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A CHARACTERIZATION OF ORDERED GROUPS BY MEANS OF SEGMENTS

JIŘÍ RACHŮNEK

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Dedicated to Prof. Miroslav Laitoch on his 60th birthday

If $G \neq \emptyset$ is a set, then any $I : G \times G \to \exp G$ is called a *binary multioperation on G*. For $a \in G$ we shall write $I(a)$ instead of $I(a, a)$. Let $G = (G, +)$ be a group, I a binary multioperation on G. Let us denote the following conditions:

(I1) $\forall\, a, b \in G\; \exists\, c \in G;\; I(a, c) \cap I(b, c) \neq \emptyset$

(I2) $\forall\, a, b, c, x, y \in G;\; c \in I(a, b) \Rightarrow x + c + y \in I(x + a + y, x + b + y)$

(I3) $\forall\, a, b \in G;\; a \neq b \Rightarrow (I(a, b) \neq \emptyset \Rightarrow I(b, a) = \emptyset)$

(I4) $\forall\, a \in G;\; a \in I(a)$

(I5) $\forall\, a, b, c \in G;\; c \in I(a, b) \Rightarrow (I(a, c) \neq \emptyset \;\&\; I(c, b) \neq \emptyset)$

Theorem 1. *Conditions I1 – I5 are mutually independent.*

Proof. a) The independence of I1: Let $(G, +)$ be an arbitrary group such that card $G \geqq 2$. Let us define I on G as follows: $x \in G \Rightarrow I(x) = \{x\}$, $x, y \in G$, $x \neq y \Rightarrow I(x, y) = \emptyset$. Then I2 – I5 are satisfied, but I has not property I1.

b) The independence of I2: Let $\mathbf{Z}_2$ be the group of numbers 0, 1 with the addition mod 2. We put $G = \mathbf{Z}_2$, $I(0) = \{0\}$, $I(1) = \{1\}$, $I(0, 1) = \{0, 1\}$, $I(1, 0) = \emptyset$. Then $1 \in I(0, 1)$, but $0 = 1 + 1 \notin I(1 + 0, 1 + 1)$. Obviously I1 and I3 – I5 are satisfied here.

c) The independence of I3 is clear if $G = \mathbf{Z}_2$ and $I(x, y) = \{0, 1\}$ for each $x, y \in G$.

d) The independence of I4: Let $\mathbf{Q}$ be the additive group of the rationals, „$<$" the relation „to be strictly less than" in the natural ordering of $\mathbf{Q}$ and let $(x, y) = \{z \in Q;\; x < z < y\}$ for $x, y \in \mathbf{Q}$, $x < y$ and $I(x, y) = \emptyset$ for $x \geqq y$. Then I1 – I3 and I5 hold, but I4 is not satisfied.

e) The independence of I5: Let $G = \mathbf{Q}$ and let $\langle x, y) = \{z \in \mathbf{Q};\; x \leqq z < y\}$ for $x, y \in \mathbf{Q}$, $x < y$. Let us define $I(x, y) = \langle x, y)$ for $x < y$, $I(x) = \{x\}$ for each x

and $I(x, y) = \emptyset$ for $y < x$. Then for $x < y$ it holds $I(x, y) \neq \emptyset$, $I(y, y) \neq \emptyset$ and $y \notin I(x, y)$. Evidently, I1 – I4 are satisfied.

For a binary multioperation I on a group $(G, +)$ let us consider also the condition

(I6) $\forall\, a, b \in G\, \exists\, c \in G\, \forall\, x \in G;\ I(c, x) = I(a, x) \cap I(b, x)$.

If $(G, +, \leqq)$ is an ordered group, then we put $\langle x, y\rangle = \{z \in G;\ x \leqq z \leqq y\}$ for $x \leqq y$ and $\langle x, y\rangle = \emptyset$ for $y \ngeqq x$.

Theorem 2. a) *Let $G = (G, +)$ be a group, I a binary multioperation on G satisfying I2 – I5. Let $\forall\, a, b \in G;\ a \leqq b \Leftrightarrow_{df} I(a, b) \neq \emptyset$. Then $(G, +, \leqq)$ is an ordered group.*

b) *A group $G = (G, +)$ admits a directed (lattice) order if and only if there exists a binary multioperation I on G satisfying I1 – I5 (I2 – I6).*

c) *Moreover, if I is a binary multioperation on a group $G=(G, +)$ satisfying I2 – I5 (I1 – I5, I2 – I6), then there exists an order (a directed order, a lattice order) $\leqq$ on $(G, +)$ such that in $(G, +, \leqq)$ it holds $\langle x, y\rangle = I(x, y)$ for each $x, y \in G$.*

Proof. a) If $a \in G$, then by I4 $a \in I(a)$, hence $a \leqq a$.

Let $a, b \in G$, $a \leqq b$, $b \leqq a$. Then $I(a, b) \neq \emptyset$, $I(b, a) \neq \emptyset$, thus by I3 $a = b$.

Let $a, b, c \in G$, $a \leqq b$, $b \leqq c$. Then $I(a, b) \neq \emptyset$, $I(b, c) \neq \emptyset$, hence by I5 $b \in I(a, c)$, therefore $a \leqq c$.

Let $a, b, x, y \in G$, $a \leqq b$ and let $c \in I(a, b)$. Then by I2 $x + c + y \in I(x + a + y, x + b + y)$, and so $x + a + y \leqq x + b + y$.

b) Let I satisfy I2 – I5 and let $\leqq$ be the order on G defined in a). If $a, b, c \in G$, $a \leqq b$, then $c \in \langle a, b\rangle$ if and only if $I(a, c) \neq \emptyset$ and $I(c, b) \neq \emptyset$. By I5, this is equivalent with $c \in I(a, b)$. Evidently for $a \nleqq b$ we have $\langle a, b\rangle = I(a, b) = \emptyset$.

If I1 – I5 are satisfied, then the ordered group $(G, +, \leqq)$ is directed.

Let I2 – I6 be satisfied and let $a, b \in G$. Let us suppose that $c \in G$ is such that $I(c, x) = I(a, x) \cap I(b, x)$ for each $x \in G$. Then $I(c, c) = I(a, c) \cap I(b, c)$, hence by I4 $I(a, c) \neq \emptyset$ and $I(b, c) \neq \emptyset$, therefore in the order $\leqq$ defined in a) it is $a \leqq c$ and $b \leqq c$. Let $y \in G$, $a \leqq y$, $b \leqq y$. Then $I(a, y) = \langle a, y\rangle$ and $I(b, y) = \langle b, y\rangle$ imply $I(a, y) \cap I(b, y) \neq \emptyset$, and so $I(c, y) \neq \emptyset$. Therefore $c \leqq y$, i.e. $c = a \vee b$. This means that the order $\leqq$ is a lattice one.

Conversely, if $(G, +, \leqq)$ is an ordered group and if we put $I(a, b) = \langle a, b\rangle$ for each $a, b \in G$, then evidently I2 – I4 are satisfied. Moreover, if $(G, +, \leqq)$ is directed, then also I1 is satisfied. Let $(G, +, \leqq)$ be lattice ordered, $a, b, x, y \in G$. Then $y \in \langle a \vee b, x\rangle \Leftrightarrow a \vee b \leqq y \leqq x \Leftrightarrow (a \leqq y \leqq x \;\&\; b \leqq y \leqq x) \Leftrightarrow y \in \langle a, x\rangle \cap \langle b, x\rangle$. Therefore in this case I2 – I6 are satisfied.

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Souhrn

CHARAKTERIZACE USPOŘÁDANÝCH GRUP POMOCÍ INTERVALŮ

JIŘÍ RACHŮNEK

V článku jsou definována uspořádání grup pomocí výsledků binárních multioperací. Speciálně jsou charakterizované grupy, které připouštějí usměrněná, resp. svazová, uspořádání. Přitom je dokázáno, že výsledky multioperací splývají s intervaly v uspořádaných grupách.

Резюме

ХАРАКТЕРИЗАЦИЯ УПОРЯДОЧЕННЫХ ГРУПП ПРИ ПОМОЩИ ИНТЕРВАЛОВ

ЙИРЖИ РАХУНЕК

В статье определяются порядки групп при помощи бинарных мультиопераций. В частности характеризованы группы, которые допускают направлённые или решёточные порядки. Притом показано, что результатами мультиопераций являются интервалы упорядоченных групп.