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Jitka Kojecká

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Katedra matematické analýzy a numerické matematiky přírodovědecké fakulty Univerzity Palackého v Olomouci
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ON PHASES OF ACCOMPANYING SPACES TO A LINEAR TWO-DIMENSIONAL SPACE OF FUNCTIONS WITH A CONTINUOUS FIRST DERIVATIVE

JITKA KOJECKÁ

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This paper deals with relations between phases of two accompanying spaces $P\varrho[\alpha, \beta]$ and $P\sigma[\gamma, \delta]$ to the space $S \subset C_1(i)$ from the point of view of Academician O. Borůvka's theory on transformations of integrals of the second order differential equations [1]. It is referred to [4] and results of [2] and [3] are applied.

Throughout this article $S \subset C_1(i)$ is assumed to be a regular two-dimensional space of a certain type and the set $S' \subset C_0(i)$ of derivatives of all functions relative to S to be a regular two-dimensional space of a certain type as well. The function $w = uv' - u'v$ is a Wronskian of functions of the basis (u, v) of the space S.

Definition 1. *Let (u, v) be a basis of the space S and (u', v') be a basis of the space S'. The function $r_1(t) = \sqrt{u^2(t) + v^2(t)}$, $t \in i$, will be called the amplitude of the basis (u, v), the function $r_2(t) = \sqrt{u'^2(t) + v'^2(t)}$, $t \in i$, will be called the second amplitude of the basis (u, v).*

Theorem 1. *Let (u, v) be a basis of the space S, r_1 be the first amplitude and A be the first phase of the basis (u, v). Then it holds for $t \in i$*

$$\begin{aligned} u(t) &= \varepsilon r_1(t) \sin A(t), \\ v(t) &= \varepsilon r_1(t) \cos A(t), \end{aligned} \tag{1}$$

where $\varepsilon = \pm 1$.

Proof: With reference to the definition of the phase, it holds for all $t \in i$, for which $v(t) \neq 0$, $\operatorname{tg} A(t) = \dfrac{u(t)}{v(t)}$, where $A(t)$ is a phase of the basis (u, v). It then follows for $t \in i$

$$\sin A(t) = q(t)\, u(t),$$
$$\cos A(t) = q(t)\, v(t).$$

On squaring and adding, we obtain $1 = q^2 r_1^2$ and therefrom $| q | = \dfrac{1}{r_1}$, i.e. the relations of (1).

Definition 2. *The phase A of the basis (u, v) relative to the space S will be called proper and improper if $\varepsilon = 1$ and $\varepsilon = -1$, respectively, in* (1).

Theorem 2. *Any two phases in the system of the first phases of the basis (u, v) relative to S differ from each other by $2k\pi$ if both are proper or both improper, and by $(2k + 1)\,\pi$ if one is proper and the other improper; k is an integer number.*

Proof: Following Theorem 2.4 [2] there exists a countable system of the first phases of the basis (u, v) whereby the individual phases differ from one another by an integral multiple π. From the periodicity of functions sin and cos we obtain for k-integer and $x \in (-\infty, +\infty)$

$$\sin(x + 2k\pi) = \sin x, \qquad \sin(x + (2k + 1)\,\pi) = -\sin x,$$
$$\cos(x + 2k\pi) = \cos x, \qquad \cos(x + (2k + 1)\,\pi) = -\cos x.$$

Thus, if A_1 and A_2 are phases of the basis (u, v) both proper or improper, there must hold $A_2 = A_1 + 2k\pi$; if A_1 is proper and A_2 improper or vice versa, then $A_2 = A_1 + (2k + 1)\,\pi$.

Corollary 1. *The proper (improper) phases form a countable subsystem in the system of the first phase of the basis (u, v).*

Remark 1. *It is obvious that Theorems* 1 *and* 2 *and Corollary* 1 *are valid for the phases of an arbitrary two-dimensional space of continuous functions being regular and a certain type. If the functions are considered without a continuous first derivative, we speak only of an amplitude or a phase of the basis relative to this space.*

In all what follows every function $y \in S$ and its derivative y' will be considered to be independent on the interval i and two accompanying spaces $P\varrho[\alpha, \beta]$ and $P\sigma[\gamma, \delta]$ to the space S will be discussed (cf. definition 1.1 [4]). Then accompanying space $P\varrho[\alpha, \beta]$ or $P\sigma[\gamma, \delta]$ is a set of all functions having the form $\varrho(\alpha y + \beta y')$ or $\sigma(\gamma y + \delta y')$, where $\alpha, \beta, \gamma, \delta$ are real constants different from zero satisfying the condition $\alpha\delta - \beta\gamma \neq 0$, and $\varrho > 0$, $\sigma > 0$ are functions continuous on the interval i. We assume the spaces $P\varrho[\alpha, \beta]$ and $P\sigma[\gamma, \delta]$ to be regular and of a certain type on i. If (u, v) is a basis of the space S, then the characteristic or the phase of the basis $(\varrho(\alpha u + \beta u'), \varrho(\alpha v + \beta v'))$ relative to the space $P\varrho[\alpha, \beta]$ will be written as $f(t)$ or $\varphi(t)$, $t \in i$, and the characteristic or the phase of the basis $(\sigma(\gamma u + \delta u'), \sigma(\gamma v + \delta v'))$ relative to the space $P\sigma[\gamma, \delta]$ will be written as $p(t)$ or $\psi(t)$, $t \in i$.

Theorem 3. *Letting $t_0 \in i$ yields $w(t_0) = 0$ exactly if either $f(t_0) = p(t_0)$ or f and p are not simultaneously defined at the point t_0.*

Proof: I. Let f and p be defined at the point t_0. It then follows from $f(t_0) = p(t_0)$ that

$$[\alpha u(t_0) + \beta u'(t_0)]\,[\gamma v(t_0) + \delta v'(t_0)] = [\alpha v(t_0) + \beta v'(t_0)]\,[\gamma u(t_0) + \delta u'(t_0)],$$

whence a brief calculation gives $(\alpha\delta - \beta\gamma)\,w(t_0) = 0$, i.e. $w(t_0) = 0$. If f and p are not simultaneously defined at t_0, then

$$\alpha v(t_0) + \beta v'(t_0) = 0,$$
$$\gamma v(t_0) + \delta v'(t_0) = 0$$

and since $\alpha\delta - \beta\gamma \neq 0$, we obtain $v(t_0) = v'(t_0) = 0$ and by Theorem 1.7 [3] finally $w(t_0) = 0$.

II. Let $w(t_0) = 0$. Then it holds with reference to the part I of the proof, that

$$[\alpha u(t_0) + \beta u'(t_0)]\,[\gamma v(t_0) + \delta v'(t_0)] = [\alpha v(t_0) + \beta v'(t_0)]\,[\gamma u(t_0) + \delta u'(t_0)].$$

If $\alpha v(t_0) + \beta v'(t_0) \neq 0$ and $\gamma v(t_0) + \delta v'(t_0) \neq 0$, then $f(t_0) = p(t_0)$; if $\alpha v(t_0) + \beta v'(t_0) = 0$, then necessarity $\gamma v(t_0) + \delta v'(t_0) = 0$ because of the regularity of the space $P\varrho[\alpha, \beta]$. Thus f and p are not defined at the point t_0.

Corollary 2. *We see that $w(t_0) \neq 0$ for every $t_0 \in i$ iff either*

(i) the functions f, p are defined at t_0 and $f(t_0) \neq p(t_0)$

or

(ii) exactly one of the functions f, p is not defined at t_0.

Theorem 4. *If $t_0 \in i$, then $\varphi(t_0) = \psi(t_0) + k\pi$, k an integer, exactly if $w(t_0) = 0$.*

Proof: With respect to Theorem 3 the statement follows from the continuity of the phases φ and ψ on i as well as of the relations

$$\operatorname{tg} \varphi(t) = f(t), \qquad \operatorname{tg} \psi(t) = p(t)$$

for all $t \in i$, for which $f(t)$ and $p(t)$ are defined.

Corollary 3. *If $t_0 \in i$, then $\varphi(t_0) - \psi(t_0) \neq k\pi$, where k is an integer exactly if $w(t_0) \neq 0$.*

Remark 2. *Let us write $s_1 = \sqrt{(\alpha u + \beta u')^2 + (\alpha v + \beta v')^2}$ and $s_2 = \sqrt{(\gamma u + \delta u')^2 + (\gamma v + \delta v')^2}$. Following Theorem 1*

$$\begin{aligned} \varrho(\alpha u + \beta u') &= \varepsilon\varrho s_1 \sin \varphi, \\ \varrho(\alpha v + \beta v') &= \varepsilon\varrho s_1 \cos \varphi, \end{aligned} \tag{2}$$

where ϱs_1 is an amplitude, φ is a phase of the basis $(\varrho(\alpha u + \beta u'), \varrho(\alpha v + \beta v'))$

relative to the space $P\varrho[\alpha, \beta]$ *and* $\varepsilon = +1$ *or* -1 *according as the phase* φ *is proper or improper and*

$$\begin{aligned} \sigma(\gamma u + \delta u') &= \varepsilon' \sigma s_2 \sin \psi, \\ \sigma(\gamma v + \delta v') &= \varepsilon' \sigma s_2 \cos \psi, \end{aligned} \tag{3}$$

where σs_2 *is an amplitude,* ψ *is a phase of the basis* $(\sigma(\gamma u + \delta u'), \sigma(\gamma v + \delta v'))$ *relative to the space* $P\sigma[\gamma, \delta]$ *and* $\varepsilon' = +1$ *or* -1 *according as the phase* ψ *is proper or improper.*

Theorem 5. *Let* $w \neq 0$ *on the interval* $j \subset i$ *and* ε, ε' *be the numbers of* (2) *and* (3). *Then it holds for any* $t \in j$ *and* k *an integer that*

$$\begin{aligned} 2k\pi < \varphi(t) - \psi(t) < (2k+1)\pi, \quad &\textit{if} \quad \varepsilon\varepsilon'(\alpha\delta - \beta\gamma)\, w(t) > 0, \\ (2k-1)\pi < \varphi(t) - \psi(t) < 2k\pi, \quad &\textit{if} \quad \varepsilon\varepsilon'(\alpha\delta - \beta\gamma)\, w(t) < 0. \end{aligned}$$

Proof: On making use of (2) and (3) we can write

$$\begin{aligned} (\alpha u + \beta u')(\gamma v + \delta v') - (\alpha v + \beta v')(\gamma u + \delta u') &= \\ = \varepsilon\varepsilon' s_1 s_2 (\sin \varphi \cos \psi - \cos \varphi \sin \psi), \end{aligned}$$

whence a simple calculation gives

$\varepsilon\varepsilon'(\alpha\delta - \beta\gamma)\, w = s_1 s_2 \sin(\varphi - \psi)$, from which the statement results.

Theorem 6. *Let* (u, v) *be a basis of the space* S, $t_1, t_2 \in i$. *Then the functions* u, v *and the points* t_1, t_2 *satisfying the equation*

$$\begin{vmatrix} \alpha u(t_1) + \beta u'(t_1) & \alpha v(t_1) + \beta v'(t_1) \\ \gamma u(t_2) + \delta u'(t_2) & \gamma v(t_2) + \delta v'(t_2) \end{vmatrix} = 0 \tag{4}$$

exactly if there exists an $y \in S$ *such that* $\alpha y(t_1) + \beta y'(t_1) = 0$ *and* $\gamma y(t_2) + \delta y'(t_2) = 0$.

Proof: I. Let (4) by satisfied. Then the system of linear equations with the unknowns a, b

$$\begin{aligned} a(\alpha u(t_1) + \beta u'(t_1)) + b(\alpha v(t_1) + \beta v'(t_1)) &= 0, \\ a(\gamma u(t_2) + \delta u'(t_2)) + b(\gamma v(t_2) + \delta v'(t_2)) &= 0, \end{aligned}$$

has a nontrivial solution a_0, b_0 and it holds for the function $y = a_0 u + b_0 v$ relative to S $\alpha y(t_1) + \beta y'(t_1) = 0$ and $\gamma y(t_2) + \delta y'(t_2) = 0$.

II. Let (u, v) be the basis of the space S and let there exist an $y \in S$, $y = a_0 u + b_0 v$, $a_0^2 + b_0^2 \neq 0$, such that $\alpha y(t_1) + \beta y'(t_1) = 0$ and $\gamma y(t_2) + \delta y'(t_2) = 0$. On substituting we get

$$\begin{aligned} a_0(\alpha u(t_1) + \beta u'(t_1)) + b_0(\alpha v(t_1) + \beta v'(t_1)) &= 0, \\ a_0(\gamma u(t_2) + \delta u'(t_2)) + b_0(\gamma v(t_2) + \delta v'(t_2)) &= 0, \end{aligned}$$

from which we get the validity of (4).

Corollary 4. *Let the points $t_1, t_2 \in i$ and let the functions of (u, v) relative to the space S satisfy equation* (4). *If $w(t_1) \neq 0$ or $w(t_2) \neq 0$, then $t_1 \neq t_2$.*

Theorem 7. *Let $t_1, t_2 \in i$ and let there exist the basis (u, v) relative to the space S such that the functions u, v and the points t_1, t_2 satisfy equation* (4). *Then any two independent functions of the space S satisfy equation* (4) *at the points t_1, t_2.*

Proof: In view of the fact that every function $y \in S$ may be expressed as a nontrivial combination of two arbitrary functions of the space S, the statement follows from proof II of Theorem 6.

Theorem 8. *Let $t_1, t_2 \in i$. Then there exists the basis (u, v) relative to the space S such that the functions u, v and the points t_1, t_2 satisfy equation* (4) *exactly if either*

(*i*) *the function f is defined at the point t_1, the function p is defined at the point t_2 and $f(t_1) = p(t_2)$,*

or

(*ii*) *the function f is not defined at the point t_1 and the function p is not defined at the point t_2.*

Proof: I. Let equation (4) be valid. If $\alpha v(t_1) + \beta v'(t_1) \neq 0$ and $\gamma v(t_2) + \delta v'(t_2) \neq$ $\neq 0$, then

$$\frac{\alpha u(t_1) + \beta u'(t_1)}{\alpha v(t_1) + \beta v'(t_1)} = \frac{\gamma u(t_2) + \delta u'(t_2)}{\gamma v(t_2) + \delta v'(t_2)}$$

whence the statement (i) follows. If $\alpha v(t_1) + \beta v'(t_1) = 0$, then because of the regularity of the space $P\varrho[\alpha, \beta]$ we have $\gamma v(t_2) + \delta v'(t_2) = 0$, whence the statement (ii) follows.

II. If f at t_1 and p at t_2 are defined and $f(t_1) = p(t_2)$, then the validity of equation (4) is evident. If $\alpha v(t_1) + \beta v'(t_1) = 0$ and $\gamma v(t_2) + \delta v'(t_2) = 0$, then equation (4) holds (by Theorem 6).

Theorem 9. *Let $t_1, t_2 \in i$. Then there exists a basis (u, v) of the space S such that the functions u, v and the points t_1, t_2 satisfy equation* (4) *exactly if $\varphi(t_1) =$ $= \psi(t_2) + k\pi$ holds, k being an integer.*

Proof: The statement follows from the continuity of φ and ψ on i and from Theorem 8.

Considering the cases of the bases of the accompanying spaces $[\alpha, \beta] = [\alpha, 0]$ and $[\gamma, \delta] = [0, \delta]$ we find that the system of phases relative to the space $P\varrho[\alpha, \beta]$ is identical with the system of the first phases A relative to the space S and the system of phases relative to the space $P\sigma[\gamma, \delta]$ is identical with the system of the second phases B relative to the space S. With reference to Remark 1 it holds for the second phase $B(t)$, $t \in i$, of the basis (u, v) relative to S

$$\begin{aligned} u'(t) &= \varepsilon' r_2(t) \sin B(t), \\ v'(t) &= \varepsilon' r_2(t) \cos B(t), \end{aligned} \tag{5}$$

where $\varepsilon' = +1$ or -1 according as $B(t)$ is a proper phase or an improper one.

This leads us to conclude that between the phases A and B the following Theorem holds:

Theorem 10. *Let A and B be, respectively, the first and the second phase of the basis (u, v) relative to the space S. Let next $w \neq 0$ hold on the interval $j \subset i$ and $\varepsilon, \varepsilon'$ be the numbers from* (1) *and* (5). *Then*

$$2k\pi < A(t) - B(t) < (2k + 1)\pi, \quad \textit{if} \quad \varepsilon\varepsilon' w(t) > 0,$$
$$(2k - 1)\pi < A(t) - B(t) < 2k\pi, \quad \textit{if} \quad \varepsilon\varepsilon' w(t) < 0$$

holds for any $t \in j$ and k being an integer.

Proof: On making use of (1) and (5) we obtain

$$w = uv' - u'v = \varepsilon\varepsilon' r_1 r_2 (\sin A \cos B - \cos A \sin B),$$

thus

$$\varepsilon\varepsilon' w = r_1 r_2 \sin (A - B),$$

whence the statement follows.

Corollary 5. *Let A and B be, respectively, the first and the second phase of the basis (u, v) relative to the space S and $w(t_0) = 0$, where $t_0 \in i$. Then*

$$A(t_0) = B(t_0) + k\pi$$

holds for k being an integer.

ФАЗЫ СОПРОВОДИТЕЛЬНЫХ ПРОСТРАНСТВ К ЛИНЕЙНОМУ ДВУХРАЗМЕРНОМУ ПРОСТРАНСТВУ ФУНКЦИЙ С НЕПРЕРЫВНОЙ ПЕРВОЙ ПРОИЗВОДНОЙ

Резюме

Пусть $P\varrho[\alpha, \beta]$ и $P\sigma[\gamma, \delta]$ сопроводительные пространства к двухразмерному пространству $S \subset C_1(i)$, где $\alpha, \beta, \gamma, \delta$ не равные нулю вещественные постоянные, $\alpha\delta - \beta\gamma \neq 0$, $\varrho > 0$ и $\sigma > 0$ непрерывные функции на интервале i. Пусть (u, v) базис пространства S, обозначим $\varphi(t)$ фазу базиса $(\varrho(\alpha u + \beta u'), \varrho(\alpha v + \beta v'))$ пространства $P\varrho[\alpha, \beta]$ и $\psi(t)$ фазу базиса $(\sigma(\gamma u + \delta u'), \sigma(\gamma v + \delta v'))$ пространства $P\sigma[\gamma, \delta]$. Функция $w = uv' - u'v$ есть определитель Вронского функций базиса (u, v) пространства S.

Для фаз φ и ψ получаем следующие теоремы:

Теорема 4. *Если $t_0 \in i$, то $\varphi(t_0) = \psi(t_0) + k\pi$, k-целое, тогда и только тогда, когда $w(t_0) = 0$.*

Теорема 5. *Пусть на интервале $j \subset i$ есть $w \neq 0$ и $\varepsilon, \varepsilon'$ числа из формул* (2) *и* (3). *Тогда для каждого $t \in j$ и k-целого имеет место*

$$2k\pi < \varphi(t) - \psi(t) < (2k + 1)\pi, \quad \textit{если} \quad \varepsilon\varepsilon'(\alpha\delta - \beta\gamma)\, w(t) > 0,$$
$$(2k - 1)\pi < \varphi(t) - \psi(t) < 2k\pi, \quad \textit{если} \quad \varepsilon\varepsilon'(\alpha\delta - \beta\gamma)\, w(t) < 0.$$

Теорема 9. *Пусть* $t_1, t_2 \in i$. *Тогда существует базис* (u, v) *пространства* S *так, что функции* u, v *и точки* t_1, t_2 *удовлетворяют уравнению* (4) *тогда и только тогда, когда имеет место* $\varphi(t_1) = \psi(t_2) + k\pi$, k-*целое*.

В заключении показаны в теореме 10 и в ее следствии соотношения между первой фазой A и второй фазой B базиса (u, v) пространства S.

FÁZE PRŮVODNÍCH PROSTORŮ K LINEÁRNÍMU DVOJROZMĚRNÉMU PROSTORU FUNKCÍ SE SPOJITOU PRVNÍ DERIVACÍ

Souhrn

Nechť $P\varrho[\alpha, \beta]$ a $P\sigma[\gamma, \delta]$ jsou průvodní prostory k lineárnímu dvojrozměrnému prostoru $S = C_1(i)$, kde $\alpha, \beta, \gamma, \delta$ jsou reálné konstanty různé od nuly, $\alpha\delta - \beta\gamma \neq 0$, a $\varrho > 0$, $\sigma > 0$ jsou funkce spojité na intervalu i. Nechť (u, v) je báze prostoru S, označme $\varphi(t)$ fázi báze $(\varrho(\alpha u + \beta u'), \varrho(\alpha v + \beta v'))$ prostoru $P\varrho[\alpha, \beta]$ a $\psi(t)$ fázi báze $(\sigma(\gamma u + \delta u'), \sigma(\gamma v + \delta v'))$ prostoru $P\sigma[\gamma, \delta]$. Funkce $w = uv' - u'v$ je wronskián funkcí báze (u, v) prostoru S.

Pro fáze φ a ψ platí tato tvrzení:

Věta 4. *Buď* $t_0 \in i$. *Pak* $\varphi(t_0) = \psi(t_0) + k\pi$, k-*celé, právě tehdy, když* $w(t_0) = 0$.

Věta 5. *Nechť na intervalu* $j \subset i$ *je* $w > 0$ *a* $\varepsilon, \varepsilon'$ *jsou čísla ze vztahů* (2) *a* (3). *Pak platí pro každé* $t \in j$ *a* k-*celé*

$$2k\pi < \varphi(t) - \psi(t) < (2k+1)\pi, \text{ je-li } \varepsilon\varepsilon'(\alpha\delta - \beta\gamma)\, w(t) > 0$$
$$(2k-1)\pi < \varphi(t) - \psi(t) < 2k\pi, \text{ je-li } \varepsilon\varepsilon'(\alpha\delta - \beta\gamma)\, w(t) < 0.$$

Věta 9. *Buďte* $t_1, t_2 \in i$. *Pak existuje báze* (u, v) *prostoru* S *tak, že funkce* u, v *a body* t_1, t_2 *splňují rovnici* (4) *právě tehdy, když platí* $\varphi(t_1) = \psi(t_2) + k\pi$, *kde* k *je celé číslo.*

Závěrem jsou ve větě 10 a jejím důsledku uvedeny vztahy mezi první fází A a druhou fází B báze (u, v) prostoru S.

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RNDr. Jitka Kojecká
katedra mat. analýzy a numerické matematiky přírodovědecké fakulty Univerzity Palackého
Gottwaldova 15
771 46 Olomouc, ČSSR

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