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HOMOMORPHIC CORRESPONDENCES OF RELATIONAL SYSTEMS

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1. Motivation of the problem

In this paper the concept of a homomorphism of a language is generalized. Our considerations are motivated by the theorems on the homomorphisms of languages which have appeared in the algebraic linguistics (see [1]). But there occur certain asymmetries in these theorems. If two languages and a homomorphism of one of them onto the other are given, then the situation of both languages is asymmetric, because in general there exists no homomorphism of the second language onto the first one. Symmetric formulations of the theorems on languages can be attained in such a way that instead of homomorphisms we take correspondences between the languages which in a certain sence preserve the correctness of the theorems.

The transition to the more general conception obviously leads to the question, at which rate the concept of a homomorphic correspondence is more general than the concept of a homomorphism. This question is solved in this paper showing that every strongly homomorphic correspondence is a superposition of a strong homomorphism and a correspondence inverse to a strong homomorphism. One of the main results is a theorem stating that two languages between which a strongly homomorphic correspondence exists have isomorphic kernels.

Obviously all these results on languages can more generally be formulated for relational structures; the languages may be considered as particular cases of them.

2. Preserving correspondences

Let M be a set, let $n \geqq 0$ be an integer. The symbol M^n denotes the set of all words of the length n (= all n-term sequences) formed from the elements of M. In particular M^o contains a unique element, the empty word o. An element $x \in M$ is identified with the word of the length 1 whose unique element is x, thus $M^1 = M$. We put $M^* = \bigcup_{i \in N} M^i$, where N is the set of all non-negative integers.

The non-empty word over M, whose terms are subsequently the elements $x_1, x_2, \ldots, x_n$ of M, is written as $x_1 x_2 \ldots x_n$. The non-empty word over M^* whose terms are subsequently $x_1, x_2, \ldots, x_n$ from M^* (words over M) is written as $x_1 \bullet x_2 \bullet \ldots \bullet x_n$. For example if $m = 2$, $x_1 = ab$, $x_2 = baa$, then $x_1 \bullet x_2 = ab \bullet \bullet baa$.

Let M, N be sets. Put $P = M \cup N$. Then the words xy of P^2 such that $x \in M$ and $y \in N$ form the Cartesion product $M \times N$. If $\varrho \subseteqq M \times N$, then ϱ is called a *correspondence between M and N*. For $X \subseteqq M \times N$ we put $\varrho[X] = \{y \in N;$ there exists $x \in X$ such that $xy \in \varrho\}$. For every correspondence ϱ between M and N we define the *inverse correspondence* ϱ^{-1} by $\varrho^{-1} = \{yx;\ x \in M,\ y \in N,\ xy \in \varrho\}$. If ϱ is a correspondence between M and N and σ a correspondence between N and O, we put $\varrho \circ \sigma = \{xr;\ x \in M, r \in O$ and there exists an $y \in N$ such that $xy \in \varrho$, $yr \in \sigma\}$; $\varrho \circ \sigma$ is called *the product or the superposition of* ϱ *and* σ. It is well-known that for arbitrary correspondences $\alpha, \beta, \beta', \gamma$ the inclusion $\beta \subseteqq \beta'$ implies the inclusion $\alpha \circ \beta \circ \gamma \subseteqq \alpha \circ \beta' \circ \gamma$. The operation $\circ$ is evidently associative. Further $(\alpha \circ \beta)^{-1} = \beta^{-1} \circ \alpha^{-1}$ holds.

If ϱ is a correspondence between M and N such that $\varrho[M] = N$ and $\varrho^{-1}[N] = M$, we say that ϱ is a *correspondence of the set M onto the set N.*

Let M be a set and $n \geqq 0$ an integer. An arbitrary set $\sigma \subseteqq M^n$ is called an *n-ary relation on M*. The number n is denoted also by $r(\sigma)$ and called the *arity* of the relation σ.

Let M, N be sets, $n \geqq 0$ an integer, σ an n-ary relation on M, τ an n-ary relation on N, ϱ a correspondence of M onto N. About the correspondence ϱ we say that

1° *it is weakly* $\sigma\tau$*-preserving if the following holds:*

a) *If* $n \geqq 1$, $x_1, \ldots, x_n \in M$ *and* $x_1 \ldots x_n \in \sigma$, *then there exist* $x'_1, \ldots, x'_n \in N$ *that* $x_i x'_i \in \varrho$ *for* $1 \leqq i \leqq n$ *and* $x'_1 \ldots x'_n \in \tau$.

b) *If* $n = 0$ *and* $\sigma = \{o\}$, *then* $\tau = \{o\}$;

2° *it is* $\sigma\tau$*-preserving if the following holds:*

a) *If* $n \geqq 1$, $x_1, \ldots, x_n \in M$, $x_1 \ldots x_n \in \sigma$, $x'_1, \ldots, x'_n \in N$ *and* $x_i x'_i \in \varrho$ *for* $1 \leqq i \leqq n$, *then* $x'_1 \ldots x'_n \in \tau$.

b) *If* $n = 0$ *and* $\sigma = \{o\}$, *then* $\tau = \{o\}$;

3° *it is semistrongly* $\sigma\tau$*-preserving if* ϱ *is* $\sigma\tau$*-preserving and* ϱ^{-1} *is weakly* $\tau\sigma$*-preserving*;

4° *it is strongly* $\sigma\tau$*-preserving if* ϱ *is* $\sigma\tau$*-preserving and* ϱ^{-1} *is* $\tau\sigma$*-preserving.*

The following lemma can be easily proved.

1. Lemma. *Let M, N be sets, $n \geqq 0$ an integer, σ an n-ary relation on M, τ an n-ary relation on N, ϱ a correspondence of M onto N. Then the following assertions hold:*

(i) *If ϱ is $\sigma\tau$-preserving, then it is also weakly $\sigma\tau$-preserving.*

(ii) *If ϱ is strongly $\sigma\tau$-preserving, then it is also semistrongly $\sigma\tau$-preserving.*

(iii) *If ϱ is strongly $\sigma\tau$-preserving, then ϱ^{-1} is strongly $\sigma\tau$-preserving.* □

2. Example. *Let $M = \{x_1, x_2\}$, $N = \{x'_1, x'_2\}$. Let us define the 1-ary relations σ, τ on M and N and the correspondence ϱ of M onto N in the following way:*

(i) $\sigma = \{x_1, x_2\}$, $\tau = \{x'_1\}$, $\varrho = \{x_1x'_1, x_2x'_2, x_2x'_1\}$,

(ii) $\sigma = \{x_1\}$, $\tau = \{x'_1, x'_2\}$, $\varrho = \{x_1x'_1, x_2x'_2\}$,

(iii) $\sigma = \{x_1\}$, $\tau = \{x'_1\}$, $\varrho = \{x_1x'_1, x_2x'_2, x_2x'_1\}$,

(iv) $\sigma = \{x_1, x_2\}$, $\tau = \{x'_1, x'_2\}$, $\varrho = \{x_1x'_1, x_2x'_2, x_2x'_1\}$.

This example implies

3. Theorem. (i) *There exist sets M, N, unary relations σ on M and τ on N and a correspondence of M onto N which is weakly $\sigma\tau$-preserving.*

(ii) *There exist sets M, N, unary relations σ on M and τ on N and a correspondence of M onto N which is $\sigma\tau$-preserving, but not semistrongly $\sigma\tau$-preserving.*

(iii) *There exist sets M, N, unary relations σ on M and τ on N and a correspondence of M onto N which is semistrongly $\sigma\tau$-preserving, but not strongly $\sigma\tau$-preserving.*

(iv) *There exist sets M, N, unary relations σ on M and τ on N and a correspondence of M onto N which is strongly $\sigma\tau$-preserving.* □

Therefore, according to 1, there hold certain implications between the introduced concepts. According to 3 there hold no inverse implications do not hold.

4. Lemma. *Let M, N, O be sets, $n \geqq 0$ an integer, σ, τ, ω n-ary relations on M, N, O, respectively. Let α be a correspondence of M onto N, β a correspondence of N onto O.*

(i) *If α is weakly $\sigma\tau$-preserving and β is weakly $\tau\omega$-preserving, then $\alpha \circ \beta$ is weakly $\sigma\omega$-preserving.*

(ii) *If α is $\sigma\tau$-preserving and β is $\tau\omega$-preserving, then $\alpha \circ \beta$ is $\sigma\omega$-preserving.*

Proof. The case $n = 0$ is trivial. Thus let $n > 0$.

Let the assumptions of (i) be fulfilled. Let $x_1, \ldots, x_n \in M$, $x_1 \ldots x_n \in \sigma$. Then there exist $y_1, \ldots, y_n \in N$ such that $x_1y_1 \in \alpha, \ldots, x_ny_n \in \alpha$ and $y_1 \ldots y_n \in \tau$. Further there exist $r_1, \ldots, r_n \in O$ such that $y_1r_1 \in \beta, \ldots, y_nr_n \in \beta$ and $r_1 \ldots r_n \in \omega$. Then $x_1r_1 \in \alpha \circ \beta, \ldots, x_nr_n \in \alpha \circ \beta$ and $\alpha \circ \beta$ is weakly $\sigma\omega$-preserving and (i) holds.

Let the assumptions of (ii) be fulfilled. Let $x_1, \ldots, x_n \in N$, $x_1 \ldots x_n \in \sigma$, $r_1, \ldots, r_n \in O$, $x_1r_1 \in \alpha \circ \beta, \ldots, x_nr_n \in \alpha \circ \beta$. Then there exist $y_1, \ldots, y_n \in N$ such that $x_1y_1 \in \alpha, \ldots, x_ny_n \in \alpha$, $y_1r_1 \in \beta, \ldots, y_nr_n \in \beta$. Therefore $y_1 \ldots y_n \in \tau$ and $r_1 \ldots r_n \in \omega$. Therefore $\alpha \circ \beta$ is $\sigma\omega$-preserving and (ii) holds. □

3. Relational systems and their homomorphic correspondences

Let M, K be sets. To each $k \in K$ let a relation σ_k on the set M be assigned. Then the ordered pair $(M, (\sigma_k)_{k \in K})$ is called a *relational system.*

Two relational systems $(M, (\sigma_\varkappa)_{\varkappa \in K})$ and $(N, (\tau_\lambda)_{\lambda \in L}$ are said to be *similar* if there exists a bijection φ of K onto L such that $r(\sigma_\varkappa) = r(\tau_{\varphi(\varkappa)})$ for each $\varkappa \in K$. Without loss of generality we shall always assume for similar relational systems $(M, (\sigma_\varkappa)_{\varkappa \in K})$ and $(N, (\tau_\lambda)_{\lambda \in L})$ that $K = L$ and $\varphi = \mathrm{id}_K$.

Let $\mathbf{G} = (M, (\sigma_\varkappa)_{\varkappa \in K})$ and $\mathbf{H} = (N, (\tau_\varkappa)_{\varkappa \in K})$ be similar relational systems, let ϱ be a correspondence of M onto N. We shall say that ϱ is

1⁰ *weakly* **GH**-*preserving if* ϱ *is weakly* $\sigma_\varkappa\tau_\varkappa$-*preserving for each* $\varkappa \in K$;

2⁰ **GH**-*preserving if* ϱ *is* $\sigma_\varkappa\tau_\varkappa$-*preserving for each* $\varkappa \in K$;

3⁰ *semistrongly* **GH**-*preserving if* ϱ *is semistrongly* $\sigma_\varkappa\tau_\varkappa$-*preserving for each* $\varkappa \in K$;

4⁰ *strongly* **GH**-*preserving if* ϱ *is strongly* $\sigma_\varkappa\tau_\varkappa$-*preserving for each* $\varkappa \in K$.

Semistrongly **GH**-preserving correspondences will be also called *homomorphic correspondences* of **G** onto **H**, strongly **GH**-preserving correspondences will be also called *strongly homomorphic correspondences* of **G** onto **H**.

From 2.1 we have

1. Lemma. *Let* $\mathbf{G} = (M, (\sigma_\varkappa)_{\varkappa \in K})$, $\mathbf{H} = (N, (\tau_\varkappa)_{\varkappa \in K})$ *be similar relational systems, let* ϱ *be a correspondence of* M *onto* N.

(i) *If* ϱ *is* **GH**-*preserving, it is also weakly* **GH**-*preserving.*

(ii) *If* ϱ *is semistrongly* **GH**-*preserving, then it is also* **GH**-*preserving.*

(iii) *If* ϱ *is strongly* **GH**-*preserving, then it is also semistrongly* **GH**-*preserving.*

(iv) *If* ϱ *is a strongly* **GH**-*preserving correspondence of* **G** *onto* **H**, *then* ϱ^{-1} *is a strongly* **HG**-*preserving correspondence of* **H** *onto* **G**. □

From 2.3 we have

2. Lemma. (i) *There exist similar relational systems* **G**, **H** *and a weakly* **GH**-*preserving correspondence which is not* **GH**-*preserving.*

(ii) *There exist similar relational systems* **G**, **H** *and a* **GH**-*preserving correspondence which is not semistrongly* **GH**-*preserving.*

(iii) *There exist similar relational systems* **G**, **H** *and a semistrongly* **GH**-*preserving correspondence which is not strongly* **GH**-*preserving.*

(iv) *There exist similar relational systems* **G**, **H** *and a strongly* **GH**-*preserving correspondence.* □

From 2.4 we have two theorems:

3. Theorem. *Let* **G**, **H**, **I** *be similar relational systems,* α *a homomorphic correspondence of* **G** *onto* **H**, β *a homomorphic correspondence of* **H** *onto* **I**. *Then* $\alpha \circ \beta$ *is a homomorphic correspondence of* **G** *onto* **I**. □

4. Theorem. *Let* **G**, **H**, **I** *be similar relational systems,* α *a strongly homomorphic* *correspondence of* **G** *onto* **H**, β *a strongly homomorphic correspondence of* **H** *onto* **I**. *Then* $\alpha \circ \beta$ *is a strongly homomorphic correspondence of* **G** *onto* **I**. $\square$

Note that among homomorphic correspondences as particular cases homomorphic mappings are included which are surjective. A homomorphic mapping is called a *homomorphism*, a strongly homomorphic mapping is called a *strong homomorphism*. Evidently every bijective homomorphism is strong; a bijective strong homomorphism is called an *isomorphism*. Then two relational systems are called *isomorphic* if there exists an isomorphism of one of them onto the other.

Now we shall introduce relations on a strongly homomorphic correspondence between two similar relational systems so that this correspondence becomes a relational system similar to both given relational systems.

Thus let $\mathbf{G} = (M, (\sigma_\varkappa)_{\varkappa \in K})$, $\mathbf{H} = (N, (\tau_\varkappa)_{\varkappa \in K})$ be similar relational systems, let ϱ be a strongly homomorphic correspondence of **G** onto **H**. For each $\varkappa \in K$ with $r(\sigma_\varkappa) \geqq 1$ we put $\omega_\varkappa = \{x_1x'_1 \bullet x_2x'_2 \bullet \ldots \bullet x_{r(\sigma_\varkappa)}x'_{r(\sigma_\varkappa)};\ x_1, \ldots, x_{r(\sigma_\varkappa)} \in M,\ x'_1, \ldots, x'_{r(\sigma_\varkappa)} \in N,\ x_1x'_1 \in \varrho,\ x_2x'_2 \in \varrho, \ldots, x_{r(\sigma_\varkappa)}x'_{r(\sigma_\varkappa)} \in \varrho,\ x_1 \ldots x_{r(\sigma_\varkappa)} \in \sigma_\varkappa\}$; for $r(\sigma_\varkappa) = 0$ we put $\omega_\varkappa = \{o\}$ if and only if $\sigma_\varkappa = \{o\}$.

Evidently $\omega_\varkappa$ is a relation on ϱ and $r(\omega_\varkappa) = r(\sigma_\varkappa)$.

The last condition $x_1 \ldots x_{r(\sigma_\varkappa)} \in \sigma_\varkappa$ in the definition of $\omega_\varkappa$ is evidently equivalent to the condition $x'_1 \ldots x'_{r(\sigma_\varkappa)} \in \tau_\varkappa$. A relational system $(\varrho, (\omega_\varkappa)_{\varkappa \in K})$ will be called an *algebraization of the strongly homomorphic correspondence* ϱ *of* **G** *onto* **H**.

5. Theorem. *Let* $\mathbf{G} = (M, (\sigma_\varkappa)_{\varkappa \in K})$, $\mathbf{H} = (N, (\tau_\varkappa)_{\varkappa \in K})$ *be similar relational systems, let* ϱ *be a strongly homomorphic correspondence of* **G** *onto* **H**, $\mathbf{I} = (\varrho, (\omega_\varkappa)_{\varkappa \in K})$ *the algebraization of* ϱ. *Further put*

$$f = \{xx' \,.\, x;\ x \in M,\ x' \in N,\ xx' \in \varrho\},$$
$$g = \{xx' \,.\, x';\ x \in M,\ x' \in N,\ xx' \in \varrho\},$$

Then f, g *are strong homomorphisms of* **I** *onto* **G** *and* **H**, *respectively, and* $\varrho = f^{-1} \circ g$ *holds.*

Proof. f and g are evidently surjective mappings. Let $\varkappa \in K$ and $r(\sigma_\varkappa) = 0$. Then $\omega_\varkappa = \{o\}$ if and only if $\sigma_\varkappa = \{o\}$. Thus let $r(\sigma_\varkappa) \geqq 1$, let $x_1, \ldots, x_{r(\sigma_\varkappa)} \in M$, $x'_1, \ldots, x'_{s(\sigma_\varkappa)} \in N$, further let $x_1x'_1 \bullet x_2x'_2 \bullet \ldots \bullet x_{r(\sigma_\varkappa)}x'_{r(\sigma_\varkappa)} \in \omega_\varkappa$. Let $y_1, \ldots, y_{r(\sigma_\varkappa)} \in M$ and let $x_1x'_1 \,.\, y_1 \in f, \ldots, x_{r(\sigma_\varkappa)}x'_{r(\sigma_\varkappa)} \,.\, y_{r(\sigma_\varkappa)} \in f$. According to the definition of f we have $y_1 = x_1, \ldots, y_{r(\sigma_\varkappa)} = x_{r(\sigma_\varkappa)}$. According to the definition of $\omega_\varkappa$ we have $y_1 \ldots y_{r(\sigma_\varkappa)} = x_1, \ldots, x_{r(\sigma_\varkappa)} \in \sigma_\varkappa$. We have proved that f is a **GH**-preserving relation.

Now let $r(\sigma_\varkappa) \geqq 1$, let $x_1, \ldots, x_{r(\sigma_\varkappa)} \in M$ and $x_1 \ldots x_{r(\sigma_\varkappa)} \in \sigma_\varkappa$. Further let $z_1, \ldots, z_{r(\sigma_\varkappa)} \in \varrho$ be such that $x_1 \,.\, z_1 \in f^{-1}, \ldots, x_{r(\sigma_\varkappa)} \,.\, z_{r(\sigma_\varkappa)} \in f^{-1}$. Then $z_1 \,.\, x_1 \in f, \ldots, z_{r(\sigma_\varkappa)} \,.\, x_{r(\sigma_\varkappa)} \in f$. According to the definition there exist $x'_1, \ldots, x'_{r(\sigma_\varkappa)} \in N$ such that $z_1 = x_1x'_1 \in \varrho, \ldots, z_{r(\sigma_\varkappa)} = x_{r(\sigma_\varkappa)}x'_{r(\sigma_\varkappa)} \in \varrho$. According to the definition of $\omega_\varkappa$ we have $z_1 \ldots z_{r(\sigma_\varkappa)} = x_1x'_1 \bullet \ldots \bullet x_{r(\sigma_\varkappa)}x'_{r(\sigma_\varkappa)} \in \omega_\varkappa$ and therefore f^{-1} is a **GH**-preserving relation.

Altogether we have proved that f is a strong homomorphism. An analogous assertion can be similarly proved for g.

Let $x \in M$, $x' \in N$ be arbitrary. Then evidently $xx' \in \varrho$ if and only if $x \,.\, xx' \in f^{-1}$ and $xx' \,.\, x' \in g$, hence if and only if $xx' \in f^{-1} \circ g$.

4. Congruences of relational systems

In the sequel the symbol $E(M)$ denotes the set of all equivalences on the set M.

Let $\mathbf{G} = (M, (\sigma_\varkappa)_{\varkappa \in K})$ be a relational system. Then an arbitrary equivalence $\Pi \in E(M)$ is a correspondence of $\mathbf{G}$ onto $\mathbf{G}$; if it is moreover $\mathbf{GG}$-preserving, we call it a *congruence* on $\mathbf{G}$. As the condition b) from the definition of a $\mathbf{GH}$-preserving correspondence is automatically fulfilled, the definition a congruence can be expressed as follows: It is an equivalence $\Pi \in E(M)$ such that for each $\varkappa \in K$ with the property $r(\sigma_\varkappa) \geqq 1$ and for arbitrary $x_1, \dots, x_{r(\sigma_\varkappa)}, y_1, \dots, y_{r(\sigma_\varkappa)}$ in M with the properties $x_1 \dots x_{r(\sigma_\varkappa)} \in \sigma_\varkappa$ and $x_1 y_1 \in \Pi, \dots, x_{r(\sigma_\varkappa)} y_{r(\sigma_\varkappa)} \in \Pi$ we have $y_1 \dots$ $\dots, y_{r(\sigma_\varkappa)} \in \sigma_\varkappa$.

As $\Pi^{-1} = \Pi$ for each $\Pi \in E(M)$, we have

1. Lemma. *Let* $\mathbf{G} = (M, (\sigma_\varkappa)_{\varkappa \in K})$ *be a relational system, let* $\Pi \in (EM)$. *Then* Π *is a congruence on* $\mathbf{G}$, *if and only if it is strongly* $\mathbf{GG}$*-preserving.* □

The symbol $S(\mathbf{G})$ denotes the set of all congruences on the relational system $\mathbf{G}$. Evidently $S(\mathbf{G}) \subseteq E(M)$. The last set is a complete lattice with respect to the relation of inclusion. It is well-known (see [1], theorem 4.7):

2. Theorem. *Let* $\mathbf{G} = (M, (\sigma_\varkappa)_{\varkappa \in K})$ *be a relational system. Then* $S(\mathbf{G})$ *is a convex complete sublattice of the lattice* $E(M)$, *i.e. for each* $\emptyset \neq K \subseteq S(\mathbf{G})$ *we have*

$\sup_{E(M)} K \in S(\mathbf{G})$, $\inf_{E(M)} K \in S(\mathbf{G})$

and $\alpha \in S(\mathbf{G})$, $\beta \in E(M)$, $\beta \subseteq \alpha$ *implies* $\beta \in S(\mathbf{G})$. □

Put $\equiv_{\mathbf{G}} = \sup_{E(M)} S(\mathbf{G})$. According to 2 $\equiv_{\mathbf{G}}$ is a greatest congruence on $\mathbf{G}$. From 2 we have

3. Corollary. *Let* $\mathbf{G} = (M, (\sigma_\varkappa)_{\varkappa \in K})$. For an arbitrary $\Pi \in E(M)$ we have $\Pi \in S(\mathbf{G})$ if and only if $\Pi \subseteq \equiv_{\mathbf{G}}$. □

Further congruences can be obtained from a given congruence and a strong homomorphism:

4. Theorem. *Let* $\mathbf{G} = (M, (\sigma_\varkappa)_{\varkappa \in K})$, $\mathbf{H} = (N, (\tau_\varkappa)_{\varkappa \in K})$ *be similar relational systems, let* f *be a strong homomorphism of* $\mathbf{G}$ *onto* $\mathbf{H}$. *Then the following assertions hold:*

(i) *If* $\alpha \in S(\mathbf{H})$, *then* $f \circ \alpha \circ f^{-1} \in S(\mathbf{G})$.

(ii) $f \circ f^{-1} \in S(\mathbf{G})$.

(iii) $f^{-1} \circ \equiv_{\mathbf{G}} \circ f \in S(\mathbf{H})$.

Proof. (1) *If* $\alpha \in S(\mathbf{H})$, then $f \circ \alpha \circ f^{-1}$ is a strongly **GG**-preserving correspondence according to 1 and 3.4. For each $x \in M$ we have $xf(x) \in f$, $f(x)f(x) \in \alpha$, $f(x)\, x \in f^{-1}$, therefore $xx \in f \circ \alpha \circ f^{-1}$. The symmetry of α implies $(f \circ \alpha \circ f^{-1})^{-1} = f \circ \alpha \circ f^{-1}$. Finally, the associative law for the operation $\circ$, the equality $f^{-1} \circ f = \mathrm{id}_N$, and the inclusion $\alpha \circ \alpha \subseteq \alpha$ (transitivity) imply $(f \circ \alpha \circ f^{-1}) \circ (f \circ \alpha \circ f^{-1}) \subseteq f \circ \alpha \circ f^{-1}$. Hence $f \circ \alpha \circ f^{-1}$ is a strongly **GG**-preserving equivalence on **G** and we have (i).

(2) As $\mathrm{id}_N \in S(\mathbf{H})$, also $f \circ f^{-1} \in S(\mathbf{G})$ according to (i), which is (ii).

(3) $f^{-1} \circ \equiv_{\mathbf{G}} \circ f$ is a strongly **HH**-preserving correspondence of the relational system **H** onto **H** according to 3.4. For each $y \in N$ there exists $x \in M$ such that $f(x) = y$. Then $yx \in f^{-1}$, $xx \in \equiv_{\mathbf{G}}$, $xy \in f$, therefore $yy \in f^{-1} \circ \equiv_{\mathbf{G}} \circ f$. The symmetry of the relation $\equiv_{\mathbf{G}}$ implies the symmetry of $f^{-1} \circ \equiv_{\mathbf{G}} \circ f$ similarly as in (i). The associative law for the operation $\circ$ (ii) and the condition $\equiv_{\mathbf{G}} \circ \equiv_{\mathbf{G}} \subseteq \equiv_{\mathbf{G}}$ (transitivity) imply $(f^{-1} \circ \equiv_{\mathbf{G}} \circ f) \circ (f^{-1} \circ \equiv_{\mathbf{G}} \circ f) \subseteq f^{-1} \circ \equiv_{\mathbf{G}} \circ f$. Hence $f^{-1} \circ \equiv_{\mathbf{G}}$ $\equiv_{\mathbf{G}} \circ f$ is a strongly **HH**-preserving equivalence on **H** and we have (iii). □

5. Corollary. *Let* **G** **H** *be similar relational systems let* f *be a strong homomorphism of* **G** *onto* **H**. *Then* $\equiv_{\mathbf{H}} = f^{-1} \circ \equiv_{\mathbf{G}} \circ f$.

Proof. As $f^{-1} \circ f = \mathrm{id}_N$ we have $\equiv_{\mathbf{H}} = f^{-1} \circ f \circ \equiv_{\mathbf{H}} \circ f^{-1} \circ f$. According to 4(i) we have $f \circ \equiv_{\mathbf{H}} \circ f^{-1} \subseteq \equiv_{\mathbf{G}}$ therefore $\equiv_{\mathbf{H}} \subseteq f^{-1} \circ \equiv_{\mathbf{G}} \circ f \subseteq \equiv_{\mathbf{H}}$ according to 4(iii). □

5. Quotient relational systems

Let $\mathbf{G} = (M, (\varrho_\varkappa)_{\varkappa \in K})$ be a relational system let $\Pi \in E(M)$. Let $\varkappa \in K$, $r(\sigma_\varkappa) \geqq 1$. For arbitrary $A_1, \ldots, A_{r(\sigma_\varkappa)} \in M/\Pi$ we put $A_1 \ldots A_{r(\sigma_\varkappa)} \in \sigma_\varkappa/\Pi$, if and only if there exist $a_1 \in A_1, \ldots, a_{r(\sigma_\varkappa)} \in A_{r(\sigma_\varkappa)}$ such that $a_1 \ldots a_{r(\sigma_\varkappa)} \in \sigma_\varkappa$. If $r(\sigma_\varkappa) = 0$ and $\sigma_\varkappa = \{o\}$ we put $\sigma_\varkappa/\Pi = \{o\}$.

Evidently $(M/\Pi, (\sigma_\varkappa/\Pi)_{\varkappa \in K})$ is a relational system similar to the relational system **G**. We denote it by $\mathbf{G}/\Pi$ and call it the *quotient* of the relational system **G** by the equivalence Π. The *natural* mapping *nat* Π is the mapping to each $x \in M$ assigning $X \in M/\Pi$ so that $x \in X$. The definition immediately implies

1. Theorem. *Let* $\mathbf{G} = (M\ (\sigma_\varkappa)_{\varkappa \in K})$ *be a relational system.* $\Pi \in E(M)$. *Then nat* Π *is a homomorphism of* **G** *onto* $\mathbf{G}/\Pi$. □

The following theorem is well-known (see [1] theorem 4.6)

2. Theorem. *Let* $\mathbf{G} = (M, (\sigma_\varkappa)_{\varkappa \in K})$ *be a relational system let* $\Pi \in E(M)$. Then $\Pi \in S(\mathbf{G})$ *if and only if nat*Π *is a strong homomorphism.* □

Let **G** be a relational system. Then the quotient $\mathbf{G}/\equiv_{\mathbf{G}}$ will be called the kernel of **G** and denoted by *ker***G**.

3. Theorem. *Let* $\mathbf{G} = (M\ (\sigma_x)_{x \in K})$, $\mathbf{H} = (N\ (\tau_x)_{x \in K})$ *be similar relational systems. Let there exist astrong homomorphism* f *of* $\mathbf{G}$ *onto* $\mathbf{H}$. *Then the corresponld-ence* $F = (nat \equiv_{\mathbf{G}})^{-1} \circ f \circ nat \equiv_{\mathbf{H}}$ *is an isomorphism of ker*$\mathbf{G}$ *onto ker*$\mathbf{H}$.

Proof. According to 2 the correspondences $nat \equiv_{\mathbf{G}}$ and $nat \equiv_{\mathbf{H}}$ are strong homomorphism. According to 3.1 and 3.4 F is strongly homomorphic. We shall prove that it is a bijection.

Really let $A \in M/\equiv_{\mathbf{G}}$, $B, C \in N/\equiv_{\mathbf{H}}$, $AB \in F$, $AC \in F$. Then there exist $a, a' \in A$, $b \in B$, $c \in C$ such that $ab \in f$, $a'c \in f$. This implies $aa' \in \equiv_{\mathbf{G}}$, therefore $bc \in f^{-1} \circ \equiv_{\mathbf{G}} \equiv_{\mathbf{G}} \circ f = \equiv_{\mathbf{H}}$ according to 4.5. This implies $B = C$, therefore F is a mapping of $M/\equiv_{\mathbf{G}}$ onto $N/\equiv_{\mathbf{H}}$.

Now let $A, D \in M/\equiv_{\mathbf{G}}$, $B \in N/\equiv_{\mathbf{H}}$, $AB \in F$, $DB \in F$. Then there exist $a \in A$, $d \in D$, $b, b' \in B$ such that $ab \in f$, $db' \in f$. We have $bb' \in \equiv_{\mathbf{H}}$, therefore $ad \in f \circ \equiv_{\mathbf{G}} \equiv_{\mathbf{H}} \circ f^{-1} \subseteq \equiv_{\mathbf{G}}$ according to 4.4(i). Hence $A = D$ and F is a bijection.

Therefore F is a bijective strong homomorphism, i.e. an isomorphism of $\mathbf{G}/\equiv_{\mathbf{G}}$ onto $\mathbf{H}/\equiv_{\mathbf{H}}$. □

4. Theorem. *Let* $\mathbf{G}$, $\mathbf{H}$ *be similar relational systems. Let there exist a strongly homomorphic correspondence of* $\mathbf{G}$ *onto* $\mathbf{H}$. *Then the kernels ker*$\mathbf{G}$ *and ker*$\mathbf{H}$ *are isomorphic.*

Proof. Let ϱ be a strongly homomorphic correspondence of $\mathbf{G}$ onto $\mathbf{H}$, $\mathbf{I}$ its algebraization. According to 3.5 there exist strong homomorphisms f of $\mathbf{I}$ onto $\mathbf{G}$ and g of $\mathbf{I}$ onto $\mathbf{H}$. According to 3 *ker*$\mathbf{I}$ is isomorphic with both *ker*$\mathbf{G}$ and *ker*$\mathbf{H}$. This implies the assertion. □

6. Applications to formal languages

Let V be a set and let $L \subseteq V^*$. Then the ordered pair (V, L) is called a (formal) *language*. The elements of V are usually interpreted as word forms and the elements of L as correct sentences of the language.

For every integer $n \geqq 0$ the symbol L_n denotes the set of all words of the length n from the set L. Then $(V, (L_n)_{n \in N})$ is the *relational system assigned to the language* (V, L).

Conversely, if a relational system $(V, (\sigma_n)_{n \in N})$ is given, where σ_n is a relation on V of the arity n, then $(V, \bigcup_{n \in N} \sigma_n)$ is a language and $(V, (\sigma_n)_{n \in N})$ is its assigned relational system.

All concepts of the theory of relational systems can now be transferred to languages. If $\mathbf{L} = (V, L)$ is a language and $\Pi \in E(V)$, then Π is called a *congruence on the language* $\mathbf{L}$, if and only if it is a congruence on $(V, (L_n)_{n \in N})$. According to the definition Π is a congruence on $\mathbf{L}$, if and only if it has the following property: If $n \geqq 0$, $x_1, \ldots, x_n, x'_1, \ldots, x'_n \in V$ and $x_1 \ldots x_n \in L$, $x_1 x'_1, \ldots, x_n x'_n \in \Pi$, then $x_1 \ldots x_n \in L$. The greatest congruence on $(V, (L_n)_{n \in N})$ will be denoted by $\equiv_{\mathbf{L}}$.

Let $\mathbf{L} = (V, L)$ be a language, let $\Pi \in E(V)$. Then to $\mathbf{L}$ the relational system $(V, (L_n)_{n \in N})$ is assigned, which defines the quotient by Π, i.e. $(V/\Pi, (L_n/\Pi)_{n \in N})$, to this quotient again the language $(V/\Pi, \bigcup_{n \in N} L_n/\Pi)$ is assigned; we call it the quotient of the language $\mathbf{L}$ by the equivalence Π and denote it by $\mathbf{L}/\Pi$. According to the definition, for each $m \geqq 0$ and $A_1, \ldots, A_m \in V/\Pi$ we have $A_1 \ldots A_m \in \bigcup_{n \in N} L_n/\Pi$ if and only if there exist $a_1 \in A_1, \ldots, a_m \in A_m$ such that $a_1 \ldots a_m \in L$. The quotient $\mathbf{L}/\equiv_{\mathbf{L}}$ is called *the kernel of the language* $\mathbf{L}$ and denoted by *ker*$\mathbf{L}$.

Finally, let $\mathbf{L} = (V, L)$ and $\mathbf{M} = (U, M)$ be languages, let ϱ be a correspondence of V onto U. We shall call it a *homomorphic correspondence* (*or a strongly homomorphic correspondence*) of the language $\mathbf{L}$ onto $\mathbf{M}$, if and only if it is a homomorphic (or strongly homomorphic respectively) correspondence of the relational system $(V, (L_n)_{n \in N})$ onto $(U, (M_n)_{n \in N})$. In the first case this means that $n \geqq 0$, $x_1, \ldots, x_n \in V$, $x_1 \ldots x_n \in L$ and $x_1 y_1 \in \varrho, \ldots, x_n y_n \in \varrho$ imply $y_1 \ldots y_n \in M$ and $y_1, \ldots, y_n \in U$, $y_1 \ldots y_n \in M$ imply the existence of $x_1, \ldots, x_n \in V$ such that $x_1 y_1 \in \varrho, \ldots, x_n y_n \in \varrho$ and $x_1 \ldots x_n \in L$. In the other case this means that for $n \geqq 0$, $x_1, \ldots, x_n \in V$, $y_1, \ldots, y_m \in U$, $x_1 y_1 \in \varrho, \ldots, x_n y_n \in \varrho$ the conditions $x_1 \ldots \ldots x_n \in L$, $y_1 \ldots y_n \in M$ are equivalent.

In the introduction we have promised symmetrical analoga to the theorems on homomorphisms of languages. Let us present some of them.

1. Theorem. *Let* $\mathbf{L}$, $\mathbf{M}$, $\mathbf{P}$ *be languages, let* α *be a* (*strongly*) *homomorphic correspondence of* $\mathbf{L}$ *onto* $\mathbf{M}$, *let* β *be a* (*strongly*) *homomorphic correspondence of* $\mathbf{M}$ *onto* $\mathbf{P}$. *Then* $\alpha \circ \beta$ *is a* (*strongly*) *homomorphic correspondence of* $\mathbf{L}$ *onto* $\mathbf{P}$.

This is a particular case of 3.3 and 3.4.

2. Theorem. *Let* $\mathbf{L}$, $\mathbf{M}$ *be languages. Let there exist a strongly homomorphic correspondence of* $\mathbf{L}$ *onto* $\mathbf{M}$. *Then the kernels ker* $\mathbf{L}$ *and ker* $\mathbf{M}$ *are isomorphic.*

This is a particular case of theorem 5.4. □

ГОМОМОРФНЫЕ ЧАСТИЧНЫЕ МУЛЬТИОТОБРАЖЕНИЯ РЕЛЯЦИОННЫХ СИСТЕМ

Резюме

В работе введено понятие гомоморфизма частичного мультиотображения реляционных систем как обобщение понятия гомоморфизма этих систем.

Результаты применены на языки, которые мы понимаем как специальные реляционные системи. Этим способом сделано обобщение понятия гомоморфизма языка и одним из основных результатов является етот теорем — два языка между которыми существует сильный гомоморфизм имеют изоморфные ядра. Притом в работе приведено доказательство, что всякое сильно гомоморфное мультиобpажение есть произведением сильного гомоморфизма и мультиотображения, обратного к сильному гомоморфизму.

HOMOMORFNÍ KORESPONDENCE RELAČNÍCH SYSTÉMŮ

Souhrn

V práci se zavádí pojem homomorfní korespondence mezi relačními systémy jako zobecnění pojmu homomorfismu těchto systémů.

Výsledky jsou aplikovány na jazyky, které se uvažují jako speciální relační systémy. Tím je zobecněn pojem homomorfismu jazyka a jedním z hlavních výsledků práce je pak věta, že dva jazyky, mezi nimiž existuje silně homomorfní korespondence, mají isomorfní jádra. Přitom je v práci dokázáno, že každá silně homomorfní korespondence je superposicí silného homomorfismu a korespondence inverzní k silnému homomorfismu.

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