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Sborník prací Přírodovědecké fakulty University Palackého v Olomouci. Matematika, Vol. 20 (1981), No. 1, 23--25

1981 — ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM — TOM 69

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A CONTRIBUTION TO THE THEORY OF DECOMPOSITIONS OF PARTIAL ALGEBRAS

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(*Received March 15th, 1980*)

In this paper, there is shown, that the least common covering of two admissible decompositions of a partial algebra need not be admissible.

1. PRELIMINARIES AND NOTATIONS

By *a partial algebra* we will mean, as usually, a system $\mathcal{G} = \langle G, f_\gamma \rangle_{\gamma \in \Gamma}$, where G is a non-empty set—the support of $\mathcal{G}$ and f_γ are partial operations on G. For our purpose we will assume that the arities of all f_γ are positive integers. For any $\gamma \in \Gamma$ we denote by p_γ the arity of f_γ and by $\mathrm{Dom} f_\gamma$ the domain of f_γ. It is $\mathrm{Dom} f_\gamma \subseteqq G^{p_\gamma}$ of course.

Let D denote the set of all decompositions on G. It is well known, that the set D ordered by the condition $\bar{A} \leqq \bar{B}$ iff $\bar{A}$ is a refinement of $\bar{B}$ ($\Leftrightarrow \bar{B}$ is a covering of $\bar{A}$) forms a complete lattice. The supremum (the least upper bound) of two decompositions on G is their *least common covering*, the infimum (the greatest lower bound) of them is their *greatest common refinement*. The least common covering (the greatest common refinement) of two decompositions $\bar{A}$, $\bar{B}$ on G will be denoted by $[A, B]$ $((A, B))$. Further, we denote by $\bar{G}_{\max}$ the greatest decomposition on G consisting of a single element, namely G. Analogously we denote by $G_{\min}$ the least decomposition on G consisting of all one-point sets $\{x\}$, $x \in G$. $\bar{G}_{\max}$ is the maximum and $\bar{G}_{\min}$ is the minimum of the lattice $(D, [.,.], (.,.))$ of all decompositions on G.

2. LATTICE OF DECOMPOSITIONS OF $\mathcal{G}$

The decompositions on G are in a $1-1$ correspondence with an equivalence relation on G. If a decomposition $\bar{G}$ on G corresponds to the congruence on $\mathcal{G}$, then $\bar{G}$ will be called *the admissible decomposition* of partial algebra $\mathcal{G}$ or breafly a decomposition of $\mathcal{G}$.

Theorem.

Let $\mathscr{G} = \langle G, f_\gamma \rangle_{\gamma \in \Gamma}$ be a partial algebra and let $(D, [.,.], (.,.))$ be the lattice of all decompositions on G. Then the set of all admissible decompositions of $\mathscr{G}$ forms a complete lower subsemilattice of $(D, (.,.))$.

Proof: Let us consider a non-empty system $\{\bar{G}_\lambda\}_{\lambda \in \Gamma}$ of decompositions of $\mathscr{G}$. Their greatest common refinement $\inf \{\bar{G}_\lambda\}$ consists of all non-empty intersections $\bigcap_{\lambda \in I} \bar{g}_\lambda$, $\bar{g}_\lambda \in \bar{G}_\lambda$. Let us choose an arbitrary partial operation f_γ and the classes $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{p_\gamma}$ of $\inf \{\bar{G}_\lambda\}$ such that the intersections $\bar{x}_1 \cap \operatorname{Dom} f_\gamma, \ldots, \bar{x}_{p_\gamma} \cap \operatorname{Dom} f_\gamma$ are non-empty. Further, let for any $i = 1, 2, \ldots, p_\gamma$, x_i, x_i' be two elements of $\bar{x}_i \cap$ $\cap \operatorname{Dom} f_\gamma$. It holds $\bar{x}_i = \bigcap_{\lambda \in I} \bar{g}_{i,\lambda}$ for a suitable system $\{\bar{g}_{i,\lambda}\}_{\lambda \in I}$, $\bar{g}_{i,\lambda} \in \bar{G}_\lambda$. Since $\bar{G}_\lambda$ is an admissible decomposition of $\mathscr{G}$ and since x_i, x_i' belong to the same class of $\bar{G}_\lambda$, then $f_\gamma(x_1, \ldots, x_{p_\gamma}), f_\gamma(x_1', \ldots, x_{p_\gamma}')$ belong to the same class of $\bar{G}_\lambda$, too. Let us denote it by y_λ. Therefore, $f_\gamma(x_1, \ldots, x_{p_\gamma})$ and $f_\gamma(x_1', \ldots, x_{p_\gamma}')$ belong to $\bigcap_{\lambda \in I} y_\lambda$, which is a class of $\inf \{\bar{G}_\lambda\}$. Consequently, $\inf \{G_\lambda\}$ corresponds with some congruence on $\mathscr{G}$, i.e. it is a decomposition of $\mathscr{G}$. Thus, we have proved our theorem.

Let us denote by Δ the set of all admissible decompositions of $\mathscr{G}$. Then $(\Delta, (.,.))$ is a complete (lower) semilattice with the maximum $\bar{G}_{\max}$. Then we may transform $(\Delta, (.,.))$ in a lattice $(D, \langle .,. \rangle, (.,.))$ in the obvious way: The supremum $\langle \bar{A}, \bar{B} \rangle$ of two admissible decompositions of $\mathscr{G}$ is defined as the infimum of all admissible decompositions being the common covering of $\bar{A}$ and $\bar{B}$. The following example shows, that $\langle .,. \rangle$ need not be the restriction of $[.,.]$ on $\triangle \times \triangle$, hence the lattice $(\triangle, \langle .,. \rangle,$ $(.,.))$ need not be a sublattice of $(D, [.,.], (.,.))$.

3. EXAMPLE

Let $G = \{a, b, c, d\}$ and let f be a partial binary operation on G with $\operatorname{Dom} f =$ $= \{(a, a), (a, c)\}$, $f(a, a) = a$ and $f(a, c) = d$. The decompositions

$$\bar{G}_1 = \{\{a, b\}, \{c\}, \{d\}\}$$

and

$$\bar{G}_2 = \{\{a\}, \{b, c\}, \{d\}\} \qquad \text{on } G$$

are obviously admissible decompositions of $\langle G, f \rangle$. The least common covering of $\bar{G}_1$ and $\bar{G}_2$ is the decomposition $\{\{a, b, c\}, \{d\}\}$. According to the definition of f we see, that this decomposition is a not admissible decomposition of $\langle G, f \rangle$, consequently it is not the supremum of $\bar{G}_1, \bar{G}_2$ in the lattice of all admissible decompositions of $\langle G, f \rangle$.

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SOUHRN

PŘÍSPĚVEK K TEORII ROZKLADŮ PARCIÁLNÍCH ALGEBER

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V předložené práci je ukázáno, že nejmenší společný zákryt dvou přípustných rozkladů dané parciální algebry není obecně přípustným rozkladem téže parciální algebry.

Dále je zde dokázáno, že systém všech přípustných rozkladů parciální algebry není podsvazem ve svazu všech rozkladů dané parciální algebry.

РЕЗЮМЕ

ЗАМЕЧАНИЕ К ТЕОРИИ РАЗЛОЖЕНИЙ ЧАСТИЧНЫХ АЛГЕБР

ЯРОСЛАВ ШВРЧЕК

В предлагаемой работе показанно, что самое малое общее накрытие двух допустимых разложений частичной алгебры в общности нет допустимым разложением.

Здесь показанно, что система всех допустимых разложений частичной алгебры не является подрешёткой решётки всех разложений данной частичной алгебры.