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NATURAL PLANAR TERNARY RINGS

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The systematical study of non—embedable projective planes formed the theory of projective planes in a special branch of mathematics. This theory has its algebraization namely the theory of planar ternary rings. Transformation one of the theories into to other one is realized by so called coordinatization (i.e. by introduction of a coordinate system and of the structure of planar ternary ring onto the coordinate domain). In this way we can assign to every projective plane **P** with a distinguished flag (**V**, **n**) a certain planar ternary ring (**S**, **t**). We may rightly expect, that the properties of (**S**, **t**) will be depended onto the plane **P** as well as onto the coordinatization. By a fixed coordinatization the corespondence between properties of **P** and of (**S**, **t**) will be one to one.

In the article, we are dealing with a coordinatization of a given projective plane **P** by a natural planar ternary ring (**S**, **t**). Such coordinatization is the most general with the following property: For any $a, b \in \mathbf{S}$ the equation $x = a$ as well as the equation $y = b$ expresses (a) a line, if x, y are considered as point—coordinates, (b) a pencil of lines, if x, y are considered as line-coordinates. Our purpose is to deduce the necessary and sufficient conditions for (**S**, **t**) so, that **P** may be (a) a vertical transitive plane (i.e. a (**V**, **n**)—transitive plane), (b) a translation plane (i.e. a **n**—transitive plane), (c) a desarguesian plane, (d) a pappian plane.

The results found will be a generalization of results of [1] for Hall planar ternary rings and of [2], [3], [4] for planar ternary rings with zero. On the other hand, there exist the generalizations of some results of ours, especially by O. Sorrace in [5] and by J. Klouda in [6].

The mentioned necessary and sufficient conditions may be obtained in two different ways. The first of them employs inner properties of the plane **P** and by using of coordinatization transforms them into properties of (**S**, **t**). The known results for Hall planar ternary rings follow from the obtained ones as special cases. The second way uses the known results for Hall planar ternary rings and by using of isotopy transforms them again into properties of (**S**, **t**). We will apply the first way.

1. AXIOMS OF NATURAL PLANAR TERNARY RINGS AND THEIR IMMEDIATE CONSEQUENCES

We define a *planar ternary ring* (abbreviation: PTR) as an ordered pair $(\mathbf{S}, \mathbf{t})$, where $\mathbf{S}$ is a set with card $\mathbf{S} \geqq 2$ and $\mathbf{t} : \mathbf{S}^3 \to \mathbf{S}$ is a ternary operation satisfying following axioms:

A 1. $\forall a, b, c \in \mathbf{S}\ \exists\,!\ x \in \mathbf{S} : \mathbf{t}(a, b, x) = c$.

A 2. $\forall a, b, c, d \in \mathbf{S},\ a \neq c\ \exists\,!\ x \in \mathbf{S} : \mathbf{t}(x, a, b) = \mathbf{t}(x, c, d)$.

A 3. $\forall a, b, c, d \in \mathbf{S},\ a \neq c\ \exists\,!\ (x, y) \in \mathbf{S}^2 : \mathbf{t}(a, x, y) = c \wedge \mathbf{t}(c, x, y) = d$.

If, in addition, it holds:

A 4. *There exist elements* $o_L, o_R \in \mathbf{S}$ (*so called left – quasizero and right – quasizero*) *such that*

$$\mathbf{t}(o_L, y_0, z^*) = z \Rightarrow \forall y \in \mathbf{S} : \mathbf{t}(o_L, y, z^*) = z,$$
$$\mathbf{t}(x_0, o_R, z^*) = z \Rightarrow \forall x \in \mathbf{S} : \mathbf{t}(x, o_R, z^*) = z,$$

then $(\mathbf{S}, \mathbf{t})$ *will be called natural PTR (abb. NPTR).*

If we replace A 4 by the axiom:

A*4: *There exists an element* $0 \in \mathbf{S}$ (so called *zero*) *such that*

$$\forall x, y, z \in \mathbf{S} : \mathbf{t}(0, y, z) = \mathbf{t}(x, 0, z) = z,$$

we get the definition of PTR *with zero* (abb.: ZPTR).

A ZPTR $(\mathbf{S}, \mathbf{t})$ will be called *Hall* PTR (abb.: HPTR), if it is fulfilled:

A*5: *There exists an element* $1 \in \mathbf{S}$ (so called *unity*) *such that*

$$\forall x, y \in \mathbf{S} : \mathbf{t}(1, y, 0) = y, \qquad \mathbf{t}(x, 1, 0) = x.$$

Let $(\mathbf{S}, \mathbf{t})$ be a NPTR. For any $z \in \mathbf{S}$, let us denote by z^* the element satisfying

$$\mathbf{t}(o_L, o_R, z^*) = z.\ ^{1})$$

We introduce in $\mathbf{S}$ the binary operation *multiplication* . by virtue of

$$a \,.\, b = \mathbf{t}(a, b, o_L^*), \qquad a, b \in \mathbf{S}.$$

It is easy to prove:

(a) $\forall a \in \mathbf{S} : a \,.\, o_R = o_L \,.\, a = o_L$,

(b) $\forall b \in \mathbf{S},\ \forall a \in \mathbf{S} \setminus \{o_L\}\ \exists\,!\ x \in \mathbf{S} : a \,.\, x = b$. *Such element* x *will be denoted by* $x = a \setminus b$.

(c) $\forall b \in \mathbf{S},\ \forall a \in \mathbf{S} \setminus \{o_R\}\ \exists\,!\ x \in \mathbf{S} : y \,.\, a = b$. *Such element* y *will be denoted by* $y = b/a$.

For every $a \neq o_L$ put $e_a = a \setminus a$ [2]), furthermore put $e_{0_L} = o_R$. Now, we can introduce further binary operation *addition* + defining

$$a + b = \mathbf{t}(a, e_a, b) \qquad a, b \in \mathbf{S}.$$

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[1]) If $(\mathbf{S}, \mathbf{t})$ is a ZPTR (especially HPTR), then $z^* = z\ \forall z \in \mathbf{S}$.

[2]) If $(\mathbf{S}, \mathbf{t})$ is a HPTR, then $\forall a \in \mathbf{S} \setminus \{0\}$: $e_a = 1$.

It is easily seen, that

(d) $\forall a \in \mathbf{S}: a + o_L = o_L + a = a$.

(e) $\forall a, b \in \mathbf{S}, \exists! x \in \mathbf{S}: a + x = b$.

We say, that $(\mathbf{S}, \mathbf{t})$ is a *linear* **N P T R**, if $\forall a, b, c \in \mathbf{S}: \mathbf{t}(a, b, c^*) = a \cdot b + c$.

2. COORDINATIZATION OF A PROJECTIVE PLANE BY A NATURAL PLANAR TERNARY RING

Let $\mathbf{P} = (\mathscr{P}, \mathscr{L})$ be a projective plane, where $\mathscr{P}$ is the set of points and $\mathscr{L}$ is the set of lines considered as subsets of $\mathscr{P}$. An ordered tripple $(\mathbf{P}, \mathbf{V}, \mathbf{n})$, where $(\mathbf{V}, \mathbf{n})$ is a flag of $\mathbf{P}$ will be called the projective plane with flag and it will be denoted $\mathbf{P}(\mathbf{V}, \mathbf{n})$. We will shortly speak about the plane $\mathbf{P}(\mathbf{V}, \mathbf{n})$.

Now, put

$$\mathscr{A} = \mathscr{P} \setminus \mathbf{n}, \qquad \mathscr{B} = \mathscr{L} \setminus \tilde{\mathbf{V}},$$

where $\tilde{\mathbf{V}}$ denotes the set of all lines o $\mathbf{P}$ containing $\mathbf{V}$ (a pencil of lines). We will use the "affine" terminology in the following sense: The points of $\mathscr{A}$ will be called *proper points*, the points of $\mathbf{n}$ *improper*. The lines different from $\mathbf{n}$ will be called *proper lines*, $\mathbf{n}$ is the *improper line*. For every proper line p put $p_a = p \cap \mathscr{A}$ and for every improper point N put $\tilde{N}_a = \tilde{N} \setminus \{\mathbf{n}\}$. The set p_a will be called the *affine line* or exactly the *affine restriction of* p; the set $\tilde{N}_a$ will be called the *direction* or the *affine restriction* of the pencil $\tilde{N}$.

The lines of $\tilde{\mathbf{V}}_a$ (and also their affine restrictions) will be called vertical lines, the lines of $\mathscr{B}$ (as well as their affine restrictions) will be called cross lines. The directions different from $\tilde{\mathbf{V}}_a$ are cross directions, $\tilde{\mathbf{V}}_a$ is the vertical direction.

As it is known

$$\operatorname{card} \mathscr{A} = \operatorname{card} \mathscr{B} = m^2, \qquad \text{where } m = \operatorname{ord} \mathbf{P}.$$

Let $\mathbf{S}$ be a set with card $\mathbf{S} = m$. An ordered double (π, λ) of bijections

$$\pi : \mathbf{S}^2 \to \mathscr{A}, \qquad \lambda : \mathbf{S}^2 \to \mathscr{B}$$

will be called *coordinate system* for $\mathbf{P}(\mathbf{V}, \mathbf{n})$ (cf. [7]), $\mathbf{S}$ is the *coordinate domain* of (π, λ).

We introduce four mappings

$$\mathbf{p}_1 : \mathscr{A} \to \mathbf{S}, \qquad \mathbf{p}_2 : \mathscr{A} \to \mathbf{S}, \qquad \mathbf{p}^1 : \mathscr{B} \to \mathbf{S}, \qquad \mathbf{p}^2 : \mathscr{B} \to \mathbf{S}$$

by virtue of

$$\mathbf{p}_1((x, y)^\pi) = x, \qquad \mathbf{p}_2((x, y)^\pi) = y, \qquad \mathbf{p}^1((u, v)^\lambda) = u, \qquad \mathbf{p}^2((u, v)^\lambda) = v.$$

The coordinate system (π, λ) will be called *halfcartesian* (cf. [7]), if following two conditions are fulfilled:

(i) Let X, Y be arbitrary two points of $\mathscr{A}$. Then X, Y lie on the same vertical line if and only if $\mathbf{p}_1(X) = \mathbf{p}_1(Y)$.

(ii) Let a, b be arbitrary two lines of $\mathscr{B}$. Then, a, b have a common improper point (we say that a, b are parallel lines) if and only if $\mathbf{p}^1(a) = \mathbf{p}^1(b)$.

Suppose, that the coordinate system (π, λ) is halfcartesian. Then, for every tripple $(x, u, w) \in \mathbf{S}^3$ there exists the unique element y such, that $(x, y)^\pi \in (u, w)^\lambda$. Therefore, we may introduce a ternary operation $\mathbf{t} : \mathbf{S}^3 \to \mathbf{S}$ by virtue of

$$y = \mathbf{t}(x, u, w) \Leftrightarrow (x, y)^\pi \in (u, w)^\lambda.$$

In this case, we can prove without difficulties, that $(\mathbf{S}, \mathbf{t})$ is a PTR. We say, that $(\mathbf{S}, \mathbf{t})$ is a PTR associated to coordinate system (π, λ) for $\mathbf{P}(\mathbf{V}, \mathbf{n})$ or that $(\mathbf{S}, \mathbf{t})$ is a PTR associated to plane $\mathbf{P}(\mathbf{V}, \mathbf{n})$.

A halfcartesian coordinate system (π, λ) will be called the *natural coordinate system,* if it holds:

(iii) There exists a direction $\mathbf{H}_a$ (so-called *horizontal direction*) different from $\mathbf{V}_a$ and having following property: Let X, Y be arbitrary two points of $\mathscr{A}$. Then X, Y lie on the same line of $\mathbf{H}_a$ if and only if $\mathbf{p}_2(X) = \mathbf{p}_2(Y)$.

(iv) There exists a vertical affine line v_a (so called vertical axis) with following property: Let a, b be arbitrary two lines of $\mathscr{B}$. Then a, b have a common point on v_a if and only if $\mathbf{p}^2(a) = \mathbf{p}^2(b)$.

We can construct a natural coordinate system for $\mathbf{P}(\mathbf{V}, \mathbf{n})$ in following way (cf. [8]): Let us choose an improper point $\mathbf{H} \neq \mathbf{V}$ and a vertical line $\mathbf{v}$; let $\mathbf{S}$ be an arbitrary set with card $\mathbf{S}$ = ord $\mathbf{P}$. Finally, let us choose four bijections:

$$\bar{\pi} : \mathbf{S} \to \mathbf{n} \setminus \{\mathbf{V}\}, \qquad \bar{\lambda} : \mathbf{S} \to \mathbf{V}_a, \qquad \pi' : \mathbf{S} \to \mathbf{v}_a, \qquad \lambda' : \mathbf{S} \to \mathbf{H}_a.$$

Now, we are able to define bijections $\pi : \mathbf{S}^2 \to \mathscr{A}$, $\lambda : \mathbf{S}^2 \to \mathscr{B}$ by

$$(x, y)^\pi = x^{\bar{\lambda}} \cap y^{\lambda'}; \qquad (u, w)^\lambda = (\widetilde{u^{\bar{\pi}}})_a \cap (\widetilde{w^{\pi'}})_a$$

It is easy seen, that (π, λ) is a natural coordinate system for $\mathbf{P}(\mathbf{V}, \mathbf{n})$. We remain to the reader the proof of the following assertion:

Let (π, λ) be a halfcartesian coordinate system for $\mathbf{P}(\mathbf{V}, \mathbf{n})$, $(\mathbf{S}, \mathbf{t})$ the PTR associated to (π, λ). Then (π, λ) is natural if and only if $(\mathbf{S}, \mathbf{t})$ is natural. In this case $v_a = \{(o_L, y)^\pi \mid y \in \mathbf{S}\}$ and $\mathbf{H}_a = \{(o_R, w)^\lambda \mid w \in \mathbf{S}\}$.

For next, let us suppose, that in given projective plane $\mathbf{P}(\mathbf{V}, \mathbf{n})$ a natural coordinate system (π, λ) is introduced and that $(\mathbf{S}, \mathbf{t})$ is the NPTR associated to (π, λ). Further let us assume, that $\mathbf{H}_a$ denotes the horizontal direction, $\mathbf{v}_a$ the vertical axis and $\mathbf{0}$ the point $(o_L, o_L)^\pi$ (so-called origin of coordinate system (π, λ)).

Finally, let us denote for every point C and every line c by $U(C, c)$ the set as well as the group of central collineations with the centre C and the axis c. We will say, that $\mathbf{P}(\mathbf{V}, \mathbf{n})$ is (C, c)-transitive if for any line $l \in \bar{C}$ the group $U(C, c)$ operates transitively on the set l except the point C and the common point of c and l. $\mathbf{P}(\mathbf{V}, \mathbf{n})$ is said to be c-transitive, if it is (C, c)-transitive for any $C \in c$. The central collineation with improper axis $\mathbf{n}$ is called homothety. If the centre C of a homothety $\boldsymbol{h}$ is a proper point, then $\boldsymbol{h}$ is said to be a dilatation, if C is an improper point, then $\boldsymbol{h}$ is said to be a translation. In the last case the direction $\bar{C}_a$ is called the direction of translation $\boldsymbol{h}$.

It is well-known, that $\mathbf{P}(\mathbf{V}, \mathbf{n})$ is (C, c)-transitive if and only if there exists any line $l \in \bar{C}$ such, that $U(C, c)$ operates transitively on l except the point C and the common point of c and l. Furthermore, $\mathbf{P}(\mathbf{V}, \mathbf{n})$ is c-transitive if and only if there exist two different points C_1, C_2 on c such, that $\mathbf{P}(\mathbf{V}, \mathbf{n})$ is (C_1, c)-transitive and (C_2, c)-transitive.

3. VERTICALLY TRANSITIVE PLANES

The plane $\mathbf{P}(\mathbf{V}, \mathbf{n})$ is said to be *vertically transitive*, if it is $(\mathbf{V}, \mathbf{n})$-transitive. It follows from the end of previous part, that $\mathbf{P}(\mathbf{V}, \mathbf{n})$ is vertically transitive if and only if for any point $A \in \mathbf{v}_a$ there exists a translation $\boldsymbol{F} : 0 \mapsto A$.

Proposition 1.

The following conditions are equivalent:

(a) $\mathbf{P}(\mathbf{V}, \mathbf{n})$ *is vertically transitive*

(b) $\forall\, a \in \mathbf{S}$ *the mapping* $\boldsymbol{f}_a : \mathscr{A} \to \mathscr{A}$ *given by*

$$(x, y)^\pi \to (x, y + a)^\pi \tag{1}$$

is the restriction of a translation $\boldsymbol{F}_a \in U(\mathbf{V}, \mathbf{n})$ *to* $\mathscr{A}$. *In the case, the grupoid* $(\mathbf{S}, +)$ *is antiisomorphic to* $U(\mathbf{V}, \mathbf{n})$ *and consequently* $(\mathbf{S}, +)$ *is a group.*

Proof: I. $(a) \Rightarrow (b)$. Let $\boldsymbol{F}_a \in U(\mathbf{V}, \mathbf{n})$ be a translation with $0 \to (o_L, a)^\pi$. Consider a point $X = (x, y)^\pi$. If $y = o_L$, then obviously $\boldsymbol{F}_a(X) = (x, a)^\pi = \boldsymbol{f}_a(X)$. Suppose that $y \neq o_L$. Put $Y = (y, y)^\pi$. From $\mathfrak{p}_2(X) = \mathfrak{p}_2(Y)$ it follows, that $\mathfrak{p}_2(\boldsymbol{F}_a(X)) = \mathfrak{p}_2(\boldsymbol{F}_a(Y))$. It satisfies to prove, that $\mathfrak{p}_2(\boldsymbol{F}_a(Y)) = y + a$. Put $q = (e_y, o_L^*)^\lambda$, then $Y \in \boldsymbol{q}$, $\boldsymbol{F}_a(Y) \in \boldsymbol{F}_a(q)$ and $\boldsymbol{F}_a(q) = (e_y, a^*)^\lambda$. Therefore $\mathfrak{p}_2(\boldsymbol{F}_a(Y)) = \mathbf{t}(y, e_y, a^*) = y + a$.

II. $(b) \Rightarrow (a)$ is obvious.

III. Let one of the equivalent conditions (a), (b) is fulfilled. Define a bijection $\varphi : \mathbf{S} \to U(\mathbf{V}, \mathbf{n})$ by $\varphi(a) = \boldsymbol{F}_a$. The restriction of $\boldsymbol{F}_a$ onto $\mathscr{A}$ may be expressed by (1). Let $a, b \in \mathbf{S}$. Then $\boldsymbol{F}_{a+b}(0) = (o_L, a + b)^\pi = \boldsymbol{F}_b((o_L, a)^\pi) = \boldsymbol{F}_b(\boldsymbol{F}_a(0)) = (\boldsymbol{F}_b \,.\, \boldsymbol{F}_a)(0)$. Hence $\boldsymbol{F}_{a+b} = \boldsymbol{F}_b \,.\, \boldsymbol{F}_a$ and φ is an antiisomorphism.

Theorem 1.

The plane $\mathbf{P}(\mathbf{V}, \mathbf{n})$ *is vertically transitive if and only if the following two conditions are fulfilled:*

(A) $(\mathbf{S}, +)$ *is a group*

(B) $(\mathbf{S}, \mathbf{t})$ *is a linear NPTR.*

Proof: I. Suppose that $\mathbf{P}(\mathbf{V}, \mathbf{n})$ is vertically transitive.

The (A) follows from proposition 1. Let a, b, c be elements of $\mathbf{S}$ and let $\boldsymbol{F}_c$ be the translation with $\boldsymbol{F}_c(0) = (o_L, c)^\pi$. We have $(a, a \,.\, b)^\pi \in (b, o_L^*)^\lambda \Rightarrow \boldsymbol{F}_c((a, a \,.\, b)^\pi) \in$

$\in (b, c^*)^\lambda$. By the proposition 1 F_c $((a, a \,.\, b)^\pi) = (a, a \,.\, b + c)^\pi$, therefore $a \,.\, b + c = \mathbf{t}(a, b, c^*)$.

II. Let (A) and (B) are valid. Denote by f_a the mapping (1). It satisfies to prove, that the map of arbitrary cross affine line under f_a is a line parallel with its original. Consider a cross line $(b, c^*)^\lambda$. Then: $(x, y)^\pi \in (b, c^*)^\lambda \Leftrightarrow y = t(x, b, c^*) \Leftrightarrow y = x \,.\, b + c \Leftrightarrow y + a = x \,.\, b + (c + a) \Leftrightarrow y + a = t(x, b, (c + a)^*) \Leftrightarrow (x, y + a)^\pi \in (b, (c + a)^*)^\lambda$. This means, that the map of $(b, c^*)^\lambda$ is the line $(b, (c + a)^*)^\lambda$.

Remark: If $(\mathbf{S}, \mathbf{t})$ is a HPTR, then we may rewritten the condition (B) in the form:

$$\forall a, b, c \in \mathbf{S}: \mathbf{t}(a, b, c) = a \,.\, b + c.$$

4. TRANSLATION PLANES

The plane $\mathbf{P}(\mathbf{V}, \mathbf{n})$ is called a *translation plane,* if it is $\mathbf{n}$-transitive. It follows from considerations at the end of the part 2., that $\mathbf{P}(\mathbf{V}, \mathbf{n})$ is a translation plane if and only if, it is $(\mathbf{V}, \mathbf{n})$-and $(\mathbf{H}, \mathbf{n})$-transitive. Hence, a vertically transitive plane $\mathbf{P}(\mathbf{V}, \mathbf{n})$ is a translation plane if and only if it is $(\mathbf{H}, \mathbf{n})$-transitive.

Proposition 2.

Let $\mathbf{P}(\mathbf{V}, \mathbf{n})$ *be a* $(\mathbf{V}, \mathbf{n})$*-transitive plane. Let there exist a translation* $F_b \in U(\mathbf{H}, \mathbf{n})$ *such, that*

$$F_b(0) = (b, o_L)^\pi. \qquad b \in \mathbf{S} \tag{2}$$

If $F_b : (a, o_L)^\pi \mapsto (c, o_L)^\pi$, $(a, c \in \mathbf{S})$, then

$$\forall m \in \mathbf{S}: a \,.\, m + b \,.\, m = c \,.\, m. \tag{3}$$

Proof: (3) is fulfilled for $m = o_R$. Suppose $m \neq o_R$ and put $X = (a, o_L)^\pi$, $Y = (c, o_L)^\pi$, $Q = = (b, o_L)^\pi$, $X' = (a, a \,.\, m)^\pi$. By assumption $Q = F_b(0)$, $Y = F_b(X)$. If we construct the point $Y' := F_b(Y)$, we obtain $\mathfrak{p}_2(Y') = a \,.\, m$ and $\mathfrak{p}_1(Y') = \mathfrak{p}_1(Y) = c$. Put l = OX', then $\mathfrak{p}^1(l)$ = m and $l' := F_b(l)$ = QY' is parallel to $l \Rightarrow \mathfrak{p}^1(l') = m$. Let $l' = (m, q^*)^\lambda$, then $o_L = b \,.\, m + q$ and $a \,.\, m = c \,.\, m + q$, hence (3).

Proposition 3.

Let $\mathbf{P}(\mathbf{V}, \mathbf{n})$ *be a vertically transitive plane and let* $(\mathbf{S}, \mathbf{t})$ *have the following property:*
(C) *Let* a, b, c *be arbitrary elements of* $\mathbf{S}$

$$Q(a, b, c) := \{m \in \mathbf{S} \mid a \,.\, m + b \,.\, m = c \,.\, m\} \Rightarrow$$
$$\Rightarrow Q(a, b, c) = \{o_R\} \vee Q(a, b, c) = \mathbf{S}.$$

Then for arbitrary $m \in \mathbf{S} \setminus \{o_R\}$ *the mapping*

$$f_b : (x, y)^\pi \mapsto (x', y)^\pi$$

such that

$$x \,.\, m + b \,.\, m = x' \,.\, m \qquad b \in \mathbf{S} \tag{5}$$

is the restriction of a translation F_b onto $\mathscr{A}$, with $F_b : 0 \mapsto (0, b)^\pi$.

Proof: Evidently, it is satisfy to prove, that for any cross affine line $p_a = (u, w^*)^\lambda$ its map under f_b is a line p'_a parallel to p_a. Let $(x', y)^\pi = f_b((x, y)^\pi)$. Then $x \,.\, m + b \,.\, m = x' \,.\, m$ and with respect to the condition (C) also

$$x \,.\, u + b \,.\, u = x' \,.\, u.$$

Now, $(x, y)^\pi \in (u, w^*)^\lambda \Leftrightarrow y = x \,.\, u + w \Leftrightarrow y = x' \,.\, u + (-(b \,.\, u) + w) \Leftrightarrow (x', y)^\pi \in (u, (-(b \,.\, u) + w)^*)^\lambda$. Hence $p'_a = (u, (-(b \,.\, u) + w)^*)_a$.

Combining the propositions 2. and 3., we obtain without troubles:

Theorem 2.

The vertically transitive plane $\mathbf{P}(\mathrm{V}, \mathbf{n})$ is a translation plane if and only if $(\mathbf{S}, \mathbf{t})$ satisfies to the condition (C).

Remark: If $(\mathbf{S}, \mathbf{t})$ is a HPTR fulfilling (A), (B), (C), then $1 \in Q(a, b, a + b)$. Hence for any $m \in \mathbf{S} : a \,.\, m + b \,.\, m = (a + b) \,.\, m$.

5. DESARGUESIAN PLANES

The plane $\mathbf{P}(\mathrm{V}, \mathbf{n})$ will be called a *desarguesian plane,* if it is $(C, \mathbf{n})$-transitive, for every proper point C. It is known, that if $\mathbf{P}(\mathrm{V}, \mathbf{n})$ is a translation plane, that it is desarguesian if and only if there exists a proper point C such, that $\mathbf{P}(\mathrm{V}, \mathbf{n})$ is $(C, \mathbf{n})$-transitive.

Proposition 4.

Let $\mathbf{P}(\mathrm{V}, \mathbf{n})$ be the desarguesian plane. Then $(\mathbf{S}, \mathbf{t})$ has the following property:
(D) *Let a, b, c be arbitrary elements of $\mathbf{S}$,*

$$R(a, b, c) := \{m \in \mathbf{S} \mid m \,.\, a + m \,.\, b = m \,.\, c\} \Rightarrow$$
$$\Rightarrow R(a, b, c) = \{o_L\} \vee R(a, b, c) = \mathbf{S}.$$

Proof: Suppose, that $m_0 \in R(a, b, c)$, $m_0 \neq o_L$. We may assume $a \neq o_R$, $b \neq o_R$. Consider an arbitrary element $m \in \mathbf{S}$, $m \neq o_L$ and the dilatation $K \in U(0, \mathbf{n})$, $K : (m_0, o_L)^\pi \mapsto (m, o_L)^\pi$. Let p_0, p be the parallel lines, $\mathfrak{p}^1(p_0) = \mathfrak{p}^1(p) =: a$ such that $p_0 \ni (m_0, o_L)^\pi$, $p \ni (m, o_L)^\pi$. Hence: $p_0 = (a, (-(m_0 \,.\, a))^*)^\lambda$, $p = (a, (-(m . a))^*)^\lambda$. Put $q_0 = (c, (-(m_0 \,.\, a))^*)^\lambda$, $q = (c, (-(m \,.\, a))^*)^\lambda$. It is easy seen that $K(q_0) = q$. Let $Y_0 \in q_0$, $Y \in q$, $\mathfrak{p}_1(Y_0) = m_0$, $\mathfrak{p}_1(Y) = m$. It is obvious, that $K(Y_0) = Y$, hence Y_0, Y lie on the same line $(r, o_L^*)^\lambda$ through 0. It follows from it

$$m_0 \,.\, c - (m_0 \,.\, a) = m_0 \,.\, r \tag{5}$$

$$m \,.\, c - (m \,.\, a) = m \,.\, r. \tag{6}$$

As $m_0 \in R(a, b, c)$, it follows from (5), that $r = b$. The relation (6) gives the proved assertion.

Lemma.

Let $(\mathbf{S}, \mathbf{t})$ *satisfy the conditions* (A)–(D). *Then for arbitrary* $a, b, c \in \mathbf{S}$, $c \neq o_L$

$$a \,.\, (c \setminus (-b)) = -(a \,.\, (c \setminus b)) \tag{7}$$

is true.

Proof: Put $k = c \setminus (-b)$, $s = c \setminus b \Rightarrow c \,.\, k + c \,.\, s = o_L$. The condition (D) gives $a \,.\, k + a \,.\, s = o_L$, which implies (7).

Proposition 5.

Let $\mathbf{P}(\mathbf{V}, \mathbf{n})$ *be the desarguesian plane. Let* $K \in U(0, \mathbf{n})$ *be the dilatation* with

$$K : (u, o_L)^\pi \mapsto (\bar{u}, o_L)^\pi, \qquad (w, o_L)^\pi \mapsto (\bar{w}, o_L)^\pi, \qquad u, \bar{u}, w, \bar{w} \in \mathbf{S} \setminus \{o_L\}.$$

Then for any $m \in \mathbf{S}$

$$w \setminus (u \,.\, m) = \bar{w} \setminus (\bar{u} \,.\, m) \tag{8}$$

is valid.

Proof: (8) is true, if $m = o_R$. Assume, that $m \in \mathbf{S} \setminus \{o_R\}$. Put $k = w \setminus (u \,.\, m)$, it is to prove

$$\bar{w} \,.\, k = \bar{u} \,.\, m. \tag{9}$$

Consider two cross parallel lines $p, \bar{p}$ such that $\mathfrak{p}^1(p) = \mathfrak{p}^1(\bar{p}) = m$, $p \ni (u, o_L)^\pi$, $\bar{p} \in (\bar{u}, o_L)^\pi$. Then $p = (m, (-(u \,.\, m))^*)^\lambda$, $\bar{p} = (m, (-(\bar{u} \,.\, m))^*)^\lambda$. Further, consider the lines $q = (k, (-(u \,.\, m))^*)^\lambda$, the $\bar{q} = (k, (-(\bar{u} \,.\, m))^*)^\lambda$. As $K(p) = \bar{p}$, it follows by definition of q and $\bar{q}$, that $K(q) = \bar{q}$. Now, $w \,.\, k = u \,.\, m \Rightarrow (w, o_L)^\pi \in q \Rightarrow (\bar{w}, o_L)^\pi \in$ $\in \bar{q} \Rightarrow (9)$.

Proposition 6.

Let $\mathbf{P}(\mathbf{V}, \mathbf{n})$ *be the desarguesian plane. Then* $(\mathbf{S}, \mathbf{t})$ *has following property:*
(E) *Let* $u, \bar{u}, w, \bar{w}$ *be arbitrary elements of* $\mathbf{S}$ *different of* o_L,

$$L(w, u, \bar{w}, \bar{u}) = \{m \in \mathbf{S} \mid w \setminus (u \,.\, m) = \bar{w} \setminus (\bar{u} \,.\, m)\} \Rightarrow$$
$$\Rightarrow L(w, u, \bar{w}, \bar{u}) = \{o_R\} \vee L(w, u, \bar{w}, \bar{u}) = \mathbf{S}.$$

Proposition 6 follows immediately from proposition 5.

For the further investigation we introduce two functions $f, g : \mathrm{S} \to \mathrm{S}$ by following way: We choose elements $u, \bar{u}, m_0 \in \mathrm{S}$ such that $u, \bar{u} \neq o_L$, $m_0 \neq o_R$ and define

$$f(x) = (\bar{u} \,.\, m_0)/(x \setminus (u \,.\, m_0)) \qquad \text{if } x \neq o_L, f(o_L) = o_L$$
$$g(x) = \bar{u} \,.\, (u/x).$$

It is easy seen, that f, g are bijections.

Proposition 7.

Let $\mathbf{P}(\mathbf{V}, \mathbf{n})$ *be the translation plane and let* $(\mathbf{S}, \mathbf{t})$ *satisfy to the conditions* (A)–(E). *Then the mapping*

$$k : (x, y)^\pi \mapsto (f(x), g(y))^\pi$$

is the restriction of a dilatation $K \in U(0, \mathbf{n})$ *onto* $\mathscr{A}$ *with* $K : (u, o_L)^\pi \mapsto (\bar{u}, o_L)^\pi$.

Proof: Essentially it is to prove, that the map of arbitrary cross affine line p_a is the line $\bar{p}_a$ parallel to p_a. Let $x \in \mathbf{S} \setminus \{o_L\}$. By definition of f

$$x \setminus (u \,.\, m_0) = f(x) \setminus (\bar{u} \,.\, m_0) \tag{10}$$

is true. It implies, that $m_0 \in L(w, u, \bar{w}, \bar{u})$, hence by (E): $L(w, u, \bar{w}, \bar{u}) = \mathbf{S}$. Let $y \in \mathbf{S}$ and let us put

$$m = u \setminus (-y), \qquad b = x \setminus (u \,.\, m). \tag{11}$$

As $m \in L(x, u, f(x), \bar{u})$, (11) and lemma imply

$$g(y) = -(f(x) \,.\, b).$$

Furthermore (11) implies

$$y = -(x \,.\, b).$$

Let $p = (r, q^*)^\lambda$, $\bar{p} = (r, g(q)^*)^\lambda$. We will prove, that $k(p_a) = \bar{p}_a$.

A. First assume, that $q = o_L^* \Rightarrow p = \bar{p}$ and $0 \in p$. Let $(x, y)^\pi \neq 0$. Then $(x, y)^\pi \in \in p_a \Leftrightarrow y = x \,.\, r \Leftrightarrow x \,.\, r + x \,.\, b = o_L \Leftrightarrow f(x) \,.\, r + f(x) \,.\, b = o_L \Leftrightarrow g(y) = f(x) \,.\, r \Leftrightarrow \Leftrightarrow k((x, y)^\pi) \in p_a$.

B. Assume, that q is an arbitrary element of $\mathbf{S}$. If $x = o_L$, then $f(x) = o_L$ and $k((x, y)^\pi) = (o_L, g(y))^\pi$. Furthermore: $(o_L, y)^\pi \in p \Leftrightarrow y = q \Leftrightarrow g(y) = g(q) \Leftrightarrow k((o_L, y)^\pi) \in \in \bar{p}$.

Let $x \neq o_L$, then $f(x) \neq o_L$. Put $c = x \setminus (-q) \Rightarrow x \,.\, c = -q \Rightarrow (x, -q)^\pi \in \in (c, o_L)^\pi$. It follows from the part A that $k((x, -q)^\pi) \in (c, o_L^*)^\lambda \Rightarrow f(x) \,.\, c = -g(q)$. Now we have: $(x, y)^\pi \in p \Leftrightarrow y = x \,.\, r + q \Leftrightarrow x \,.\, r + x \,.\, b = x \,.\, c \Leftrightarrow f(x) \,.\, r + f(x) \,.\, b = = f(x) \,.\, c \Leftrightarrow f(x) \,.\, r + g(q) = g(y) \Leftrightarrow k((x, y)^\pi) \in \bar{p}_a$.

Combining the propositions 4., 6., 7., we get:

Theorem 3.

The translation plane $\mathbf{P}(\mathbf{V}, \mathbf{n})$ *is desarguesian if and only if* $(\mathbf{S}, \mathbf{t})$ *satisfies to the conditions* (D) *and* (E).

Remark: Let $(\mathbf{S}, \mathbf{t})$ be a HPTR fulfilling (A)−(E). Let $a, b \in \mathbf{S} \setminus \{0\}$. As $1 \in L(a, a \,.\, b, 1, b)$ then arbitrary $c \in \mathbf{S}$ belongs to $L(a, a \,.\, b, 1, b)$. It implies $a \setminus \setminus ((a \,.\, b) \,.\, c) = 1 \setminus (b \,.\, c) \Rightarrow a \,.\, (b \,.\, c) = (a \,.\, b) \,.\, c$. The last relation is evidently true also for $a = 0$ or $b = 0$.

As $1 \in R(a, b, a + b)$, it follows from (D) that for any c

$$c \,.\, (a + b) = c \,.\, a + c \,.\, b$$

is true.

6. PAPPIAN PLANES

The plane $\mathbf{P}(\mathbf{V}, \mathbf{n})$ will be called pappian if it is desarguesian and for any proper poin C the group $U(C, \mathbf{n})$ is Abelian. It is known, that the desarguesian plane $\mathbf{P}(\mathbf{V}, \mathbf{n})$ is pappian if and only if, there exists a point $C \in \mathscr{A}$ such that $U(C, \mathbf{n})$ is Abelian.

Theorem 4.

The desarguesian plane $\mathbf{P}(\mathbf{V}, \mathbf{n})$ *is pappian if and only if* $(\mathbf{S}, \mathbf{t})$ *has the following property:*

(F) $\forall\, a, b, c \in \mathbf{S},\ c \neq o_L, o_R : a\,.\,(c \setminus (b\,.\,c)) = b\,.\,(c \setminus (a\,.\,c)).$ (12)

Proof: (13) is true, if $a = o_L$ or $b = o_L$. Assume that $a \neq o_L$ and $b \neq o_L$. Consider two dilatations $K_1, K_2 \in U(0, \mathbf{n})$ such that $K_1 : (c, o_L)^\pi \mapsto (b, o_L)^\pi$, $K_2 : (c, o_L)^\pi \mapsto (a, o_L)^\pi$.

Put $Z = (o_L, c, c)^\pi$. It follows from proposition 1:

$$\mathfrak{p}_2((K_2\,.\,K_1)(Z)) = a\,.\,(c \setminus (b\,.\,c))$$
$$\mathfrak{p}_2((K_1\,.\,K_2)(Z)) = b\,.\,(c \setminus (a\,.\,c)).$$

As $\mathfrak{p}_1((K_2\,.\,K_1)(Z) = \mathfrak{p}_1((K_1\,.\,K_2)(Z)) = o_L$, then $\mathbf{P}(\mathbf{V}, \mathbf{n})$ is pappian if and only if $\forall\, a, b \in \mathbf{S} \setminus \{o_L\}$, $\forall\, c \in \mathbf{S} \setminus \{o_L, o_R\}$:

$$\mathfrak{p}_2((K_2\,.\,K_1)(Z)) = \mathfrak{p}_2((K_1\,.\,K_2)(Z))$$

$\Rightarrow$ (12).

Remark: Let $(\mathbf{S}, \mathbf{t})$ be HPTR fulfilling (A)–(E). Putting in (12) $c = 1$, we obtain $a\,.\,b = b\,.\,a$ for any $a, b \in \mathbf{S}$.

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SOUHRN

PŘIROZENÉ PLANÁRNÍ TERNÁRNÍ OKRUHY

DALIBOR KLUCKÝ

V článku se studuje projektivní rovina **P** s privilegovanou vlajkou (V, **n**) v níž je zavedena soustava souřadnic tak, že odpovídající souřadnicový obor má strukturu přirozeného planárního ternárního okruhu, tj. planární ternárního okruhu s levou a pravou kvasinulou. Jsou nalezeny nutné a postačující podmínky pro tento ternární okruh, aby rovina **P** byla (a) (V, **n**)-transitivní, (b) **n**-transitivní, (c) desarguessovská, (d) pappovská. Tím jsou zobecněny výsledky obsažené v lit. [2] a [3].

РЕЗЮМЕ

ЕСТЕСТВЕННЫЕ ТЕРНАРНЫЕ КОЛЬЦА

ДАЛИБОР КЛУЦКИ

В статье рассмотривается проективная плоскость **P** вместе с выделённым флагом (V, **n**) в которой определена сыстема координат такая, что соответствующая область координат является тернарным кольцом (**S**, **t**) обладающим левым и правым квазинулем — так называемым естественным тернарным кольцом. В статье выведены необходимые и достаточные условия, которые дольжны быть исполнены естественным тернарным кольцом (**S**, **t**) для того, что бы **P** являлась (а) (V, **n**) — транситивной, (б) **n** — транситивной, (в) Дезарговой, (г) Папповой плоскостью. Темже самым обобщены результаты из [2] и [3].