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ON THE FUNDAMENTAL INEQUALITY IN LOCALLY MULTIPLICATIVELY CONVEX ALGEBRAS

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In this paper we will show a characterization of a complete Hermitian locally multiplicatively convex in general non commutative algebra with a not necessarily continuous or projective involution.

1. Introduction

The notion of seminormed algebra was introduced by R. Arens as a natural generalization of Banach algebras. They are called locally multiplicatively convex algebras by E. A. Michael [3]. Several properties of Banach algebras have been proved also for semi-normed algebras [3], [7], [8], [10]. V. Pták [6] recognized the importance of the function $p(x) = |x^*x|^{1/2}$, the square root of the spectral radius of the element x^*x. The inequality $|x|_\sigma \leqq p(x)$ as V. Pták proved gives a full characterization of Hermitian Banach algebras with an involution. The present author generalized this result for seminormed algebras with a projective involution in [8]. The aim of this paper is to obtain the same result if the projectiveness of the involution is not assumed.

2. Preliminaries

The reader is assumed to be familiar with the basic concepts concerning topological algebras, namely the Banach algebras, including spectra, Gelfand representation theory etc. All of them, as well as proofs, can be found in [1] for Banach algebras and in [3], [10], for the semi-normed case. Let us recall now some notations and facts which we shall use in this paper. An invodution defined on algebra A is a mapping $x \longrightarrow x^*$ of A onto itself such that the following for each pair $x, y \in A$ and for each complex λ holds:

(i) $x^{**} = x$,
(ii) $(\lambda x)^* = \bar{\lambda} x^*$,
(iii) $(x + y)^* = x^* + y^*$,
(iv) $(xy)^* = y^* x^*$.

A *-algebra is an algebra endowed by an involution. An element $x \in A$ is said to be regular, (selfadjoint) respectively, if it holds that there exists an inverse to x ($x = x^*$), respectively. A topological algebra is said to be seminormed, or locally multiplicatively convex if its topology can be given by means of a family $\{q_\alpha\}_{\alpha \in \Sigma}$ of submultiplicative semi-norms on A which separate points of A. The class of all locally multiplicatively convex algebras will be denoted by LMC. The spectrum of an element $x \in A$ will be denoted by $\sigma(x)$. If it is necessary to specify the algebra with respect to which the spectrum is taken, we shall use the notation $\sigma(A, x)$. The spectral radius of an element $x \in A$ is denoted by $| x |_\sigma$ and it is defined as $| x |_\sigma = \sup \{| \mu | : \mu \in \sigma(x)\}$. Let us mention that the last expression is not necessarily finite if A is semi-normed. The unit element of A will be denoted by e and will be ommited in expressions like $\lambda - x$. If we set $N_\alpha = \ker q_\alpha$ for some $\alpha \in \Sigma$ we obtain a closed ideal in A. Let A_α denote the Banach algebra obtained by the completion of the normed algebra $(A/N_\alpha, q_\alpha)$. By π_α we denote the natural homomorphism from A into A_α. Let us denote by π the mapping $\pi : A \to \to \prod_{\alpha \in \Sigma} A_\alpha$, $\pi(x) = (\pi_\alpha(x))_{\alpha \in \Sigma}$ where $\prod_{\alpha \in \Sigma} A_\alpha$ is the Cartesian product of spaces $\{A_\alpha\}_{\alpha \in \Sigma}$ endowed by the product topology and coordinatewise defined operations. This map is a topological isomorphism. If A is complete, the image $\pi(A)$ is a closed subalgebra in $\prod_{\alpha \in \Sigma} A_\alpha$.

Let now A be a complete algebra from LMC with a system of seminorms $\{q_\alpha\}_{\alpha \in \Sigma}$ as mentioned above. Write $\alpha < \beta$ for each pair $\alpha, \beta \in \Sigma$ if q_α is continuous with respect to q_β. This relation makes from Σ a directed set. If $\alpha < \beta$ we define a map $\pi_{\alpha\beta}$ from the algebra $(A/N_\beta, q_\beta)$ into $(A/N_\alpha, q_\alpha)$ by $\pi_{\alpha\beta}(\pi_\beta(x)) = \pi_\alpha(x)$. This map is a continuous homomorphism of A/N_β onto A/N and thus it can be extended by the unique way to continuous homomorphism of A_β into A . This extended mapping will be also denoted by $\pi_{\alpha\beta}$. It is obvious that for each $\alpha, \beta, \gamma \in \Sigma$ such that $\alpha < \beta < \gamma$ yields $\pi_{\alpha\gamma} = \pi_{\alpha\beta}\pi_{\beta\gamma}$. So we obtained a projective system of Banach algebras $(A_\alpha, \alpha \in \Sigma)$ with respect to the set of continuous homomorphisms $(\pi_{\alpha\beta}, \alpha < \beta)$ and it is a welknown fact [3], [10] that $\pi(A) = \varprojlim A_\alpha$ where the last term denotes the projective, or inverse limit of the system $\{A_\alpha\}_{\alpha \in \Sigma}$. We can obviously identify A and the last projective limit. This yields that an element $x \in A$ is regular in A iff for each $\alpha \in \Sigma$ its projection $\pi_\alpha(x)$ is regular in A_α and so the equality $\sigma(x, A) = \bigcup_{\alpha \in \Sigma} \sigma(x, A_\alpha)$ holds. For the spectral radius $| x |_\sigma = \sup_\alpha | \pi_\alpha(x) |_\sigma$ where the last is taken for each $\alpha \in \Sigma$ in the algebra A_α. We got that the spectrum $\sigma(x, A)$ is a nonempty, in general unbounded set of the complex plane. The mentioned topological isomorphism yields also that a generalized sequence $(x_\alpha)_{\alpha \in \Sigma} \in \prod_{\alpha \in \Sigma} A_\alpha$ belongs to A iff for each pair $\alpha, \beta < \Sigma$ such that $\alpha < \beta \pi_{\alpha\beta}(x_\beta) = x_\alpha$.

The set of all regular, selfadjoint, normal elements of a *-algebra A will be denoted by $R(A)$, $H(A)$, $N(A)$, respectively. For any set $S \subset A$ let $S^* = \{x^* \in A : x \in S\}$. If the elements of $S \cup S^*$ are pairwise commuting we say S is a normal set. We shall make a substantial use of the welknown fact that each normal subset S of the LMC *-algebra A is contained in a maximal closed commutative *-subalgebra $C \subset A$ and for each $x \in C$ $\sigma(x, C) = \sigma(x, A)$.

3. Characterization of Hermitian complete LMC algebras

In this part we shall characterize the Hermitian complete LMC algebras possessing the unit element by a set of inequalities concerning spectra which correspond to the V. Pták's fundamental inequality for the case of the algebra being a Banach algebra.

3.1. Definition: The *-algebra A is said to be Hermitian if the spectrum $\sigma(x)$ is real for each $x \in H(A)$.

3.2. Note: Let A be a LMC *-algebra, $\{q_\alpha\}_{\alpha \in \Sigma}$ the corresponding family of seminorms, x an arbitrary element of A. If no confusion is possible we shall use the following notations: for each $\alpha \in \Sigma \pi_\alpha(x) = x_\alpha$, $| \pi_\alpha(x) |_\sigma = | x |_\sigma^\alpha$, $p(\pi_\alpha(x)) = p_\alpha(x)$. Recall now a version of a square root lemma which we shall use in our next considerations. For the proof and other facts on square roots we refer the reader to [9].

3.3. Lemma: Let A be a Banach *-algebra and let $x \in A$. Let us suppose that $\sigma(x) > 0$ and $| x |_\sigma < 1$. Then there exists the unique square root $a \in A$ with a positive spectrum and, moreover, there is $a \in B(x)$, where $B(x)$ denotes the least closed commutative subalgebra of A generated by the element x.

Now we are able to state the main result.

3.4. Theorem: Let A be a complete LMC *-algebra possessing the unit element e. Let $\{q_\alpha\}_{\alpha \in \Sigma}$ be the corresponding family of seminorms on A. Then the following conditions are equivalent:

(i) the algebra A is Hermitian,

(ii) $| \pi_\alpha(x) |_\sigma \leqq p(\pi_\alpha(x)) = p_\alpha(x)$ for each $x \in N(A)$ and for all $\alpha \in \Sigma$.

Proof:

(i) → (ii):

Obviously, it is sufficient to show that for each $y \in N(A)$ and each complex λ such that $|\lambda| > p_\alpha(y)$ the element $(\lambda - y)_\alpha$ is regular in A_α. Without any loss of generality it is sufficient to show that if for any $x \in N(A)$ there is $p_\alpha(x) < 1$ then the element $(e - x)_\alpha$ is regular in A_α. It can be easily seen that for each $\alpha \in \Sigma p_\alpha(x) = p_\alpha(x^*)$. Actually, by the definition we have

$$p_\alpha^2(x) = | x^*x |_\sigma^\alpha \qquad \text{and} \qquad p_\alpha^2(x^*) = | xx^* |_\sigma^\alpha, \tag{1}$$

where both $(x^*x)_\alpha$, $(xx^*)_\alpha$ belong to A_α and so for their spectra

$$\sigma((x^*x)_\alpha)\setminus 0 = \sigma((x^*)_\alpha\, x_\alpha)\setminus 0 = \sigma((x)_\alpha\,(x_\alpha^*))\setminus 0 = \sigma((xx^*)_\alpha)\setminus 0$$

and (1) immediately follows. Now we shal show that $(e - x)_\alpha$ is left regular and also right regular. This is necessary, because unlike to the case of Banach algebras the left regularity of x_α does not imply the right regularity of $(x^*)_\alpha$ and converse as the involution need not be projective. The elements $(e - xx^*)_\alpha$, $(e - x^*x)_\alpha$ are positive by assumption and so we have

$$\begin{aligned}(e + x^*)_\alpha\,(e - x)_\alpha &= e_\alpha - x_\alpha^* x_\alpha + x_\alpha^* - x_\alpha = \\ = w_\alpha^2 + x_\alpha^* - x_\alpha &= w_\alpha(e_\alpha + (w_\alpha)^{-1}\,(x_\alpha^* - x_\alpha)\,(w_\alpha)^{-1})\,w_\alpha,\end{aligned} \tag{2}$$

where w_α is the regular and positive square root of $(e - x^*x)_\alpha$ $w_\alpha \in B((e - x^*x)_\alpha) \subset A_\alpha$, existing by 3.3. lemma. By the same way we get

$$\begin{aligned}(e - x)_\alpha\,(e - x^*)_\alpha &= (e_\alpha - x_\alpha x_\alpha^*) + x_\alpha^* - x_\alpha = (w_\alpha')^2 + x_\alpha^* - x_\alpha = \\ &= w_\alpha'(e_\alpha + (w_\alpha')^{-1}\,(x_\alpha^* - x_\alpha)\,(w_\alpha')^{-1})\,w_\alpha',\end{aligned} \tag{3}$$

where again the $w_\alpha' \in B((e - xx^*)_\alpha) \subset A_\alpha$ is the regular positive square root of $(e - xx^*)_\alpha$. Now, we can by a unique way write $x = h + ik$ for $h, k \in H(A)$ and the fact $x \in N(A)$ implies $hk = kh$. Further we have $x^* - x = -2ik$ and so for each $\alpha \in \Sigma$ the spectrum $\sigma(x_\alpha^* - x_\alpha)$ is purely imaginary as A is Hermitian. We can easily see that

$$B(e_\alpha - x_\alpha x_\alpha^*) = B(e_\alpha - (h_\alpha^2 + k_\alpha^2)) = B((h_\alpha^2 + k_\alpha^2)). \tag{4}$$

This immediately implies that both square roots w_α, w_α' are limits of corresponding sequences of polynomials of type $\sum_{l=m}^{n} c_l(h_\alpha^2 + k_\alpha^2)^l$ for suitable integers m, n. As k_α commutes with each expression of such a type so does w_α and w_α'. Now we shall use a simple application of the Gelfand transform theory for algebras $C(k_\alpha, w_\alpha)$, $C(k_\alpha, w_\alpha')$, the maximal commutative subalgebras containing the sets $\{k_\alpha, w_\alpha\}$, $\{k_\alpha, w_\alpha'\}$, respectively. From the welknown fact that the spectrum of each element a in a Banach commutative algebra X coincides with the set of all values $f(a)$ where f runs over the set $M(X)$ where the last term denotes the set of all multiplicative functionals on X. It easily follows that the spectra

$$\sigma(w^{-1}(x_\alpha^* - x_\alpha)\,w_\alpha^{-1}) = \{f(w_\alpha^{-1})\,.\,f(-2ik_\alpha)\,.\,f(w_\alpha^{-1}) : f \in M(C(k_\alpha, w_\alpha))\},$$

$$\sigma(w_\alpha')^{-1}(x_\alpha^* - x_\alpha)(w_\alpha')^{-1} = \{f((w_\alpha')^{-1} \,.\, f(-2ik_\alpha) \,.\, f((w_\alpha')^{-1}) : f \in M(C(k_\alpha, w_\alpha'))\},$$

are purely imaginary. This implies the regularity of both expressions in (2) and (3) and so we proved the left and right regularity of $(e - x)_\alpha$ required.

Q. E. D.

(ii) – (i):

Now let $h \in H(A)$ and let us suppose that $(\alpha + i\beta) \in \sigma(h)$ for some $\beta \neq 0$, β real. It immediately follows that $i \in \sigma(a)$, where $a = \dfrac{h - \alpha e}{\beta} \in H(A)$. For each $\tau > 0$ is $(a + \tau ie) \in N(A)$ and

$$\sigma(a + \tau ie) \in (\tau + 1)\, i.$$

As $\sigma(a + \tau ie) = \bigcup_{\alpha \in \Sigma} \sigma_\alpha(a + ie)$ there exists an $\alpha \in \Sigma$ such that

$$|(\tau + 1)\, i|^2 \leqq (|a + \tau ie|_\sigma^\alpha)^2 \leqq |(a + \tau ie)^* (a + \tau ie)|_\sigma^\alpha = |(a + \tau ie)_\alpha^* (a + \tau ie)_\alpha|_\sigma =$$
$$= |(a^2 + \tau^2 e)_\alpha|_\sigma = |a^2 + \tau^2 e|_\sigma^\alpha \leqq |a^2|_\sigma^\alpha + \tau^2 \qquad \text{for arbitrary } \tau > 0.$$

As the inequality $2\tau + 1 \leqq |a^2|_\sigma^\alpha$ does not hold for each $\tau > 0$ we got a contradiction with the compactness of the spectrum in a Banach algebra.

Q. E. D.

ОБ ФУНДАМЕНТАЛЬНОМ НЕРАВЕНСТВЕ В ПОЛУНОРМИРОВАННЫХ КОЛЬЦАХ

Резюме

В настоящей работе характеризуются польные полунормированные впольне симметрические кольца с единицей. Доказывается, что кольцо вполне сумметрическо тогда и только тогда, если для каждого нормального элемента x вышольняется система неравенств

$$p_a(x) = |x_\alpha^* x_a|_\sigma^{1/2}\, |x|_a,$$

где α пробегает множество индексов любой отделяющей системы полунорм данного кольца, p_a значит отвечающую спектральную полунорму и $|\ |_a$ отвечающий спектральный радиус. В работе не предполягается коммутативность инволюции.

O FUNDAMENTÁLNÍ NEROVNOSTI V LOKÁLNĚ MULTIPLIKATIVNĚ KONVEXNÍCH ALGEBRÁCH

Souhrn

V předložené práci jsou charakterizovány obecně nekomutativní úplné lokálně multiplikativně konvexní hermiteovské algebry, které mají jednotkový prvek. Přitom se nepředpokládá spojitost ani projektivnost involuce. Dokazuje se, že nutnou a postačující podmínkou k tomu, aby algebra

byla hermiteovskou, je skutečnost, že pro každý normální prvek x je splněn systém nerovností

$$| x |_{\sigma}^{\alpha} \leqq p_{\alpha}(x) = (| x^{*}x |_{\sigma}^{\alpha})^{1/2},$$

kde α probíhá indexovou množinu libovolného vhodného oddělujícího systému pseudonorem dané algebry.

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