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TERNARY RINGS WITH A LEFT QUASI-ZERO BELONGING TO TRANSLATION PLANES

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The coordinatization of a projective plane and expressing of "analytical" conditions to transitive operating of a homothetic group or of some its significant subgroups has become subject to a large number of mathematical studies recently. Most frequently they are devoted to investigating conditions under which the coordinatizing structure (commonly a planar ternary ring) belongs to a translation plane (i.e. to a projective plane being transitive with respect to a privileged "improper" line and to all its points). These conditions depend both on establishing the coordinate system and on the very coordinatizing structure (we have f.i. Hall and Hughes coordinatizations). The results of Hall [1], already classical nowadays, (the coordinatizing structure is given by a planar ternary ring with zero and unity – the projective plane is translation plane iff the ring considered is a quasi-field with respect to induced addition and multiplication) have been generalized to a case in which the coordinatizing structure is given by a planar ternary ring with zero – not necessary with unity – ([3], [6]) and to some cases of more general planar ternary rings, commonly non-isotopic, with Hall planar ternary rings [4], [5].

This article investigates a coordinatization of a projective plane by means of a planar ternary ring with left quasi-zero and with a system of relative right quasi-zeros. There are derived necessary and sufficient conditions for such a planar ternary ring to coordinatize a translation plane.

§ 1. Hall Coordinatization of a Projective Planer

Throughout by $P(V, n)$ we mean a chosen projective plane P with a privileged flag (V, n). The lines of $P(V, n)$ are understood to be the sets of points. The line n is called an improper line and the point $V \in n$ denotes a vertical direction. In other respects we use the terminology of affine geometry in its most commonly used sence (proper point, improper point, proper line, parallel lines etc.). Proper lines having a point V are termed *vertical lines*, other proper lines are called *skew lines*. We write $\mathbf{A}$ for a set of all proper points of $P(V, n)$ and $\tilde{\mathbf{A}}$ for a set of all skew lines of $P(V, n)$. $\mathbf{S}$ is written for a set for which card $\mathbf{S}$ = ord $P(V, n)$. Then also card $(\mathbf{S} \times \mathbf{S})$ = card $\mathbf{A}$ = card $\tilde{\mathbf{A}}$.

Every pair of bijections

$$(\ ) : \mathbf{S} \times \mathbf{S} \to \mathbf{A},^{1)}$$
$$[\ ] : \mathbf{S} \times \mathbf{S} \to \tilde{\mathbf{A}},$$

will be called *a coordinate system in* $P(V, n)$ if

(a) $\forall\, x, y, x', y' \in \mathbf{S}$ the points (x, y), (x', y') lie on the same vertical line $\Leftrightarrow x = x'$;

(b) $\forall\, a, b, a', b' \in \mathbf{S}$ the lines $[a, b]$, $[a', b']$ are parallel $\Leftrightarrow a = a'$.

As a consequence of (a)

$$\{(x, y) \in \mathbf{A} \mid x = x_0; \qquad x_0 \in \mathbf{S}\} \cup \{V\}$$

is a vertical line written as $[x_0]$. Likewise, as a consequence of (b)

$$\{[a, b] \in \tilde{\mathbf{A}} \mid a = a_0;\ a_0 \in \mathbf{S}\} \cup \{n\}$$

is a pencil of lines with an improper centre. This centre will be written as (a_0).

It is well-known, if the ternary operation $\mathbf{t} : \mathbf{S}^3 \to \mathbf{S}$ is introduced on $\mathbf{S}$ by means of

$$y = \mathbf{t}(x, a, b) \Leftrightarrow (x, y) \in [a, b],$$

then the structure $(\mathbf{S}, \mathbf{t})$ satisfies the axioms below:

A1: $\forall\, a, b, c \in \mathbf{S}\ \exists!\ x \in \mathbf{S} : \mathbf{t}(a, b, x) = c$,

A2: $\forall\, a, b, c, d \in \mathbf{S},\ a \neq c\ \exists!\ x \in \mathbf{S}$: $\mathbf{t}(x, a, b) = \mathbf{t}(x, c, d)$,

A3: $\forall\, a, b, c, d \in \mathbf{S},\ a \neq b\ \exists!\ (x, y) \in \mathbf{S} \times \mathbf{S}$: $\mathbf{t}(a, x, y) = c$, $\mathbf{t}(b, x, y) = d$.[2])

The structure $(\mathbf{S}, \mathbf{t})$ is thus a planar ternary ring (in short **PTR**) (see f.i. [2]). We say, the structure $(\mathbf{S}, \mathbf{t})$ is *a* **PTR** *belonging to a chosen coordinate system.*

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[1]) For every $x, y \in$ S will (x, y) mean a proper point of $P(V, n)$, and also an ordered pair of elements of S and thus also an element of the Cartezian product $\mathbf{S} \times \mathbf{S}$. It becomes clear from the context, however, which of the above three objects is actually concerned.

[2]) The uniqueness of the ordered pair (x, y) is just a consequence of A2.

This article investigates a coordinate system in $\boldsymbol{P}(\boldsymbol{V}, \boldsymbol{n})$ having besides (a), (b) another two properties:

(c) $\exists\, 0 \in \mathbf{S}\ \forall\, a, b, c, d \in \mathbf{S}$ so that the lines $[a, b]$, $[c, d]$ have a point in common on the vertical line $[0] \Leftrightarrow b = d$.

The line $[0]$ is called *a vertical axis* which we also denote by $\boldsymbol{v}$;

(d) $\forall\, a \in \mathbf{S},\ a \neq 0\ \exists\, n_a \in \mathbf{S}\ \forall\, y, y' \in \mathbf{S}$ such that the points $(0, y)$, (a, y') lie on the same line of direction $(n_a) \Leftrightarrow y = y'$. The element $n_a \in \mathbf{S}$ is called *a right a-quasi-zero.*

Now we want to find out what properties of the coordinatizing **PTR(S, t)** correspond to ours given above under (c), (d). Property (c): Let us choose an arbitrary $a \in \mathbf{S}$ and define a permutation

$$*_a : \mathbf{S} \to \mathbf{S}, \qquad *_a : b \mapsto b^{*a}$$

by the condition

$$\forall\, b \in \mathbf{S} : (0, b) \in [a, b^{*a}].$$

If u is another arbitrary element of $\mathbf{S}$, then, of course,

$$\forall\, b \in \mathbf{S} : (0, b) \in [u, b^{*u}].$$

In consequence of (c) is $b^{*u} = b^{*a}$ for every b, so that

$$*_a = *_u .$$

Putting for an arbitrary $a \in \mathbf{S}$

$$* = *_a ,$$

then $*$ is independent of the element a which implies that $(\mathbf{S}, \mathbf{t})$ satisfies

A4: $\exists\, 0 \in \mathbf{S}$ and permutation $* : \mathbf{S} \to \mathbf{S}(* : b \to b^*)$ such that $\forall\, u, b \in \mathbf{S}$: $\mathrm{t}(0, u, b^*) = b$.

Conversely, if $(\mathbf{S}, \mathbf{t})$ satisfies A4 and if $[a, b^*]$, $[c, d^*]$ are skew lines, then $\mathrm{t}(0, a, b^*) = b$, $\mathrm{t}(0, c, d^*) = d$. The above lines have thus one point in common on the vertical axis $[0]$ iff $b = d \Leftrightarrow b^* = d^*$. Hence we see that our coordinate system has the property (c).

Let us now make the convention to denote by $\times$ the inverse permutation to $*$ relative to A4. Hence for every $u, b \in \mathbf{S}$

$$\mathrm{t}(0, u, b) = b^{\times}.$$

Property (d): Let $a \in \mathbf{S}$, $a \neq 0$ and let b be an arbitrary element of $\mathbf{S}$. Then the points $(0, b)$, (a, b) lie on the same line of direction (n_a); however it is the line $[n_a, b^*]$ so that $b = \mathrm{t}(a, n_a, b^*)$. From this we see the validity of

A5: $\forall\, a \in \mathbf{S},\ a \neq 0,\ \exists\, n_a \in \mathrm{S},\ \forall\, b \in \mathbf{S},\ b = \mathbf{t}(a, n_a, b^*)$.

Conversely, let A5 and A4 hold and let $y, y' \in \mathbf{S}$. One and only one line of direction n_a, namely the line $[n_a, y^*]$, passes through the point $(0, y)$. The point (a, y') then lies on $[n_a, y^*]$ iff $y' = \mathbf{t}(a, n_a, y^*) = y$. Hence it holds:

1.1. Theorem: *The coordinate system in a projective plane $P(V, n)$ has besides the properties (a), (b) also:*

(A) *the property (c) when and only when the* **PTR** *belonging to it satisfies the axioms A4, and*

(B) *the properties (c) and (d) when and only when the* **PTR** *belonging to it satisfies the axioms A4, A5.*

§ 2. *Planar ternary Rings with a Left Quasi-zero.*

2.1. Definition: The planar ternary ring $(\mathbf{S}, \mathbf{t})$ satisfying the A4 stated in § 1. is called *the planar ternary ring with a left quasi-zero* (in short **L-PTR**). The element 0 from A4 is called *the left quasizero of* **L-PTR(S, t)**.

2.2. Remark: If the **PTR(S, t)** has a left zero i.e. an element 0 such that for all $u, b \in \mathbf{S}$ $\mathbf{t}(0, u, b) = b$, then 0 is a left quasi-zero for $* = 1_{\mathbf{S}}$.

Every **PTR(S, t)** coordinatizes a projective plane $P(V, n)$, say its canonical plane [2]. This property will be utilized in some proofs.

2.3. Proposition: *Let* $(\mathbf{S}, \mathbf{t})$ *be a* **L-PTR**. *Then*

(a) $(\mathbf{S}, \mathbf{t})$ *has exactly one left quasi-zero,*

(b) $\forall\, a, b, c \in \mathbf{S},\ a \neq 0,\ \exists!\, x \in \mathbf{S} : t(a, x, b) = c.$

Proof: (a) Be $a, b, c \in \mathbf{S}$, $a \neq b$. The element $z \in \mathbf{S}$ is a left quasi-zero iff the vertical line $[z]$ passes through the point of intersection of lines $[a, c]$, $[b, c]$; such a vertical line is naturally exactly one.

(b) Let $a, b, c \in \mathbf{S}$, $a \neq 0$, then $\mathbf{t}(a, x, b) = c \Leftrightarrow (a, c) \in [x, b]$. Since for an arbitrary $x \in \mathbf{S}$ the line $[x, b]$ passes through the point $(0, b^*)$, so the line $[x, b]$ and thus also the element x are uniquely determined by the condition $(a, c) \in$ $\in [x, b]$.

Let $a \in \mathbf{S}$, $a \neq 0$. According to 2.3. (b) there exists exactly one $e_a \in \mathbf{S}$ such that $\mathbf{t}(a, e_a, 0^*) = 0$. Let us set $e_a = 0$ for $a = 0$. Then the ternary operation induces two binary operations called addition (+) and multiplication (.) defined by means of the functions

$$\begin{aligned} a + b &= \mathbf{t}(a, e_a, b^*), \\ a\ .\ b &= \mathbf{t}(a, b, 0^*) \end{aligned} \tag{1}$$

respectively.

In place of $a\,.\,b$ we generally write only ab.

2.4. Proposition: *The addition and multiplication operations in the* **L-PTR(S, t)** *have the following properties:*

(α) $\forall a \in \mathbf{S} : a + 0 = 0 + a = a$,

(β) $\forall a, b \in \mathbf{S}\ \exists!\ x \in \mathbf{S} : a + x = b$ and thus $\forall a, x, y \in \mathbf{S} : a + x = a + y \Rightarrow \Rightarrow x = y$,

(γ) $\forall a \in \mathbf{S} : 0 \,.\, a = 0$,

(δ) $\forall a, b \in \mathbf{S},\ a \neq 0\ \exists!\ x \in \mathbf{S} : ax = b$, thus $\forall a, x, y \in \mathbf{S},\ a \neq 0 : ax = ay \Rightarrow \Rightarrow x = y$,

(ε) $\forall a \in \mathbf{S} : ae_{\mathrm{a}} = a$.

The proof is trivial.

2.5. Definition: The $\mathbf{L}$-$\mathbf{PTR}(\mathbf{S}, \mathbf{t})$ is said *to be linear* (or *to satisfy the linearity property*) if $\forall a, b, c \in \mathbf{S} : \mathbf{t}(a, b, c^*) = ab + c$.

§ 3. The **Lr**-*planar ternary Rings.*

3.1. Definition: The $\mathbf{L}$-planar ternary ring $(\mathbf{S}, \mathbf{t})$ satisfying also A5 is called *the* $\mathbf{Lr}$-*planar ternary ring* (in short $\mathbf{Lr}$-$\mathbf{PTR}$). The element n_{a} of A5 is called *the right* (relative) *a-quasi-zero*.

3.2. Proposition: *Let* $(\mathbf{S}, \mathbf{t})$ *be a* $\mathbf{Lr}$-$\mathbf{PTR}$. *Then for every* $a \in \mathbf{S}$, $a \neq 0$ *there exists exactly one right a-quasi-zero. For every element* $a \in \mathbf{S}$, $a \neq 0$ *holds* $an_{\mathrm{a}} = 0$.

Proof: Let $(\mathbf{S}, \mathbf{t})$ belong to a certain coordinate system in a projective plane $\boldsymbol{P}(V, \boldsymbol{n})$. Let $a \in \mathbf{S}$, $a \neq 0$; for an arbitrary $n \in \mathbf{S}$ the line $[n, 0^*]$ passes through the point $(0, 0) \in [0]$. Then n is a right a-quasi-zero iff $(a, 0) \in [n, 0^*]$. However the line $[n, 0^*]$ and thus also the element n are uniquely determined by this condition.

3.3. Remark: If the $\mathbf{PTR}(\mathbf{S}, \mathbf{t})$ has besides the left zero 0_{L} also a right zero 0_{R} (i.e. $\forall u, b \in \mathbf{S}$ holds $\mathbf{t}(u, 0_{\mathrm{R}}, b) = b$), then is $(\mathbf{S}, \mathbf{t})$ a $\mathbf{LR} - \mathbf{PTR}$. Indeed, according to Remark 2.2., $(\mathbf{S}, \mathbf{t})$ is a $\mathbf{L} - \mathbf{PTR}$ with $* = 1_{\mathrm{S}}$; putting $n_{\mathrm{a}} = 0_{\mathrm{R}}$ for every $a \in \mathbf{S}$, $a \neq 0$, then $\mathbf{t}(a, n_{\mathrm{a}}, n^*) = \mathbf{t}(a, 0_{\mathrm{R}}, b) = b$.

3.4. Proposition: *If the* $\mathbf{L} - \mathbf{PTR}(\mathbf{S}, \mathbf{t})$ *is linear, then it is* $\mathbf{Lr} - \mathbf{PTR}$.

Proof: Let $a \in \mathbf{S}$, $a \neq 0$. Let us determine $n_{\mathrm{a}} \in \mathbf{S}$ so that $\mathbf{t}(a, n_{\mathrm{a}}, 0^*) = 0 \Rightarrow \Rightarrow an_{\mathrm{a}} = 0$. Let $b \in \mathbf{S}$, then $\mathbf{t}(a, n_{\mathrm{a}}, b^*) = an_{\mathrm{a}} + b = 0 + b = b$.

§ 4. Vertical transitive Planes.

Throughout the next two sections there are assumed a chosen firm projective plane $\boldsymbol{P}(V, \boldsymbol{n})$, a coordinate system with properties (a)–(d) in it, and a **PTR** belonging to it, which must be a **Lr** – **PTR**. From here on this **PTR** will be referred to as $(\mathbf{S}, \mathbf{t})$.

For every direction $\boldsymbol{U} \in \boldsymbol{n}$ let T_U stand for both a set and a group of all translation of $\boldsymbol{P}(V, \boldsymbol{n})$. Here the group operation is represented by the composition of translations.

For the purpose of simplifying our notation, we will understand under the term *affine line* p a set of proper points of p for every proper line p.

4.1. Definition: Let $\boldsymbol{U}$ be a direction in the projective plane $\boldsymbol{P}(V, \boldsymbol{n})$. The $\boldsymbol{P}(V, \boldsymbol{n}$ is called $(\boldsymbol{U}, \boldsymbol{n})$ *transitive* if the group $\mathrm{T}_{\boldsymbol{U}}$ operates transitively on every affine line of the direction $\boldsymbol{U}$. A $(V, \boldsymbol{n})$-transitive plane is said to be *vertically transitive*.

4.2. Remark: If p is a affine line of the direction $\boldsymbol{U}$ and Q is its arbitrary point, then the $\boldsymbol{P}(V, \boldsymbol{n})$ is $(\boldsymbol{U}, \boldsymbol{n})$-transitive exactly if there exists a translation $\tau : Q \mapsto A$ (it is obvious) for every point $A \in p$.

In what follows we will identify the translations of the plane $\boldsymbol{P}(V, \boldsymbol{n})$ with their restrictions on the set $\mathbf{A}$ of all proper points relative to the $\boldsymbol{P}(V, \boldsymbol{n})$.

4.3. Lemma: *Let the $\boldsymbol{P}(V, \boldsymbol{n})$ be vertically transitive. Then* $\forall\, a, b \in \mathbf{S},\ a \neq b,$ $\exists\, n_{ab} \in \mathbf{S}\ \forall\, v \in \mathbf{S}\ \mathbf{t}(a, n_{ab}, v) = \mathbf{t}(b, n_{ab}, v)$.

Proof: The Lemma is evident if either $a = 0$, or $b = 0$. Let us assume that a, b are different from 0. Let $[n_{ab}, u^*]$ be a line passing through the points $\mathrm{A} = (a, 0)$, $\mathrm{B} = (b, 0)$, thus $[n_{ab}, u^*] = \mathrm{AB}$. Putting $\mathrm{Q} = (0, 0)$, then $\mathrm{QA} = [n_a, 0^*]$, $\mathrm{QB} = [n_b, 0^*]$. Let us choose an arbitrary $v \in \mathbf{S}$ and investigate the translation $\tau : (0, u) \to (0, v^x)$. According to our assumption, such a translation exists. Let $\tau(\mathrm{Q}) = (0, q)$, $\tau(\mathrm{A}) = (a, a')$, $\tau(\mathrm{B}) = (b, b')$. Then $\tau(\mathrm{Q})\,\tau(\mathrm{A}) \parallel \mathrm{QA}$ and since $\tau(\mathrm{Q})$ is a point on a vertical axis, we get $\tau(\mathrm{Q})\,\tau(\mathrm{A}) = [n_a, q^*]$ and likewise $\tau(\mathrm{Q})\,\tau(\mathrm{B}) = [n_b, q^*]$. Since $(0, u) \in \mathrm{AB}$, then $(0, v^*) \in \tau(\mathrm{A})\,\tau(\mathrm{B})$ whereby $\tau(\mathrm{A})\,\tau(\mathrm{B}) \parallel \mathrm{AB} \Rightarrow \tau(\mathrm{A})\,\tau(\mathrm{B}) = [n_{ab}, v]$. $\tau(\mathrm{A}) \in [n_a, q^*]\ \mathbf{t}\ a' = \mathbf{t}(a, n_a, q^*) = q$; $\tau(\mathrm{B}) \in [n_b, q^*] \Rightarrow b' = \mathbf{t}(b, n_b, q^*) = q$ therefore $a' = b'$.

Since $\tau(\mathrm{A})\,\tau(\mathrm{B}) = [n_{ab}, v]$, it holds $\mathbf{t}(a, n_{ab}, v) = a' = b' = \mathbf{t}(b, n_{ab}, v)$.

4.4. Proposition: *The following conditions are equivalent:*

(a) *the $\boldsymbol{P}(V, \boldsymbol{n})$ is vertically transitive*

(b) $\forall\, b \in \mathbf{S}$ *is the mapping* $\varphi_b : \mathbf{A} \mapsto \mathbf{A}$, $\varphi_b : (x, y) \mapsto (x, y + b)$ *a translation. In this case is* $(\mathbf{S}, +)$ *antiisomorphic with the group* T_V. *Hence it is a group as well.*

Proof: 1. (a) $\Rightarrow$ (b).

Let $b \in \mathbf{S}$ and τ_b be a vertical translation in which $\tau_b : (0, 0) \mapsto (0, b)$. We now prove that $\tau_b = \varphi_b$. We choose $(x, y) \in \mathbf{A}$ and assume first $x = y$. If $y = 0$, then $\tau_b(0, 0) = (0, b) = (0, 0 + b) = \varphi_b(0, 0)$. If $y \neq 0$, then the points $(0, 0)$, (y, y) are lying on the line $[e_y, 0^*]$ and their images $\tau_b(0, 0)$, $\tau_b(y, y)$ then on the line $[e_y, b^*]$. If $\tau_b(y, y) = (y, y')$, then $y' = \mathbf{t}(y, e_y, b^*) = y + b$.

Let $x \neq y$. According to 4.3. there exists an element $n_{xy} \in \mathbf{S}$ such that $\forall v \in \mathbf{S}$ we have $\mathbf{t}(x, n_{xy}, v) = \mathbf{t}(y, n_{xy}, v)$. Hence the points (x, y) and (y, y) lie on the line p of the direction (n_{xy}), consequently $p = [n_{xy}, v^*]$, $v \in \mathbf{S}$. Their images $\tau_b(x, y)$, $\tau_b(y, y)$ lie on the line $p' \parallel p$ and therefore $p' = [n_{xy}, w^*]$. However $\tau_b(y, y) = (y, y + b)$, thus

$$y + b = \mathbf{t}(y, n_{xy}, u^*). \tag{2}$$

Let $\tau_b(x, y) = (x, y')$. Then $y' = \mathbf{t}(x, n_{xy}, w^*) = \mathbf{t}(y, n_{xy}, w^*) = y + b$. Hence for every $(x, y) \in \mathbf{A}$ holds $\varphi_b(x, y) = \tau_b(x, y)$.

2. (b) $\Rightarrow$ (a).

Let $A = (0, b)$, $Q = (0, 0)$. Then $Q, A \in \boldsymbol{v}$ (a vertical axis). The translation φ_b has a vertical direction and evidently $\varphi_b : Q \mapsto A$. Let $\varrho : \mathbf{S} \to \mathbf{T}_V$ be a mapping where $\varrho(b) = \varphi_b$ for every $b \in \mathbf{S}$. If for both elements $b, b' \in \mathbf{S}$ $\varrho(b) = \varrho(b')$ or $\varphi_b = \varphi_{b'}$, then for every $(x, y) \in \mathbf{A}$

$$(x, y + b) = (x, y + b') \tag{3}$$

Putting $y = 0$ in (3), we obtain $b = b' \Rightarrow \varrho$ is an injection, evidently surjective.

Let $a, b \in \mathbf{S}$. Then

$$\varphi_{a+b}(0, 0) = (0, a + b) = \varphi_b(0, a) = \varphi_b(\varphi_a(0, 0)) = (\varphi_b \circ \varphi_a)(0, 0) \Rightarrow$$
$$\Rightarrow \varphi_{a+b} = \varphi_b \circ \varphi_a \text{ or } \varrho(a + b) = \varrho(b) \circ \varrho(a).$$

4.5. Remark: If $(\mathbf{S}, +)$ is a group, then naturally, its neutral element is its left quasi-zero 0. For every $a \in \mathbf{S}$; let $-a$ stand for the opposite element to a. Thus $a + (-a) = (-a) + a = 0$. If b is the next element of the set $\mathbf{S}$, we shall also write $b - a$ in place of $b + (-a)$.

4.6. Theorem: *The projective plane $P(V, n)$ is vertically transitive exactly if $(\mathbf{S}, \mathbf{t})$ possesses the following properties:*

(A) $(\mathbf{S}, +)$ *is a group,*

(B) $(\mathbf{S}, \mathbf{t})$ *is linear.*

Proof: I. Let the $P(V, n)$ be vertically transitive. According to 4.4. there holds (A). Let $a, b, c \in \mathbf{S}$. Then the point $A = (a, ab)$ lies on the line $p = [b, 0^*]$. Let τ_c be a translation with $\tau_c : (0, 0) \mapsto (0, {}_c)$. According to 4.4. there holds $\tau_c = \varphi_c$ such that $\tau_c(A) = (a, ab + c)$. Further $\tau_c(p) = [b, c^*]$ and since $\tau_c(A) \in \tau_c(p)$ we get $ab + c = \mathbf{t}(a, b, c^*)$, which implies that (B) is true.

II. Conversely, let us assume that (A) and (B) hold. According to 4.4. it suffices to prove that for every $b \in \mathbf{S}$ the mapping

$$\varphi_b : (x, y) \mapsto (x, y + b) \tag{4}$$

is a translation of a vertical direction. With respect to (A) we see that $\varphi_b : \mathbf{A} \to \mathbf{A}$ is a bijection reproducing every affine vertical line. Let us consider a skew line $p = [b, c^*]$. Then $(x, y) \in p \Leftrightarrow y = \mathbf{t}(x, b, c^*) \Leftrightarrow y = xb + c \Leftrightarrow y + b = (xb + c) + b \Leftrightarrow y + b = xb + (c + b) \Leftrightarrow (x, y + b) \in [b, (c + b)^*] \Leftrightarrow \varphi_b(x, y) \in [b, (c + b)^*]$. Thus if we put $p' = [b, (c + b)^*]$, then on the one hand $p \parallel p'$, on the other hand $b' = \varphi_b(p)$, such that φ_b is a translation of a vertical direction.

§ 5. *Translation Planes.*

As is well-known, the projective plane $P(V, n)$ is called translation plane if the group $\mathbf{T}$ of all translations operates transitively on the set $\mathbf{A}$ of all proper points relative to the plane $P(V, n)$. Obviously, the plane $P(V, n)$ is translation one if $\forall (b, b') \in \mathbf{A}$ there exists a translation $\tau : (0, 0) \mapsto (b, b')$.

5.1. Proposition: *Let the $P(V, n)$ be vertically transitive and for every $b \in \mathbf{S}$ let the translation $\sigma_b : (0, 0) \mapsto (b, 0)$ exist. Then the $P(V, n)$ is translation plane.*

Proof: Let $(b, b') \in \mathbf{A}$; $\sigma_b, \tau_{b'}$ be translations with $\sigma_b : (0, 0) \mapsto (b, 0)$, $\tau_{b'} : (b, 0) \mapsto (b, b')$. Both these translations exist. Then $\tau_{b'} \circ \sigma_b : (0, 0) \mapsto (b, b')$.

5.2. Proposition: *Let the $P(V, n)$ be vertically transitive and let there exist a translation $\sigma_b : (0, 0) \mapsto (b, 0)$. Let $h = [n_b, 0^*]$, thus $(0, 0) \in h$, $(b, 0) \in h$. If $(x, y) \in h$ and $\sigma_b(x, y) = (x', y')$, then $\forall m \in \mathbf{S}$:*

$$x'm - bm - xm = x'n_b - xn_b \text{ [3]} \tag{5}$$

Proof: Clearly $(x', y') \in h$. The equality (5) holds for $m = n_b$ (3.2. Proposition). Let $m \neq n_b$. For the determination of the point (x', y') let us direct a vertical line $[x]$ through the point (x, y) and a skew line $[m, 0^*]$ through the point $(0, 0)$. The point (x, xm) is the point of intersection for both above lines. A parallel line h' to the line h be directed through this point, where $h' = [n_b, q^*]$, $q \in \mathbf{S}$ and $xm = xn_b + q$ i.e.

$$q = -(xn_b) + xm. \tag{6}$$

The point $\sigma_b(x, xm)$ lies on the line h'; if $p = [m, 0^*]$ and p' is a parallel line to p passing through the point $(b, 0)$; then the point $\sigma_b(x, xm)$ lies on the line p'.

We write $-bm$ instead of $-(bm)$.

Simultaneously, the points (x', y') and $\sigma_b(x, xm)$ lie on the same vertical line. Hence $\sigma_b(x, xm) = (x', y'')$ where it holds for y''

$$y'' = x'n_b + (xn_b) + xm \tag{7}$$

on the one hand and

$$y'' = x'm + v \tag{8}$$

on the other hand with $[m, 0^*] = p'$. Since $(b, 0) \in p'$ we can evaluate v from the relation $0 = bm + v$ or

$$v = -bm. \tag{9}$$

After substituing (9) in (8) and comparing with (7) we get (5).

Now let us write for every three elements $a, b, c \in \mathbf{S}$

$$\mathbf{Q}(a, b, c) = \{m \in \mathbf{S} \mid cm - bm - am = cn_b - an_b\}.$$

Let $h = [n_b, 0^*]$. If there exists a translation $\sigma : (0, 0) \mapsto (b, 0), (a, an_b) \mapsto (c, cn_b)$, then according to 5.2. we have $\mathbf{Q}(a, b, c) = \mathbf{S}$.

5.3. Proposition: *Let the $P(V, n)$ be vertically transitive and let for every three elements $a, b, c \in \mathbf{S}$ hold*

$$\mathbf{Q}(a, b, c) = \{n_b\} \vee \mathbf{Q}(a, b, c) = \mathbf{S}. \tag{10}$$

Then there exists a translation $\sigma_b : (0, 0) \mapsto (b, 0)$.

Proof: Let $b \in \mathbf{S}$. Let us choose an arbitrary $m \in \mathbf{S}$, $m \neq n_b$ and define a mapping

$$\psi : \mathbf{S} \to \mathbf{S}, \qquad \psi : x \mapsto x'$$

via the condition

$$x' = \psi(x) \Leftrightarrow x'm - bm - xm = x'n_b - xn_b. \tag{11}$$

As a consequence of $m \neq n_b$, the element x' is uniquely determined through the element x of (11). Rewriting (11) in the form

$$x' = \psi(x) \Leftrightarrow xm + bm - x'm = xn_b - x'n_b \tag{12}$$

yields an uniquely determination of x by x' which implies that ψ is a bijection and evidently $\psi(0) = b$. Let us make a mapping $\sigma_b : \mathbf{A} \to \mathbf{A}$, $\sigma_b : (x, y) \mapsto (\psi(x), y')$, with the points (x, y), $(\psi(x), y')$ lying on the line of the direction (n_b). It is immediate now that

I. the definition of the mapping σ_b is correct,

II. σ_b is a bijection transforming every vertical line in a vertical line again

III. σ_b reproduces every line of direction (n_b),

IV. $\sigma_b(0, 0) = (b, 0)$.

Let us now prove that σ_b is a translation. Be $[u, v^*]$ an arbitrary skew line, (x, y) be its point, $(x', y') = \sigma_b(x, y)$. Then

$$m \in \mathbf{Q}(x', b, x)$$

such that also $u \in \mathbf{Q}(x', b, x)$ as assumed, and therefore

$$x'u - bu - xu = x'n_b - xn_b. \tag{13}$$

There exists an element $q \in \mathbf{S}$ such that $(x, y) \in [n_b, q^*]$, $(x', y') \in [n_b, q^*]$ and

$$y = xn_b + q \tag{14}$$

$$y' = x'n_b + q \tag{15}$$

and finally $(x, y) \in [u, v^*] \Rightarrow$

$$y = xu + v. \tag{16}$$

Substitution of (14) in (15) gives $y' = x'n_b - xn_b + y$ and applying (16) and (13) we can write $y' = x'n_b - xn_b + xu + v = x'u - bu - xu + xu + v = x'u - bu + v$ hence $(x', y') \in [u, (-bu + v)^*]$ which is a line parallel to $[u, v^*]$.

5.4. Theorem: *The projective plane $P(V, n)$ is translation plane if and only if the* **Lr**-*planar ternary ring* $(\mathbf{S}, \mathbf{t})$ *possesses the following properties:*

(A) $(\mathbf{S}, +)$ *is a group*

(B) $(\mathbf{S}, \mathbf{t})$ *is linear*

(C) $\forall\, a, b, c \in \mathbf{S} : \mathbf{Q}(a, b, c) = \{n_b\} \vee \mathbf{Q}(a, b, c) = \mathbf{S}$.

Proof: I. If the $P(V, n)$ is translation one, then it is vertically transitive, too. We see that according to 4.6. Theorem the $(\mathbf{S}, \mathbf{t})$ possesses the properties (A), (B). Let there exists an $\overline{m} \in \mathbf{Q}(a, b, c)$ such that $\overline{m} \neq n_b$. We set $p = [n_b, 0^*]$ and let be $m \in \mathbf{S}$. Then there exists a translation $\tau : (0, 0) \to (b, 0)$. Next let there be $B \in p$, $B = (a, q)$, $B' = \tau(B) \Rightarrow B' \in p$; $B' = (c', q')$. According to 5.2. Proposition we have

$$c'\overline{m} - b\overline{m} - a\overline{m} = c'n_b - an_b, \tag{17}$$

and since $\overline{m} \in \mathbf{Q}(a, b, c)$ it holds also

$$c\overline{m} - b\overline{m} - a\overline{m} = cn_b - an_b, \tag{18}$$

and $c = c'$. For every $m \in \mathbf{S}$, with respect to 5.2. Proposition, we have

$$cm - bm - am = cn_b - an_b,$$

with $\mathbf{Q}(a, b, c) = \mathbf{S}$.

II. Let the $(\mathbf{S}, \mathbf{t})$ have the properties (A), (B), (C). According to 4.6. Theorem the $P(V, n)$ is vertically transitive. According to 5.3. Proposition there exists a translation $\sigma_b : (0, 0) \mapsto (b, 0)$ for an arbitrary $b \in \mathbf{S}$. By 5.1. Proposition the $P(V, n)$ is a translation plane.

5.5. Remark: If the $(\mathbf{S}, \mathbf{t})$ possesses both a left and a right zero written as 0_L and 0_R respectively, then there is for every $a \in \mathbf{S}$ $n_b = 0_R$ such that

$$\forall\, a, b, c \in \mathbf{S} \qquad \mathbf{Q}(a, b, c) = \{m \in \mathbf{S} \mid am + bm = cm\}.$$

Consequently, the $P(V, n)$ in this case is a translation plane exactly if for arbitrary $a, b, c \in \mathbf{S}$ $am + bm = cm$ either only for $m = n_b$ or for all $m \in \mathbf{S}$ holds ([6], [7]).

§ 6. *Example.*

Let $\mathbf{E}_2$ be a Euclidean plane, $\boldsymbol{P}$ be its projective extension, $(\boldsymbol{V}, \boldsymbol{n})$ a flag in $\boldsymbol{P}$ such that $\boldsymbol{n}$ is an improper line. Let us choose a Cartesian coordinate system with an origin 0 and orthonormal vectors $\mathbf{e}_1, \mathbf{e}_2$ such that $V = [\mathbf{e}_2]$ Let us choose a function $f: \mathbf{R} \to \mathbf{R}$ such that $f(0) = 0$ and introduce the coordinate system in $\boldsymbol{P}(V, \boldsymbol{n})$ as follows:

$$(x, y) = 0 + x\mathbf{e}_1 + (y + f(x))\mathbf{e}_2.$$

$[u, v]$ is a line possessing the equation $y = ux + v$ in the Cartesian system. Naturally, our coordinate system possesses the properties (a), (b), (c) as stated in § 1. The vertical axis presents a line given by equation $x = 0$. Let $a \in \mathbf{R}$, $a \neq 0$ and consider the points $(0, b) = 0 + b\mathbf{e}_2$ and $(a, b') = 0 + a\mathbf{e}_1 + (b' + f(a))\,\mathbf{e}_2$. For the line $p = [u, v]$ with the points $(0, b)$ and (a, b') holds $u = [b' + f(a) - b]/a$. Herefrom $b = b' \Leftrightarrow u = f(a)/a$. Thus, our coordinate system possesses even the property (d) for $n_a = f(a)/a$, $a \neq 0$. The corresponding $\mathbf{PTR}(\mathbf{S}, \mathbf{t})$ is thus a $\mathbf{Lr} - \mathbf{PTR}$; for every $x, u, v \in \mathbf{R}$ we have

$$\mathbf{t}(x, u, v) = xu + v - f(x).$$

The left quasi zero is the number 0 being at the same time also a left zero.

Since n_a depens on a, the $(\mathbf{S}, \mathbf{t})$ possesses no right zero. The plane $\boldsymbol{P}(V, \boldsymbol{n})$ is a Pappian plane and the more so translation one $\Rightarrow (\mathbf{S}, \mathbf{t})$ possesses the properties (A), (B) and (C).

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Souhrn

TERNÁRNÍ OKRUHY S LEVOU KVAZINULOU PŘÍSLUŠNÉ TRANSLAČNÍM ROVINÁM

DALIBOR KLUCKÝ, LIBUŠE MARKOVÁ

V článku se nejprve vyšetřuje soustava souřadnic v projektivní rovině $\boldsymbol{P}$ s význačnou vlajkou $(V, \boldsymbol{n})$ taková, že:

(1) každé dva vlastní body (tj. body neležící na $\boldsymbol{n}$) mají touž první souřadnici právě když leží na téže přímce směru V;

(2) každé dvě šikmé přímky (tj. přímky neobsahující V) mají touž první souřadnici právě když mají společný bod na $\boldsymbol{n}$;

(3) existuje přímka $\boldsymbol{v}$, $V \in \boldsymbol{v}$, $\boldsymbol{v} \neq \boldsymbol{n}$ té vlastnosti, že libovolné dvě šikmé přímky mají společný bod na $\boldsymbol{v}$ právě když mají touž druhou souřadnici;

(4) pro každou přímku a, $V \in a$, $a \neq \boldsymbol{n}$, $\boldsymbol{v}$ existuje bod (směr) $N_a \in \boldsymbol{n}$ takový, že spojnice libovolných dvou bodů přímek $\boldsymbol{v}$, a má směr N_a právě když tyto body mají touž druhou souřadnici.

Planární ternární okruh $(\mathbf{S}, \mathbf{t})$ příslušný k takové soustavě souřadnic se nazývá **Lr**-planárním ternárním okruhem. V článku jsou dále odvozeny nutné a postačující podmínky pro **Lr**- planární ternární okruh $(\boldsymbol{S}, \mathbf{t})$, aby rovina $\boldsymbol{P}$ byla translační (tj. X, $\boldsymbol{n}$)-transitivní pro všechna $X \in \boldsymbol{n}$).

Резюме

ТЕРНАРЫ С ЛЕВЫМ КВАЗИНУЛЕМ ПРИНАДЛЕЖАЩИЕ К ПРОЕКТИВНЫМ ПЛОСКОСТЯМ ТРАНЗИТИВНЫМ ПО ОТНОШЕНИЮ К ОДНОЙ ПРЯМОЙ

ДАЛИБОР КЛУЦКИЙ, ЛИБУШЕ МАРКОВА

В работе сначала изучается координатная система в проективной плоскости $\boldsymbol{P}$ с значительным флагом $(V, \boldsymbol{n})$ такая, что

(1) произвольные неособые точки (это значит точки нележащие на V) обладают одинаковой первой координатой тогда и только тогда они лежат на той же самой прямой направления V;

(2) произвольные косые прямые (это значит прямые несодержащие V) обладают той же самой первой координатой тогда и только тогда, если бы хотя одна их общая точка лежала на $\boldsymbol{n}$;

(3) существует прямая $\mathbf{v}$, $V \in \mathbf{v}$, $\mathbf{v} \neq \boldsymbol{n}$ обладающая следующим свойством: две косые прямые имеют общую точку на $\mathbf{v}$ тогда и только тогда если они имеют одинаковую вторую координату;

(4) для произвольной прямой а, $V \in$ а, а $\neq \boldsymbol{n}$, а $\neq \mathbf{v}$ существует точка (направление) $\boldsymbol{N}_a \in \boldsymbol{n}$ такая, что прямая соединяющая любую пару точек прямых $\boldsymbol{v}$ и а направлена в $\boldsymbol{N}_a$ тогда и только тогда эти точки имеют общую вторую координату.

Тернар (S, -) принадлежащий к такой координатной системе называется Lr-тернар. В статье выведены необходимые и достаточные условия для Lr-тернара (S, -), чтобы плоскость $\boldsymbol{P}$ являлась транзитивной плоскостью в отношению к любой неособой точке.