) := \int_0^t ds/(u^2(s, \lambda) + v^2(s, \lambda)), \qquad (t, \lambda) \in \mathbf{R} \times \mathbf{R}.$$

Then $\alpha = \alpha(t\ \lambda)$ is a phase of (q + λ) having the properties (i)–(iii).

Lemma 2. *Let $\varphi_n = \varphi_n(t, \lambda)$ be the central dispersion of* (q + λ) *with the index n defined on $\mathbf{D} \subset \mathbf{R} \times \mathbf{R}$. Then φ_n has on* **D** *continuous partial derivatives up to and including order three.*

Proof. Let $\alpha = \alpha(t. \lambda)$ be a phase of (q + λ) having the properties (i)–(iii) stated in Lemma 1. Then α has continuous partial derivatives on $\mathbf{R} \times \mathbf{R}$ up to and including order three. Let us put $\varepsilon := \operatorname{sign} \alpha'(t, \lambda)$, $F(t, \lambda, z) := \alpha'(z, \lambda) - \alpha(t, \lambda) - n\pi\varepsilon$ for $(t, \lambda, z) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}$. Then the function F has continuous partial derivatives in the definition domain up to and including order three, $\dfrac{\partial F(t, \lambda, z)}{\partial t} = \alpha'(t, \lambda) \neq 0$ on $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$ and $F(t, \lambda, \varphi_n(t, \lambda)) = 0$ for $(t, \lambda) \in \mathbf{D}$. Thus, following the Theorem on implicit functions $\varphi_n = \varphi_n(t, \lambda)$ has on **D** continuous partial derivatives up to and including order three.

Remark 1. The continuity of the central dispersion of (q + λ) with the index n with respect to parameter λ was proved in [4].

Remark 2. Let λ_0 be a number occurring in (2). Then it holds for the set **D** in Lemma 2 that $\mathbf{D} = \mathbf{R} \times (-\infty, \lambda_0)$.

Lemma 3. *Let $[b, c]$, $b < c$, be an instability interval of* (q + λ). *Then there exists a positive integer n such that* $(1, n)$ *is the category of* (q + λ) *for* $\lambda \in [b, c]$.

Proof. The equation (q + λ) is oscillatory for $\lambda \in [b, c]$. Let $\varphi_m(t, \lambda)$ be the central dispersion of (q + λ) with the index m. The above function is surely defined on $\mathbf{R} \times [b, c]$. It follows from Lemma 2 and from the Sturm comparison theorem that φ_m is a continuous function on $\mathbf{R} \times [b, c]$, it is a decreasing function of the variable λ at a firm t and $\varphi_m(t + \pi, \lambda) = \varphi_m(t, \lambda) + \pi$ for $(t, \lambda) \in \mathbf{R} \times [b, c]$. Let $(1, n)$ be the category of (q + c). Assume that (q + λ) has no category $(1, n)$ for $\lambda \in [b, c]$. Clearly, there exists a $\lambda \in [b, c]$ such that the equation $\varphi_n(t, \lambda) - t - \pi = 0$ has a solution on **R**. Let $\bar{\lambda}$ be the least number of the given property. Evidently $\bar{\lambda} \in (b, c]$. It follows from [13] that for any $\lambda \in [b, c)$ the equation $\varphi_{n+1}(t, \lambda) - t - \pi = 0$ must have a solution on **R**. Let $\bar{\bar{\lambda}}$ be the greatest number of the given property. Then necessarily $\bar{\bar{\lambda}} < \bar{\lambda}$ and naturally there is $\varphi_n(t, \lambda) < t + \pi < \varphi_{n+1}(t, \lambda)$ $(t \in \mathbf{R})$ for $\lambda \in (\bar{\bar{\lambda}}, \bar{\lambda})$. Hence (q + λ) has for $\lambda \in (\bar{\bar{\lambda}}, \bar{\lambda})$ complex characteristic multipliers (see [13]) which, however, conflicts with the fact that (q + λ) has real characteristic multipliers for $\lambda \in [b, c]$.

Lemma 4. *Let (b, c) be a stability interval of* (q + λ). *Then there exists an integer m such that for any $\lambda \in (b, c)$ the equation* (q + λ) *has category* $(2, m)$.

Proof. The equation $(q + \lambda)$ is oscillatory for $\lambda \in (b, c)$. Thus, if we denote by $\varphi_n(t, \lambda)$ the central dispersion of $(q + \lambda)$ with the index n then, φ_n is defined on $\mathbf{R} \times (b, c)$. Furthermore, $(q + \lambda)$ has for $\lambda \in (b, c)$ complex characteristic multipliers, thus the character of $(q + \lambda)$ is of type $(2, k)$, where k is an integer. We have to show that the value of the number k is independent of the choice of the parameter λ in the interval (b, c). Let $\lambda^* \in (b, c)$. Then there exists a phase α_0 of $(q + \lambda^*)$, $a \in (0, 1)$, and an integer m such that

$$\alpha_0(t + \pi) = \alpha_0(t) + (2m + a)\pi, \qquad t \in \mathbf{R}.$$

Let us put $\nu := \operatorname{sign} \alpha_0'$. Since

$$\alpha_0(\varphi_{2m\nu}(t, \lambda^*)) = \alpha_0(t) + 2m\pi < \alpha_0(t) + (2m + a)\pi = \\ = \alpha_0(t + \pi) < \alpha_0(t) + (2m + 1)\pi = \alpha_0(\varphi_{(2m+1)\nu}(t, \lambda^*)), \qquad t \in \mathbf{R},$$

there is for $\nu = 1$ (necessarily $m \geqq 0$)

$$\varphi_{2m}(t, \lambda^*) < t + \pi < \varphi_{2m+1}(t, \lambda^*), \qquad t \in \mathbf{R}, \tag{3}$$

and for $\nu = -1$ (necessarily $m < 0$)

$$\varphi_{-2m}(t, \lambda^*) > t + \pi > \varphi_{-2m-1}(t, \lambda^*), \qquad t \in \mathbf{R}. \tag{4}$$

It follows from the continuity of $\varphi_k(t, \lambda)$ on $\mathbf{R} \times (b, c)$ that in case of (3) we obtain

$$\varphi_{2m}(t, \lambda) < t + \pi < \varphi_{2m+1}(t, \lambda) \qquad \text{for } (t, \lambda) \in \mathbf{R} \times (b, c), \tag{5}$$

and in case of (4) then

$$\varphi_{-2m}(t, \lambda) > t + \pi > \varphi_{-2m-1}(t, \lambda) \qquad \text{for } (t, \lambda) \in \mathbf{R} \times (b, c). \tag{6}$$

If (5) or (6) were impaired, then the equation $(q + \lambda_1)$ would have real characteristic multipliers for any $\lambda_1 \in (b, c)$, which would lead to a contradiction. It becomes evident that if (3) holds, then (5) holds as well, and if (4) holds, then (6) holds, too.

Remark 3. From Lemmas 3 and 4 and from their proofs we are led to: Let $\{\lambda_i\}_{i=0}^{\infty}$, $\{\lambda_i'\}_{i=1}^{\infty}$ be the sequences of numbers relative to $(q + \lambda)$ satisfying (2), discussed before. Then $(q + \lambda)$ has the categories $(2, 0)$, $(1, 1)$, $(2, -1)$, $(1, 2)$, $(2, 2)$, $(1, 3)$, $(2, -2)$, $(1, 4)$, ... on the intervals (λ_1', λ_0), $[\lambda_2', \lambda_1']$, (λ_1, λ_2'), $[\lambda_2, \lambda_1]$, (λ_3', λ_2), $[\lambda_4', \lambda_3']$, (λ_3, λ_4'), $[\lambda_4, \lambda_3]$, ..., respectively.

3. Main theorem

Theorem 1. *Let $\Delta(\lambda)$ be the discriminant of* $(q + \lambda)$. *Then equation* (1) *has the discriminant also equal to $\Delta(\lambda)$ if there exists a function $c = c(t, \lambda)$ defined on $\mathbf{R} \times \mathbf{R}$ such that the function c, at the firm value of the parameter λ, is an elementary* *phase and*

$$s(t, \lambda) = c'^2(t, \lambda)\,[q(c(t, \lambda)) + \lambda] - c'''(t, \lambda)/(2c'(t, \lambda)) + \\ + (3/4)\,(c''(t, \lambda)/c'(t, \lambda))^2 \qquad \text{for } (t, \lambda) \in \mathbf{R}\times\mathbf{R}. \tag{7}$$

The converse is valid, too. Let $c = c(t, \lambda)$ be an arbitrary function defined on $\mathbf{R}\times\mathbf{R}$ *such that c at the firm value of the parameter λ is an elementary phase and the function occurring on the right side of* (7) *is continuous on* $\mathbf{R}\times\mathbf{R}$. *Then the equation* (1), *where s is defined by* (7) *has the discriminant equal to* $\Delta(\lambda)$.

Proof. ($\Rightarrow$) Let equation (1) have the discriminant equal to $\Delta(\lambda)$. Let $\{\lambda_i\}_{i=0}^{\infty}$ and $\{\lambda_i'\}_{i=1}^{\infty}$ be sequences relative to (q + λ) whose properties were treated in part 2 of the paper. Then the equations (q + λ*) and $y'' = s(t, \lambda^*)\,y$ are for $\lambda^* \in \in (-\infty, \lambda_0)$ of the same behaviour (see [12]). Let α_0 be a phase of $(q + \lambda^*)$ and α_1 be a phase of $y'' = s(t, \lambda^*)\,y$. From Theorem in [12] then follows the existence of an elementary phase $c = c(t, \lambda^*)$ such that

$$\alpha_1(t) = \alpha_0[c(t, \lambda^*)], \qquad t \in \mathbf{R}. \tag{8}$$

Since $s(t, \lambda^*) = -\{\alpha_1, t\} - \alpha_1'^2(t)$, we get from (8) and from $\{\alpha\beta, t\} = \{\alpha, \beta(t)\} \times \times \beta'^2(t) + \{\beta, t\}$ (see [2], p. 8):

$$s(t, \lambda^*) = -\{\alpha_0, c(t, \lambda^*)\}\,c'^2(t, \lambda^*) - \{c, (t, \lambda^*)\} - \\ - \alpha_0'^2[c(t, \lambda^*)]\,c'^2(t, \lambda^*) = c'^2(t, \lambda^*)\,[q(c(t. \lambda^*)) + \lambda^*] - \\ - c'''(t, \lambda^*)/(2c'(t, \lambda^*)) + (3/4)\,(c''(t, \lambda^*)/c'(t, \lambda^*))^2,$$

hence (7) is valid for $(t, \lambda) \in \mathbf{R}\times(-\infty, \lambda_0)$.

Equations (q + λ) and (1) are for $\lambda = \lambda_0$ specially disconjugate and according to [14] they are of the same behaviour. Let β_0 be a parabolic phase of $(q + \lambda_0)$ and β_1 be a parabolic phase of $y'' = s(t, \lambda_0)\,y$. By Theorem 4 in [14] there exists an elementary phase $c = c(t, \lambda_0)$ such that $\beta_1(t) = \beta_0[c(t, \lambda_0)]$, $t \in \mathbf{R}$. From the equalities $q(t) + \lambda_0 = -\{\beta_0, t\}$, $s(t, \lambda_0) = -\{\beta_1, t\}$ we get

$$s(t, \lambda_0) = -\{\beta_1, t\} = -\{\beta_0, c(t, \lambda_0)\}\,c'^2(t, \lambda_0) - \\ - \{c, (t, \lambda_0)\} = c'^2(t, \lambda_0)\,[q(c(t, \lambda_0) + \lambda_0] - \\ - c'''(t, \lambda_0)/(2c'(t, \lambda_0)) + (3/4)\,(c''(t, \lambda_0)/c'(t, \lambda_0))^2.$$

Equations (q + λ*) and (1) are for $\lambda^* \in (\lambda_0, \infty)$ pure disconjugate and according to [14] they are of the same behaviour. Let γ_0 or γ_1 be hyperbolic phases of (q + λ*) or $y'' = s(t, \lambda^*)\,y$. Then, by Theorem 2 in [14] there exists an elementary phase $c = c(t, \lambda^*)$ such that $\gamma_1(t) = \gamma_0[c(t, \lambda^*)]$. Herefrom and from the equalities $q(t) + \lambda^* = -\{\gamma_0, t\} + \gamma'^2(t), s(t, \lambda^*) = -\{\gamma_1, t\} + \gamma'^2(t)$ we get

$$s(t, \lambda^*) = -\{\gamma_0, c(t, \lambda^*)\}\,c'^2(t, \lambda^*) - \{c, (t, \lambda^*)\} + \\ + \gamma_0'^2[c(t, \lambda^*)]\,c'^2(t, \lambda^*) = c'^2(t, \lambda^*)\,[q(c(t, \lambda^*) + \lambda^*] - \\ - c'''(t, \lambda^*)/(2c'(t, \lambda^*) + (3/4)\,(c''(t, \lambda^*)/c'(t, \lambda^*))^2.$$

Hence (7) is valid for $(t, \lambda) \in \mathbf{R}\times(\lambda_0, \infty)$.

($\Leftarrow$) Let $c = c(t, \lambda)$ be an arbitrary function defined on $\mathbf{R}\times\mathbf{R}$ such that $c(t, \lambda)$ at the firm value of the parameter λ is an elementary phase and the function occurring on the right side of (7) is continuous on $\mathbf{R}\times\mathbf{R}$. Let $s = s(t, \lambda)$ be defined by (7). Let (q + λ) be oscillatory for $\lambda \in (-\infty, \lambda_0)$ and let this equation be disconjugate for $\lambda \in [\lambda_0, \infty)$. Finally, let $\lambda^* \in (-\infty, \lambda_0)$ and α_0 be a phase of (q + λ*). Then it follows from (7) that the function $\alpha_1(t) := \alpha_0[c(t, \lambda^*)]$, $t \in \mathbf{R}$, is a phase of $y'' = s(t, \lambda^*)\, y$ and since $c(t, \lambda^*)$ is an elementary phase, it follows from [12] that both equations are of the same behaviour and thus they have the same characteristic multipliers. For $\lambda^* = \lambda_0$ or $\lambda^* \in (\lambda_0, \infty)$ we proceed in the same manner as above except for considering parabolic or hyperbolic phases instead of phases. Following the results in [14] it can be shown that the equations (q + λ*) and $y'' = s(t, \lambda^*)\, y$ have the same characteristic multipliers.

This proves our assertion that $\Delta(\lambda)$ is the discriminant of equation (1).

Example. Let $c(t, \lambda) := t + (1/\pi) \operatorname{arctg} \lambda \,.\, \sin 2t$ for $(t, \lambda) \in \mathbf{R}\times\mathbf{R}$. Then, at the firm value of the parameter λ the function c is an elementary phase and it follows from Theorem 1 that the equations (q + λ) and $y'' = s(t, \lambda)\, y$, where

$$s(t, \lambda) := [1 + (2/\pi) \operatorname{arctg} \lambda \,.\, \cos 2t]^2\, [q(t + (1/\pi) \operatorname{arctg} \lambda \,.\, \sin 2t) + \lambda] - \\ - 4 \operatorname{arctg} \lambda \,.\, \cos 2t/(\pi + 2 \operatorname{arctg} \lambda \,.\, \cos 2t) + \\ + (3/4)\, [4 \operatorname{arctg} \lambda \,.\, \sin 2t/(\pi + 2 \operatorname{arctg} \lambda \,.\, \cos 2t)]^2$$

for $(t, \lambda) \in \mathbf{R}\times\mathbf{R}$, have the same discriminant.

СТРУКТУРА ЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ВТОРОГО ПОРЯДКА С ПЕРИОДИЧЕСКИМИ КОЭФФИЦИЕНТАМИ КОТОРЫЕ ИМЕЮТ ОДИНАКОВЫЙ ДИСКРИМИНАНТ

Резюме

Пусть $\Delta = \Delta(\lambda)$ — дискриминант уравнения

$$y'' = (q(t) + \lambda)\, y,\; q \in C^0(\mathbf{R}),\; q(t + \pi) = q(t),\; \lambda \in \mathbf{R},\; t \in \mathbf{R}.$$

С помощью теории фаз и теории дисперсий показаны в работе все уравнении типа

$$y'' = s(t, \lambda)\, y,\; s \in C^0(\mathbf{R} \times \mathbf{R}),\; s(t + \pi, \lambda) = s(t, \lambda),\; (t, \lambda) \in \mathbf{R} \times \mathbf{R},$$

которые имеют дискриминант $\Delta(\lambda)$.

STRUKTURA LINEÁRNÍCH DIFERENCIÁLNÍCH ROVNIC 2. ŘÁDU S PERIODICKÝMI KOEFICIENTY, KTERÉ MAJÍ STEJNÝ DISKRIMINANT

Souhrn

Nechť $\Delta = \Delta(\lambda)$ je diskriminant rovnice $y'' = (q(t) + \lambda)\,y$, $q(t + \pi) = q(t)$ pro t, $\lambda \in \mathbf{R}$. V práci jsou užitím teorie fází a teorie dispersí nalezeny všechny rovnice typu $y'' = s(t, \lambda)\,y$, $s \in C^0(\mathbf{R} \times \mathbf{R})$, $s(t + \pi, \lambda) = s(t, \lambda)$ pro t, $\lambda \in \mathbf{R}$, jejichž diskriminant je roven $\Delta(\lambda)$.

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