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OPTIMAL DISTRIBUTION OF BREAK POINTS AT A DIODE FUNCTION GENERATOR

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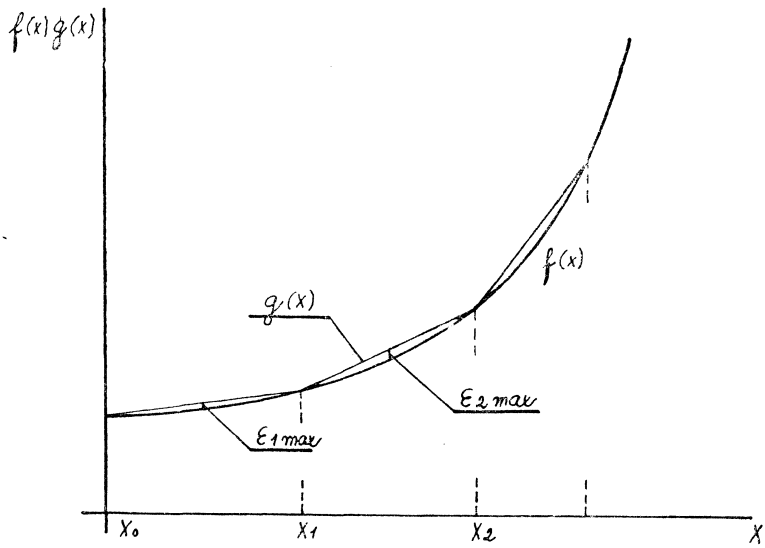
In computing nonlinear problems by an analog computer or by a hybrid system with diode function generators the accuracy can be increased by an appropriate distribution of break points (knots of approximation) if an approximating function dependence is generated by a function transformator (see figure 1). In most cases the



uniform distribution of break points is evidently far from optimal. Its disadvantage lies in the fact that the error of approximation ε_j can be substantially larger in one segment than in the other segments. Thus we find the distribution of break points respecting the requirement of the best uniform approximation to be for the total accuracy of computation of greater advantage because the maximal absolute errors here are equal in all segments. The appropriate distribution of break points can be determined by a graphic method at a random search for such a broken line approximating the given function within the tolerance, or we choose an analytic way for calculating the distribution of break points. Applying the graphic method we meet with difficulties in determining the maximal error arising at the approximation of the function (respecting the requirement of the best uniform approximation) through a certain number of linear segments. Choosing the analytic way in determining the break points leads to a system of nonlinear algebraic or transcendental equations.

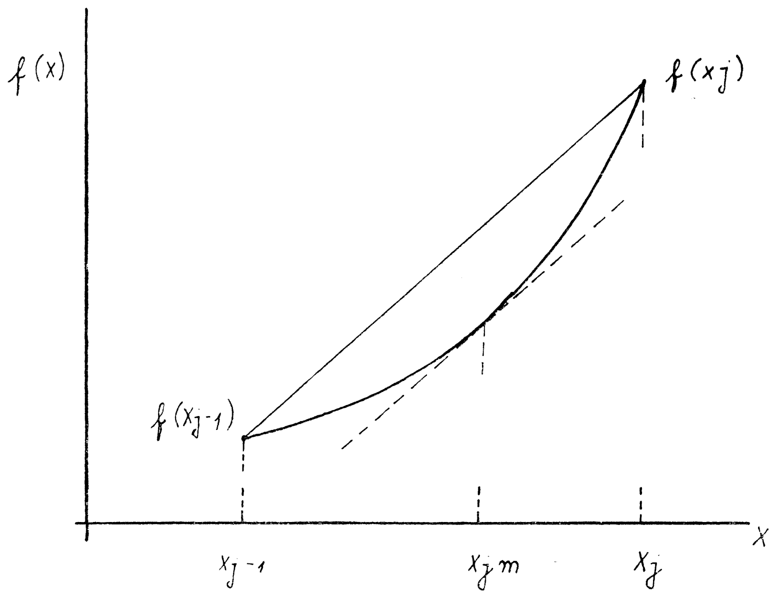
This article presents an analytic determination of break points distribution respecting the requirement of the best uniform approximation.

A. The precise machine computing of the break point distribution.

It holds

$$| \varepsilon_1(x) |_{\max} = | \varepsilon_2(x) |_{\max} = \dots = | \varepsilon_k(x) |_{\max}, \tag{1}$$

for the best uniform approximation, where k stands for the number of segments. In figures 1 and 2 the problem is described by the following system of equations:



$$| \varepsilon_j(x) | = \max \left| \frac{f(x_j) - f(x_{j-1})}{x_j - x_{j-1}} (x_{jm} - x_{j-1}) + f(x_{j-1}) - f(x_{jm}) \right|,$$

$$f'(x_{jm}) = \frac{f(x_j) - f(x_{j-1})}{x_j - x_{j-1}}, \qquad j = 1, 2, \dots, k-1, k,$$

$$| \varepsilon_j(x) |_{\max} = | \varepsilon_{j+1}(x) |_{\max}, \qquad j = 1, 2, \dots, k-1.$$

System (2) has for $f(x) = x^3$, $x \in \langle 0; 1\rangle$, $k = 5$ the form

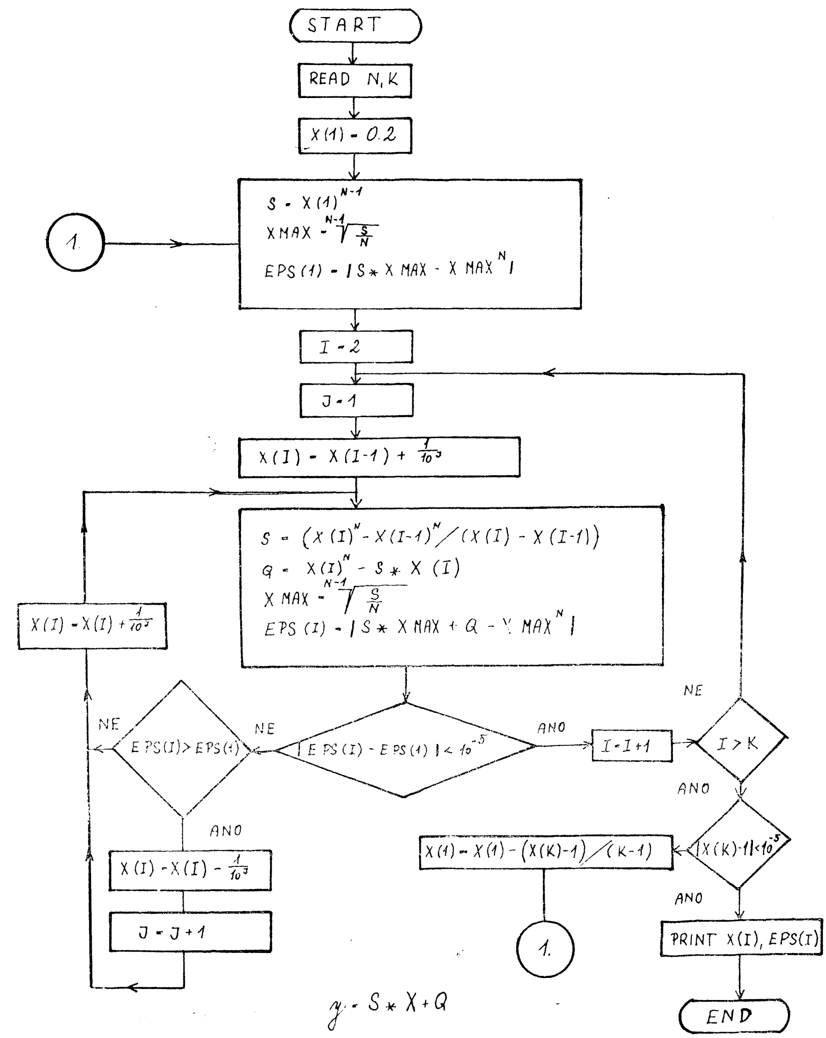
$$|\varepsilon_j(x)|_{\max} = \frac{x_j^3 - x_{j-1}^3}{x_j - x_{j-1}}(x_{jm} - x_{j-1}) + x_{j-1}^3 - x_{jm}^3, \tag{2a}$$

i.e.

$$x_{jm} = \sqrt{\frac{1}{3}\,\frac{x_j^3 - x_{j-1}^3}{x_j - x_{j-1}}}\,.$$

In computing the system of equations

$$|\varepsilon_j(x)|_{\max} = |\varepsilon_{j+1}(x)|_{\max}, \qquad j = 1, 2, 3, 4$$



gives

$$\begin{aligned} \varepsilon_j(x)_{\max} &= 0{,}013\,73 \\ x_1 &= 0{,}329\,23 \\ x_2 &= 0{,}534\,95 \\ x_3 &= 0{,}706\,67 \\ x_4 &= 0{,}859\,58 \\ x_5 &= 0{,}999\,99\ (\doteq 1), \end{aligned} \tag{2b}$$

figure 3 shows a flow chart for computing the system (2). Table 1 presents a break point distribution in some functions for $k = 5$. Table 2 presents a break point distribution of equal functions for $k = 10$.

x_j	x^3	x^4	x^5	e^{-x}	$\ln x$
x_0	0,000	0,000	0,000	0,000	0,368
x_1	0,329	0,427	0,501	0,164	0,549
x_2	0,535	0,621	0,681	0,342	0,818
x_3	0,706	0,768	0,808	0,539	1,221
x_4	0,859	0,891	0,912	0,756	1,822
x_5	1,000	1,000	1,000	1,000	2,718
$\varepsilon_{\max}$	0,0137	0,0157	0,0169	0,0031	0,0199

Tab. 1.

x_j	x^3	x^4	x^5	e^{-x}	$\ln x$
x_0	0,000	0,000	0 000	0,000	0,368
x_1	0,206	0,300	0,377	0,080	0,449
x_2	0,335	0,436	0,513	0,164	0,548
x_3	0,443	0,540	0,609	0,251	0,670
x_4	0,539	0,626	0,687	0,342	0,818
x_5	0,626	0,703	0,753	0,438	1,000
x_6	0,709	0,771	0,811	0,539	1,221
x_7	0,786	0,834	0,864	0,644	1,492
x_8	0,860	0,893	0,913	0,756	1,822
x_9	0,931	0,947	0,958	0,874	2,225
x_{10}	1,000	1,000	1,000	1,000	2,718
$\varepsilon_{\max}$	0,0034	0,0038	0,0041	0,0007	0,0050

Tab. 2.

B. An approximate determination of break point distribution

1. A METHOD BASED ON THE ESTIMATE OF THE UPPER BOUND OF THE ERROR

In the polynomial approximation of the function $f(x)$ by the function $g(x)$ in the interval of the approximation $a; b$ there is the inaccuracy of the approximation $(x) = f(x) - g(x)$ given by the relation

$$\varepsilon(x) = \frac{f(\zeta)^{(n+1)}}{(n+1)!}(x - x_0)(x - x_1), \ldots, (x - x_n), \tag{3}$$

where $x^0, x_1, \ldots, x_n$ are the knots of the approximation and ζ a certain point of the interval $\langle a; b\rangle$, n is the degree of the approximating function. The maximal absolute error of the approximation satisfies the inequality

$$|\varepsilon(x)|_{\max} = \frac{M_{n+1}}{(n+1)!}(x, x_0, x_1, \ldots, x_n), \tag{4}$$

where $M_{n+1} = \max |f(x)^{(n+1)}|$,

$$\varphi(x, x^0, x_1, \ldots, x_n) = \max_{x \in \langle a,b\rangle} |(x - x^0)(x - x_1) \ldots (x - x_n)|.$$

In the approximation of the function $f(x)$ through the the linear segments is $n = 1$ and the error in the interval $x \in \langle a; b\rangle$ (according to (3)) is given by the relation

$$\varepsilon(x) = \frac{f''(\zeta)}{2!}(x - x_{j-1})(x - x_j), \tag{5}$$

$\zeta \in \langle x_{j-1}; x_j\rangle$, $j = 1, 2, \ldots, k$, where k stands for the number of linear segments.

For the estimate of the maximal absolute error (according to (2)) it is necessary to find the extreme of the expression $v = (x - x_{j-1})(x - x_j)$. From the condition for the extreme value of the function v

$$\frac{dv}{dx} = x - x_j + x - x_{j-1} = 0,$$

we determine the value x, at which the function v assumes its extreme. Hence

$$x = \frac{x_{j-1} + x_j}{2}. \tag{5a}$$

Inserting these values into the right-hand side of equation (2) shows that

$$|\varepsilon(x)|_{\max} \leqq \frac{M_2}{2!}\left(\frac{x_j - x_{j-1}}{2}\right)^2, \tag{6}$$

holds for the upper bound of the absolute inaccuracy in the interval $\langle x_{j-1}; x_j\rangle$. Then it suffices to choose the length of the corresponding step $h_j = x_j - x_{j-1}$ as follows

$$h_j \leqq \sqrt{\frac{8 \mid \varepsilon(x) \mid}{M_2}}. \tag{7}$$

By means of (6) we can then with a sufficient accuracy determine the computation of the break point distribution if we approximate the function through linear segments respecting the requirement of the best uniform approximation while the maximal absolute errors are equally large in all segments.

$$\mid \varepsilon_1(x) \mid_{\max} = \mid \varepsilon_2(x)_{\max} = \dots = \mid \varepsilon_k(x) \mid_{\max}. \tag{8}$$

Substituting this into (8) according to (6) enables us to carry over the equality (6) to a system of equations for $x_1, x_2, \dots, x_{k-1}$. We approximate the function in the interval $\langle a; b \rangle$. If there is $M_{2j} \doteq$ constant, then we may consider an equality in (6) and the system of equations for $x_1, x_2, \dots, x_{k-1}$ becomes the form

$$\begin{aligned} M_{21}(x_1 - a)^2 &= M_{22}(x_2 - x_1)^2 \\ M_{22}(x_2 - x_1)^2 &= M_{23}(x_3 - x_2)^2 \\ &\vdots \\ M_{2k-1}(x_{k-1} - x_{k-2})^2 &= M_{2k}(b - x_{k-1})^2, \end{aligned}$$

where M_{2j} is the maximal absolute value of the second derivative of the function in the j-th segment where a is the beginning and b the end of the interval of the approximation. In monotonic functions $f(x) = x^n$, $\ln x$, e^x etc. there is M_{2j} at the beginning or at the end of the corresponding segment which is to be decided in each individual case first. For $f(x) = x^3$, $k = 5$, $a = 0$, $b = 1$ the system (9) has the form

$$\begin{aligned} x_1^3 &= x_2(x_2 - x_1)^2 \\ x_2(x_2 - x_1)^2 &= x_3(x_3 - x_2)^2 \\ x_3(x_3 - x_2)^2 &= x_4(x_4 - x_3)^2 \\ x_4(x_4 - x_3)^2 &= (1 - x_4)^2, \end{aligned} \tag{9a}$$

and

$$\begin{aligned} x_1 &= 0{,}286 \qquad & x_3 &= 0{,}684 \\ x_2 &= 0{,}500 \qquad & x_4 &= 0{,}848 \end{aligned} \tag{9b}$$

is the solution of the system. The sizes of the maximal errors $\mid \varepsilon_j(x) \mid_{\max}$ are

$$\begin{aligned} \varepsilon_1(x)_{\max} &= 0{,}0090 \qquad & \varepsilon_3(x)_{\max} &= 0{,}0150 \\ \varepsilon_2(x)_{\max} &= 0{,}0135 \qquad & \varepsilon_4(x)_{\max} &= 0{,}0155 \\ & & \varepsilon_5(x)_{\max} &= 0{,}0160. \end{aligned} \tag{9c}$$

The above determination of the break points based on the relations of (9) without computer application is a relatively laborious task. The system (7) is a generally nonlinear system of algebraic or transcendental equations whose solution is difficult to be found without computer. It is relatively easy to find the solution of the system (7) in case of $f(x) = \sqrt{x}$ (cf. [1]) and $f(x) = \ln x$.

In computing the break point distribution using the approximation of the function $f(x) = \ln x$ with the number of segments k we proceed according to (7) as follows: Seeing that $f''(x) = \frac{1}{x^2}$, the maximal absolute value of the second derivative of the function $f(x) = \ln x$ will be always at the beginning of the corresponding segment, i.e. $M_{2j} = \frac{1}{x_{j-1}^2}$. The system of equations of (7) will have the form

$$\begin{aligned} \frac{1}{a^2}(x_1 - a)^2 &= \frac{1}{x_1^2}(x_2 - x_1)^2, \\ \frac{1}{x_1^2}(x_2 - x_1)^2 &= \frac{1}{x_2^2}(x_3 - x_2)^2, \\ &\vdots \\ \frac{1}{x_{j-2}^2}(x_{j-1} - x_{j-2})^2 &= \frac{1}{x_{j-1}^2}(x_j - x_{j-1})^2, \\ &\vdots \\ \frac{1}{x_{k-2}^2}(x_{k-1} - x_{k-2})^2 &= \frac{1}{x_{k-1}^2}(b - x_{k-1})^2. \end{aligned} \tag{9}$$

Extracting these equation we get the following system of equations

$$\frac{1}{x_{j-2}}(x_{j-1} - x_{j-2}) = \frac{1}{x_{j-1}}(x_j - x_{j-1}),$$
$$j = 2, 3, \ldots, k, \qquad x_0 = a, x_k = b. \tag{10a}$$

From (10a) we calculate x_2 for $j = 2$,

$$\frac{x_1}{a} - 1 = \frac{x_2}{x_1} - 1,$$

i.e.

$$x_2 = \frac{x_1^2}{a}. \tag{11}$$

Inserting this into (10a) for $j = 3$ instead of x_2 we get

$$\frac{1}{x_1}\left(\frac{x_1^2}{a} - x_1\right) = \frac{a}{x_1^2}\left(x_3 - \frac{x_1^2}{a}\right),$$

i.e.

$$x_3 = \frac{x_1^3}{a^2}. \tag{11a}$$

Repeating the above process gives

$$\frac{1}{\frac{1}{a}x_1^2}\left(\frac{x_1^3}{a^2} - \frac{1}{a} - x_1^2\right) = \frac{1}{\frac{1}{a^2}x_1^3}\left(x_4 - \frac{1}{a^2}x_1^3\right),$$

with the modification $\left(\text{multiplying by } \frac{1}{a} x_1^3\right)$ we get

$$x_4 = \frac{x_1^4}{a^2}. \tag{11b}$$

If we compare the relations (11), (11a) and (11b) we get

$$x_j = \frac{1}{a^{j-1}} x_1^j, \tag{11c}$$

i.e.

$$b = x_k = \frac{1}{a^{k-1}} x_1^k,$$

$$x_1 = \sqrt[k]{a^{k-1} b}. \tag{11d}$$

Insering into (11c) gives the values x_j, $j = 2, 3, \ldots, k - 1$.

For $k = 5$, $x \in \left\langle \frac{1}{e}; e \right\rangle$ is the distribution of break points as follows:

$$x_1 = 0{,}549, \qquad x_2 = 0{,}816, \qquad x_3 = 1{,}215, \qquad x_4 = 1{,}807. \tag{11e}$$

The distribution of break points by (11e) agrees with that obtained by the digita computer in solving system (2), where $x_1 = 0{,}5488$, $x_2 = 0{,}8187$, $x_3 = 1{,}2213$,l $x_4 = 1{,}8221$.

System (9) is generally calculated by the trial-and-error method (of successive approximations). Assume again a monotonic function of $f(x)$, k the number of linear segments, $x \in \langle a, b \rangle$. If we first assume a uniform segmentation, i.e. the length of the segment $h = \frac{b-a}{k}$, then we can determine from (7) the error with which the function will be approximated through the given number of linear segments with a uniform distribution of break points, i.e.

$$|\varepsilon(x)|_{\max} = \frac{1}{8}\left(\frac{b-a}{k}\right)^2 M_2, \tag{12}$$

$$M_2 = \max |f''(x)|, \qquad x \in \langle a, b \rangle.$$

Because by nonequidistant distribution of the break points the maximal error will be smaller than that determined by relation (12). Thus we reduce the determined error by the estimate (f.i. by 50 per cent according to the type or degree of the function approximated). Let us now start from the point $x_j \in \langle a; b \rangle$, where $f''(x)$ maximal (which is in monotonic functions mostly at the beginning or at the end of the interval $\langle a; b \rangle$ and following (5) we search for the bound of the segment i.e. the point x_{j+1} or x_{j-1}. We proceed analogous from this point further on till exhausting all k segments.

If

$$\sum_{j=1}^{k} h_j > b - a,$$

then we reduce the error, in the case of

$$\sum_{j=1}^{k} h_j < b - a,$$

the error will be enlarged. This procedure is repeated till we obtain

$$\sum_{j=1}^{k} h_j \doteq b - a. \tag{13}$$

Let us show the given procedure on the approximation of the function $f(x) = x^3$, $x \in \langle 0, 1 \rangle$, i.e. $a = 0, b = 1, k = 5$. By (12) the maximal absolute error in the uniform distribution of break points may be equal to 0,030. Let us assume that we shall have in the nonuniform distribution of break points $|\varepsilon(x)| \max = 0{,}020$. The maximal absolute value $f''(x)$ in the interval of approximation is at the point $x_5 = b = 1$, so that we start the calculation of the break point distribution from the value $x_5 = 1$ towards $x_0 = a = 0$. The length of the segment is by (5)

$$h_5 = \frac{0{,}16}{6} = 0{,}163.$$

The maximal values $f''(x)$ in the segment h_4 is at the point $x_4 = 1 - 0{,}163 = 0{,}837$ so that

$$h_4 = \frac{0{,}16}{6\,.\,0{,}837} = 0{,}179$$

and $x_3 = x_4 - h_4 = 0{,}658$. Likewise $h_3 = 0{,}210$, $x_2 = 0{,}448$, $h_2 = 0{,}244$, $x_1 = x_2 - h_2 = 0{,}204$, $h_1 = 0{,}363$. Because of $h_1 > x_1$ (i.e. $\sum . h_j > 1$) we reduce the estimated maximal error and will repeat the calculation. In $|\varepsilon(x)| \max = 0{,}0173$ we obtain

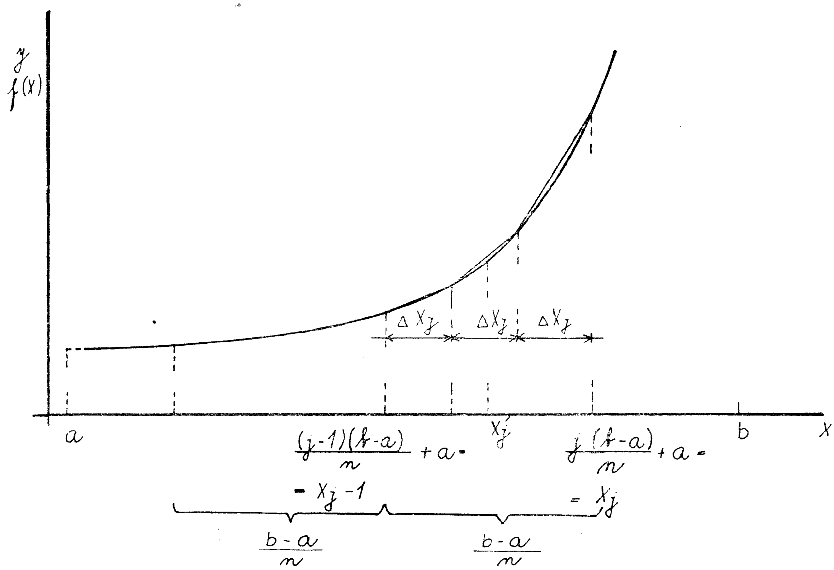
$$\begin{array}{ll} h_5 = 0{,}152 & x_4 = 0{,}848 \\ h_4 = 0{,}164 & x_3 = 0{,}684 \\ h_3 = 0{,}184 & x_2 = 0{,}500 \\ h_2 = 0{,}214 & x_1 = 0{,}286 \\ h_1 = 0{,}284 \doteq x_1 . & \end{array} \tag{14}$$

The calculation is proved satisfactory, condition (13) is satisfied with a sufficient accuracy ($\sum h_j = 0{,}998$).

The above procedure is just a certain modification of solving the system of (7a), as it is appeared from the results of (7b) and (14).

2. THE METHOD BASED ON A NUMERICAL INTERPRETATION OF THE SECOND DERIVATIVE

Likewise the followng approximative method is based on using approximative numerical methods and on the understanding that the second derivative of the approximated functions is approximately constant, i.e. $f''(x) \doteq$ const, $x \in \langle x_j, x_{j+1} \rangle$ (see figure 4). Devide the interval $\langle a; b \rangle$ into n equal subintervals where n is chosen



so that in any subinterval $f''(x) \doteq$ const. holds. By the required accuracy of approximation, each of these subintervals is approximated by an appropriate number of linear segments. Then the total number of segments k is by given the relation

$$k = \sum_{j=1}^{N} \frac{b-a}{N} \cdot \frac{1}{\Delta x_j} = \frac{b-a}{n} \sum_{j=1}^{N} \frac{1}{\Delta x_j}. \tag{15}$$

Inserting the point x_j' into the middle of the interval, i.e. if $x' - x_{j-1} = \dfrac{x_j - x_{j-1}}{2}$, holds, then we have there for y''

$$y_j'' \doteq \frac{y(x_{j-1}) + y(x_j) - 2y(x_j')}{\left(\dfrac{x_j - x_{j-1}}{2}\right)^2} =$$

$$= \frac{4N^2(y(x_{j-1}) + y(x_j) - 2y(x_j'))}{(b-a)^2} = \frac{4N^2 \Delta^2 y_j}{(b-a)^2}, \tag{16}$$

where $\Delta^2 y_j$ is th second difference in the j-th subinterval. If $y'' \doteq$ const. then we have in (5) an equality and for the lenght of the linear segment in the j-th subinterval

$$(\Delta x_j)^2 = \frac{8}{y''_j} = \frac{8\varepsilon(b - a)^2}{4N^2 \, |\Delta^2 y_j|} = \frac{2\varepsilon(x_j - x_{j-1})^2}{y(x_{j-1}) + y(x_j) - 2y(x'_j)}, \tag{17}$$

$$\frac{1}{\Delta x_j} = \frac{N\sqrt{|\Delta^2 y_j|}}{\sqrt{2\varepsilon}(b - a)}. \tag{17a}$$

Inserting the last relation into (15) gives

$$k = \frac{b - a}{N} \sum_{j=1}^{N} \frac{n\sqrt{|\Delta^2 y_j|}}{\sqrt{2\varepsilon}(b - a)} = \frac{1}{\sqrt{2\varepsilon}} \sum_{j=1}^{N} \sqrt{|\Delta^2 y_j|}. \tag{18}$$

Rewriting the last equation into the form

$$k^2 \varepsilon = \frac{1}{2} \Big(\sum_{j=1}^{N} \sqrt{|\Delta^2 y_j|}\Big)^2, \tag{19}$$

then the equation (19) expresses the relation between the number of segments and the error of approximation, i.e. it determines the error of approximation ε if k is given, or it determines the number of linear segments needed, if the error ε is given.

According to [2] it holds for the second derivative y'' (cf. 16)) in the j-th subinterval

$$y''_j = \frac{y(x_{j-1}) + y(x_j) - 2y(x'_j)}{\left(\dfrac{x_j - x_{j-1}}{1}\right)^2} - \frac{\left(\dfrac{x_j - x_{j-1}}{2}\right)^2}{12} y^{(4)}(\zeta), \tag{20}$$

$$\zeta \in \langle x_{j-1}; x_j \rangle.$$

The expression $\frac{1}{12}\left(\frac{x_j - x_{j-1}}{2}\right)^2 y^{(4)}(\zeta) = \frac{1}{12} h^2 y^{(4)}(\zeta)$ gives the estimate of the error of approximation of the second derivative by the expression (16) at the middle point x'_j of the subinterval. According to equation (17) it holds $\frac{1}{8}(\Delta x_j)^2 y''_j = \varepsilon$, if the second derivative is expressed by the error $\frac{1}{12} h^2 y^{(4)}(\zeta)$, then

$$\frac{1}{8}\left[(\Delta x_j)^2 \left(y''_j - \frac{1}{12} h^2 y^{(4)}(\zeta)\right)\right] = \varepsilon + \Delta\varepsilon, \tag{21}$$

where $\Delta\varepsilon = -(\Delta x)^2 \frac{1}{96} h^2 y^{(4)}(\zeta)$ is the deviation of the error caused by the error of approximation of the second derivative. Let us choose $\Delta x = \frac{b - a}{k}$ (analogous to the uniform distribution of break points). Then

$$| \Delta\varepsilon | = \left| \left(\frac{b-a}{k}\right)^2 \cdot \frac{1}{96} h^2 y^{(4)}(\zeta) \right| = \left| \left(\frac{b-a}{k}\right)^2 \cdot \frac{1}{96} \left(\frac{b-a}{2N}\right)^2 y^{(4)}(\zeta) \right| =$$

$$= \frac{(b-a)^4}{4k^2N^2} \cdot \frac{1}{96} y^{(4)}(\zeta). \tag{22}$$

From the tolerated change of error $\Delta\varepsilon$ we determine the value N and thus the coordinates of points, at which we will perform the preparatory calculation i.e.

$$N \geqq \frac{(b-a)^2 \sqrt{| y^{(4)} | \max}}{k \sqrt{384 \Delta\varepsilon}} \doteq \frac{(b-a)^2 \sqrt{| y^{(4)} | \max}}{20k \sqrt{\Delta\varepsilon}}. \tag{23}$$

At the boundary points x_{j-1}, x_j of the subintervals is the estimate for the error of approximation of the second derivative given by the relation $| \Delta y'' | = | hy'''(\zeta) |$, so that

$$| \Delta\varepsilon | = \left| \frac{1}{8} \left(\frac{b-a}{k}\right)^2 \cdot \frac{b-a}{2N} y'''(\zeta) \right|, \tag{24}$$

i.e.

$$N \geqq \frac{(b-a)^3 y'''(\zeta)}{16k^2}. \tag{25}$$

The procedure of the calculation:

Given the number of segments k, we determine the maximal absolute error $| \varepsilon(x) | \max$ on the basis of (12). Since the maximal error will be smaller in the nonuniform distribution of break points than that determined by relation (12), we reduce this error (say by 50 m according to the type of the approximated function) and choose the tolerated change of error $\Delta\varepsilon$. We determine the number N of subintervals from (23) or (25). We determine the function values of the approximated function at the end points x_{j-1}, x_j and at the middle point x_j' of any subinterval, calculate the second differences and their roots which we insert into relation (19).

On the basis of (17) we have

$$\frac{\Delta x_j}{x_j - x_{j-1}} = \sqrt{\frac{2\varepsilon}{y(x_{j-1}) - 2y(x_j') + y(x_j)}} = g. \tag{26}$$

If we choose the distribution of break points, then we make it more precise by means of (26). The break points are then given by $x_1 = x_0 + \Delta x_1, x_2 = x_1 + \Delta x_2, \ldots, x_1 = x_{1-1} + \Delta x_1$.

Below we give an example of an approximation of the function $y = x^3$ in five linear segments, i.e. $k = 5$, in the interval $x \in \langle 0; 1 \rangle$. By (12) we have $| \varepsilon(x) |_{\max} = 0{,}030$. Let us choose $\Delta\varepsilon = 0{,}002$, (by (23)), where $y'''(\zeta) = 6$, and $N = 7{,}5$. Choose $N = 10$.

$$\sum_{i=1}^{N} \sqrt{\Delta_i^2} = 0{,}819\,28, \qquad k^2\varepsilon = \frac{(0{,}819\,28)^2}{2} = 0{,}335\,61.$$

l	x	y	Δ^1	$\lvert\Delta^2\rvert$	$\sqrt{\lvert\Delta^2\rvert}$
	0,00	0,000 00			
			0,000 12		
	0,05	0,000 12		0,000 76	0,027 56
			0,000 88		
1	0,10	0,001 00			
			0,002 37		
	0,15	0,003 37		0,002 26	0,047 53
			0,004 63		
2	0,20	0,008 00			
			0,007 62		
	0,25	0,015 62		0,003 76	0,061 31
			0,011 38		
3	0,30	0,027 00			
			0,015 87		
	0,35	0,042 87		0,005 26	0,072 52
			0,021 13		
4	0,40	0,064 00			
			0,027 12		
	0,45	0,091 12		0,006 76	0,082 21
			0,033 88		
5	0,50	0,125 00			
			0,041 37		
	0,55	0,166 37		0,008 26	0,090 88
			0,049 63		
6	0,60	0,216 00			
			0,058 62		
	0,65	0,274 62		0,009 76	0,098 79
			0,068 38		
7	0,70	0,343 00			
			0,078 87		
	0,75	0,421 87		0,011 62	0,106 11
			0,090 13		
8	0,80	0,512 00			
			0,102 12		
	0,85	0,614 12		0,012 76	0,112 96
			0,114 88		
9	0,90	0,729 00			
			0,128 37		
	0,95	0,857 37		0,014 26	0,119 41
10	1,00	1,000 00	0,142 63		

Tab. 3.

For $k = 5$ the error $\varepsilon = 0{,}0134$. The size of the error agrees with that gives by relation (9b). For simplicity let us assume a uniform distribution of break points, i.e. $x_l - x_{l-1} = 0{,}2$. By relation (26) we have

$$\Delta x_1 = 0{,}2\sqrt{\frac{0{,}0268}{y(0) - 2y(0,1) + y(0,2)}} = 0{,}2\sqrt{\frac{0{,}0268}{0{,}0060}} \doteq 0{,}422$$

$$x_1 = 0{,}422,$$

$$\Delta x_2 = 0{,}2\sqrt{\frac{0{,}0268}{y(0{,}422) - 2y(0{,}522) + y(0{,}622)}} = 0{,}2\sqrt{\frac{0{,}0268}{0{,}0313}} \doteq 0{,}185$$

$$x_2 = x_1 + \Delta x_2 = 0{,}607,$$

$$\Delta x_3 = 0{,}2\sqrt{\frac{0{,}0268}{y(0{,}607) - 2y(0{,}707) + y(0{,}807)}} = 0{,}2\sqrt{\frac{0{,}0268}{0{,}0425}} \doteq 0{,}158$$

$$x_3 = x_2 + \Delta x_3 = 0{,}765$$

$$\Delta x_4 = 0{,}2\sqrt{\frac{0{,}0268}{y(0{,}765) - 2y(0{,}865) + y(0{,}965)}} = 0{,}2\sqrt{\frac{0{,}0268}{0{,}0518}} \doteq 0{,}143$$

$$x_4 = x_3 + \Delta x_4 = 0{,}908$$

For checking let us determine another point $x_5 = x_4 + \Delta x_5$, where

$$\Delta x_5 = 0{,}2\sqrt{\frac{0{,}0268}{y(0{,}908) - 2y(1{,}008) + y(1{,}108)}} = 0{,}2\sqrt{\frac{0{,}0268}{0{,}0604}} \doteq 0{,}133$$

$$x_5 = x_4 + \Delta x_5 = 1{,}041 \doteq 1.$$

Since the computed coordinate x_5 slightly deviates from the correct value $x_5 = 1$, it is necessary to amend the distribution of break points by multiplying out by a convenient coefficient so that $x_5 = 1$; in our case by $c = \frac{1}{1{,}041}$. Then the coordinates of break points are

$$x_1 = 0{,}405; \quad x_2 = 0{,}583; \quad x_3 = 0{,}734; \quad x_4 = 0{,}872; \quad x_5 = 1 \quad (27)$$

The computation of break points was performed on the assumption of a uniform distribution of break points, i.e. $x_l - x_{l-1} = 0{,}2$. Since the first segment is (by (27)) substantially larger than the other segments, we define exactly the computation through the assumption of the following distribution of points:

$$x_1 - x_0 = 0{,}4; \qquad x_l - x_{l-1} = 0{,}2, \qquad l = 2, \ldots, 5$$

$$\Delta x_1 = 0{,}4\sqrt{\frac{0{,}0268}{y(0) - 2y(0{,}2) + y(0{,}4)}} = 0{,}4\sqrt{\frac{0{,}0268}{0{,}0480}} = 0{,}298,$$

$$x_1 = 0{,}298,$$

$$\Delta x_2 = 0{,}2\sqrt{\frac{0{,}0268}{y(0{,}298) - 2y(0{,}398) + y(0{,}498)}} = 0{,}2\sqrt{\frac{0{,}0268}{0{,}0230}} = 0{,}215,$$

$$x_2 = x_1 + \Delta x_2 = 0{,}513,$$

$$\Delta x_3 = 0{,}2\sqrt{\frac{0{,}0268}{y(0{,}513) - 2y(0{,}613) + y(0{,}713)}} = 0{,}2\sqrt{\frac{0{,}0268}{0{,}0370}} = 0{,}170,$$

$$x_3 = x_2 + \Delta x_3 = 0{,}683,$$

$$\Delta x_4 = 0{,}2\sqrt{\frac{0{,}0268}{y(0{,}683) - 2y(0{,}783) + y(0{,}883)}} = 0{,}2\sqrt{\frac{0{,}0268}{0{,}0460}} = 0{,}152,$$

$$x_4 = x_3 + \Delta x_4 = 0{,}835,$$

$$\Delta x_5 = 0{,}2\sqrt{\frac{0{,}0268}{y(0{,}835) - 2y(0{,}935) + y(1{,}035)}} = 0{,}2\sqrt{\frac{0{,}0268}{0{,}0560}} = 0{,}138,$$

$$x_5 = x_4 + \Delta x_5 = 0{,}973.$$

Since we slightly deviate from the correct value $x_5 = 1$, we amend the coordinates of break points by multiplying out by the coefficient $c = \frac{1}{0{,}973}$. Then the coordinates of break points are

$$x_1 = 0{,}306; \quad x_2 = 0{,}527; \quad x_3 = 0{,}702; \quad x_4 = 0{,}858; \quad x_5 = 1. \quad (28)$$

The maximal errors are in the particular segments as follows:

$$\varepsilon_1(x)_{max} = 0{,}0110, \quad \varepsilon_2(x)_{max} = 0{,}0153, \quad \varepsilon_3(x)_{max} = 0{,}0141;$$
$$\varepsilon_4(x)_{max} = 0{,}0142; \quad \varepsilon_5(x)_{max} = 0{,}0141. \quad (29)$$

We see that the distribution of break points by (26) and the size of the errors by (29) in good accuracy comply with the correct distribution of break points by (2b).

Approximating functions at which the curvature (the second derivative) less changes, we obtain somewhat better result as regards the resulting coordinate of the end point in choosing the division into an equal number of subintervals.

For instance, in approximating the function $f(x) = e^{-x}$ in the interval $x \in \langle 0; 1 \rangle$ where $N = 10$ and $k = 5$, the coordinates of break points and the maximal deviates are computed in the particular segments:

$$x_1 = 0{,}166; \quad x_2 = 0{,}342; \quad x_3 = 0{,}530; \quad x_4 = 0{,}752; \quad x_5 = 1{,}001 \doteq 1. \quad (30)$$
$$\varepsilon_1(x)_{max} = 0{,}0031, \quad \varepsilon_2(x)_{max} = 0{,}0030, \quad \varepsilon_3(x)_{max} = 0{,}0028,$$
$$\varepsilon_4(x)_{max} = 0{,}0032, \quad \varepsilon_5(x)_{max} = 0{,}0032.$$

On the basis of (2) the theoretical value of x_j and the error are

$$x_1 = 0{,}164, \quad x_2 = 0{,}342, \quad x_3 = 0{,}538, \quad x_4 = 0{,}756,$$
$$(x_0 = 0, \quad x_5 = 1), \quad \varepsilon(x)_{max} = 0{,}0031. \quad (30a)$$

In the above case the end point well complies with the end point $x = 1$ of the interval. In these functions there is no need to divide the interval into a great number of subintervals. (Cf. (25)).

This method is not absolutely accurate since we use approximate numerical methods. The accuracy may be increased by using a larger number of subintervals which, however, alongates the calculation. The advantage of this method is the fact that we obtain the result comparatively quickly without using a computer.

Valuation of these methods.

The given problem is described by a system of equations (2). Due to complexity of this system, the solution is understood to be done on a computer. The program for computing this problem is a comparatively laborious task.

The numerical method B1 in some cases and B2 do not require any computer. They are sufficently accurate for the practical determination of break points. The maximal deviation at $f(x) = x^3$ is in B1 0,0047, i.e. 36 % from theoretical maximal error (by (2b) and (9c)). Very good accuracy is at function $f(x) = \ln x$ (see (11e) and the next text.

In B2 is the maximal deviation 0,0027, i.e. 20 % (by (2b) and (29)). At the function $f(x) = e^{-x}$ is the maximal deviation of the theoretical error 0,0002, i.e. 6 % in B2 (by (30) and (30a)).

Using of ideal diods without considering nonlinearities at the initial segments of a $V-A$ characteristic is assumed in this calculation.

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SOUHRN

DIODOVÉ FUNKČNÍ TRANSFORMÁTORY S OPTIMÁLNÍM ROZLOŽENÍM BODŮ ZLOMŮ

KAREL BENEŠ

Práce se zabývá určením rozložení bodů zlomů podle požadavků nejlepší stejnoměrné aproximace při aproximaci funkce lomenou čarou. Je uveden přesný způsob výpočtu s použitím číslicového počítače (A) i přibližné metody založené na předpokladu $f''(x) \doteq$ konst (B1) a na numerickém vyjádření $f''(x)$ (B2). Při metodách B1 (v některých případech) a B2 není třeba číslicového počítače. Je provedeno srovnání výsledků u funkcí $f(x) = x^3$ a $f(x) = e^{-x}$. Lepší výsledky dává metoda B2. Předpokládá se použití ideálních diod.

РЕЗЮМЕ

ДИОДНЫЕ ФУНКЦИОНАЛЬНЫЕ ПРЕОБРАЗОВАТЕЛИ С ОПТИМАЛЬНЫМ РАСПРЕДЕЛЕНИЕМ ТОЧЕК ИЗЛОМА

КАРЕЛ БЕНЕШ

В статье описан способ вычисления распределения точек излома при аппроximации функции зломной кривой по требованию найлучшей аппроximации. Описан точный способ вычисления с использованием вычислительной машины (А) приблизительные методы основанне на предположении $\phi''(x) \doteq$ конст (В1), и на нумерическом высказынию $\phi''(x)$ (Б2). При методах Б1 и Б2 не надо вычислительной машины. Показано сравнение результатов у функций $\phi(x) = x^3$ и функций $\phi(x) = e^{-x}$. Лучшие результаты давает метод Б2.