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ON ONE DEFINITION OF GRAMMATICAL CATEGORIES

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Introduction

The concept of grammatical categories introduced in traditional linquistics by means of formal and semantic features has a great number of defects. There are many exceptions in the system of these categories, they are not suitable for machine processing e.t.c. That is why the analytical model of language was introduced in algebraic linquistics; one of the main tasks of these models is to provide a formal definition of grammatical categories.

In this paper we are giving a method of generating "large" grammatical categories corresponding approximately to parts of speech.

From the mathematic point of view, we deal with a free monoid having a marked subset and an equivalence relation on the set of generators. We define a binary operation on the set ($\mathbf{E}(\mathbf{V})$) of all equivalence relations of the language vocabulary which assignes equivalence relation to any ordered pair of equivalence relations. In the literature dealing with the algebraic linquistics we meet several operators mapping the set of all equivalence relations of the language vocabulary into itself. Among these operators there are also Kulagina's and Trybulec's operators. We prove that our binary operation assigns to any pair of equal equivalence relations an equivalence relation that is obtained from the given one by means of Kulagina's operator. Trybulec's operator is also included as a special case in our operation. It is obtained in choosing the second equivalence of an ordered pair to be the identity.

It is demonstrated on examples that the introduced operation is neither commutative nor associative. It has several simple properties described in Chapter 2.

1. Basic concepts

1.1. *Let* A, B *be sets, f a surjection of* A *onto* B. *The relation $f^{-1}f$ is an equivalence relation on* A *and f can be factorized in the form $f = ie$ where e is the canonical surjection of* A *onto* A$/f^{-1}f$ *and i is a bijection of* A$/f^{-1}f$ *onto* B.

We denote by $\mathbf{E}(\mathbf{A})$ the set of all equivalence relations on the set $\mathbf{A}$. The set $\mathbf{E}(\mathbf{A})$ is ordered by inclusion: it is a complete lattice with respect to this order relation.

Let $\alpha, \beta \in \mathbf{E}(\mathbf{A})$, $\alpha \subseteq \beta$. For every $x \in \mathbf{A}/\alpha$ we denote by $f(x)$ the set $y \in \mathbf{A}/\beta$ with the property $x \subseteq y$. Then f is a surjection of $\mathbf{A}/\alpha$ onto $\mathbf{A}/\beta$. It will be called the *natural surjection.*

Let $\mathbf{A}$ be a set, $\alpha \in \mathbf{E}(\mathbf{A})$, $\beta \in \mathbf{E}(\mathbf{A})$, $\alpha \subseteq \beta$, k the natural surjection of $\mathbf{A}/\alpha$ onto $\mathbf{A}/\beta$. As we have seen, $k^{-1}k$ is an equivalence relation on $\mathbf{A}/\alpha$ and k can be factorized in the form $k = ie$, where e is the canonical surjection of $\mathbf{A}/\alpha$ onto $(\mathbf{A}/\alpha)/k^{-1}k$ and i is a bijection of $(\mathbf{A}/\alpha)/k^{-1}k$ onto $\mathbf{A}/\beta$.

We put:

$$\beta/\alpha = k^{-1}k$$

Let $\mathbf{A}$ be a set. We denote by $\mathbf{A}^*$ the set of all finite sequences of elements of $\mathbf{A}$; we suppose that the empty sequence Λ is an element of $\mathbf{A}^*$, too. The set $\mathbf{A}^*$ together with the binary operation of catenation is called the *free monoid over* $\mathbf{A}$. The elements of the set $\mathbf{A}^*$ are called *strings.* We identify one element sequences of $\mathbf{A}^*$ with elements of $\mathbf{A}$; thus $\mathbf{A} \subseteq \mathbf{A}^*$ holds true. If $x \in \mathbf{A}^*$ and $x = x_1x_2 \dots x_n$ where n is a natural number and $x_i \in \mathbf{A}$ for $i = 1, 2, \dots, n$, we put $|x| = n$, further we put $|\Lambda| = 0$. The natural number $|x|$ the length of the string x.

If n is a natural number and if $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$ are subsets of $\mathbf{A}^*$ we denote by $\mathbf{A}_1\mathbf{A}_2 \dots \mathbf{A}_n$ the set of all elements of the form $x_1x_2 \dots x_n$ where $x_i \in \mathbf{A}_i$ for every $i = 1, 2, \dots, n$.

Let $\mathbf{A}, \mathbf{B}$ be sets, f a surjection of $\mathbf{A}$ onto $\mathbf{B}$. Then there exists a unique homomorphism f_* of $\mathbf{A}^*$ onto $\mathbf{B}^*$ such that $f_*/\mathbf{A} = f$. This homomorphism is defined as follows: for every $x \in \mathbf{A}^*$, $x = x_1x_2 \dots x_n$ where $n \geqq 0$ and $x_i \in \mathbf{A}_i$ for $i = 1, 2, \dots, n$, we put $f(x) = f(x_1)f(x_2) \dots f(x_n)$. Clearly, $f_*(\Lambda) = \Lambda$. Let us suppose $x \in \mathbf{B}^*$ for $i = 1, 2, \dots, n$. Then there exist the elements $x_1, x_2, \dots, x_n$ in $\mathbf{A}^*$ with the properties $x = x_1x_2 \dots x_n$ and $f_*(x_i) = y_i$ for every $i = 1, 2, \dots, n$.

Clearly, $|x| = |f_*(x)|$ for every $x \in \mathbf{A}^*$.

Let $\alpha \in \mathbf{E}(\mathbf{A})$ be an equivalence relation on $\mathbf{A}$, f the canonical surjection of $\mathbf{A}$ onto $\mathbf{A}/\alpha$ with the property $f_*/\mathbf{A} = f$. We put $\alpha_* = f_*^{-1}f_*$; α_* is a congruence relation on $\mathbf{A}^*$ with the property $\alpha_* \cap (\mathbf{A} \times \mathbf{A}) = \alpha$. This congruence relation is defined as follows: for every $x \in \mathbf{A}^*$, $y \in \mathbf{A}^*$, $x = x_1x_2 \dots x_m$, $y = y_1y_2 \dots y_n$ where m, n are natural numbers and $x_i \in \mathbf{A}$ for every $i = 1, 2, \dots, m$, $y_j \in \mathbf{A}$ for every $j = 1, 2, \dots, n$ we put $(x, y) \in \alpha_*$ iff $m = n$ and $(x_i, y_i) \in \alpha$ for every $i = 1, 2, \dots, n$. Clearly, $(x, \Lambda) \in \alpha_*$, iff $x = \Lambda$.

We have $\mathbf{A}^*/\alpha_* = (\mathbf{A}/\alpha)^*$.

1.2. *The ordered pair* $(\mathbf{V}, \mathbf{L})$ *where* $\mathbf{V}$ *is a set and* $\mathbf{L}$ *a subset of* $\mathbf{V}^*$ *is called a language. The elements of* $\mathbf{L}$ *are called correct sentences of the language* $(\mathbf{V}, \mathbf{L})$. *The set* $\mathbf{V}$ *is called the vocabulary of* $(\mathbf{V}, \mathbf{L})$.

Let $(\mathbf{V}, \mathbf{L})$, $(\mathbf{U}, \mathbf{M})$ be languages, f a surjection of $\mathbf{V}$ onto $\mathbf{U}$. The *mapping* f is called a *weak homomorphism* of $(\mathbf{V}, \mathbf{L})$ onto $(\mathbf{U}, \mathbf{M})$ iff $f_*(\mathbf{L}) = \mathbf{M}$. The mapping f is called a *strong homomorphism* of $(\mathbf{V}, \mathbf{L})$ onto $(\mathbf{U}, \mathbf{M})$ iff $f_*^{-1}(\mathbf{M}) = \mathbf{L}$. Clearly, every strong homomorphism is a weak one and every bijective weak homomorphism is strong. A bijective strong homomorphism of $(\mathbf{V}, \mathbf{L})$ onto $(\mathbf{U}, \mathbf{M})$ is called an *isomorphism.* If i is an isomorphism of $(\mathbf{V}, \mathbf{L})$ onto $(\mathbf{U}, \mathbf{M})$ then i^{-1} is an isomorphism of $(\mathbf{U}, \mathbf{M})$ onto $(\mathbf{V}, \mathbf{L})$.

Let $(\mathbf{V}, \mathbf{L})$ be a language, $\alpha \in \mathbf{E}(\mathbf{V})$, e the canonical surjection of $\mathbf{V}$ onto $\mathbf{V}/\alpha$. We put $\mathbf{L}(\mathbf{V}/\alpha) = e_*(\mathbf{L})$. Then $(\mathbf{V}/\alpha, \mathbf{L}(\mathbf{V}/\alpha))$ is called the *factor language* on $\mathbf{V}/\alpha$. Clearly, e is a weak homomorphism of $(\mathbf{V}, \mathbf{L})$ onto $(\mathbf{V}/\alpha, \mathbf{L}(\mathbf{V}/\alpha))$.

Let $(\mathbf{V}, \mathbf{L})$ be a language. For every $x \in \mathbf{V}$ we define the binary relation $\sigma(x) = \{(u, v); (u, v) \in \mathbf{V}^* \times \mathbf{V}^*, uxv \in \mathbf{L}\}$ on $\mathbf{V}^*$ and its elements are called *contexts* accepting the symbol x. Let $\alpha \in \mathbf{E}(\mathbf{V})$, $x \in \mathbf{V}$. We denote by $\alpha[x]$ the set $\mathrm{X} \in \mathbf{V}/\alpha$ with the property $x \in \mathrm{X}$. We put: $\Sigma_\alpha(x) = \bigcup_{t \in \alpha[x]} \sigma(t)$. $\Sigma_\alpha(x)$ is again a binary relation on $\mathbf{V}^*$. The elements of $\Sigma_\alpha(x)$ are all contexts accepting the symbol x and all contexts accepting elements that are α-equivalent with x.

Let $(\mathbf{V}, \mathbf{L})$ be a language, $\alpha \in \mathbf{E}(\mathbf{V})$ an equivalence relation such that the canonical surjection of $\mathbf{V}$ onto $\mathbf{V}/\alpha$ is a strong homomorphism of $(\mathbf{V}, \mathbf{L})$ onto $(\mathbf{V}/\alpha, \mathbf{L}(\mathbf{V}/\alpha))$. Then α is called a *strong congruence relation* on $(\mathbf{V}, \mathbf{L})$. We denote by $\mathbf{S}(\mathbf{V}, \mathbf{L})$ the set of all strong congruence relations on $(\mathbf{V}, \mathbf{L})$. We have $\mathbf{S}(\mathbf{V}, \mathbf{L}) \subseteq \mathbf{E}(\mathbf{V})$ and $\mathbf{S}(\mathbf{V}, \mathbf{L})$ is ordered as it is a subset of an ordered set.

1.3. Theorem. *For* $\alpha \in \mathbf{E}(\mathbf{V})$ *we have* $\alpha \in \mathbf{S}(\mathbf{V}, \mathbf{L})$ *iff, for every* $x, y \in \mathbf{V}$ *with the property* $(x, y) \in \alpha_*$, *the condition* $x \in \mathbf{L}$ *implies the condition* $y \in \mathbf{L}$.

1.4. Theorem. *Let* $(\mathbf{V}, \mathbf{L})$ *be a language. Then* $\mathbf{S}(\mathbf{V}, \mathbf{L})$ *is a complete lattice which is a complete convex sublattice of* $\mathbf{E}(\mathbf{V})$.

It follows that there exists the greatest strong congruence relation on every language $(\mathbf{V}, \mathbf{L})$. We denote by $\vartheta(\mathbf{V}, \mathbf{L})$ the equivalence relation on $\mathbf{V}$ defined as follows: $\vartheta(\mathbf{V}, \mathbf{L}) = \{(x, y); x \in \mathbf{V}, y \in \mathbf{V}, \sigma(x) = \sigma(y)\}$. Clearly, if $x \in \mathbf{L}$, $(x, y) \in \vartheta(\mathbf{V}, \mathbf{L})$, then $y \in \mathbf{L}$; therefore $\vartheta(\mathbf{V}, \mathbf{L})$ is a strong congruence relation on $(\mathbf{V}, \mathbf{L})$. Let $\alpha \in \mathbf{S}(\mathbf{V}, \mathbf{L})$ be arbitrary. Let $x, y \in \mathbf{V}^*$ be such that $(x, y) \in \alpha$. Let $u, v \in \mathbf{V}^*$, $(u, v) \in \sigma(x)$ be arbitrary. Then by 1.3, we have $(u, v) \in \sigma(y)$. Thus, $(x, y) \in \vartheta(\mathbf{V}, \mathbf{L})$ and $\alpha \subseteq \vartheta(\mathbf{V}, \mathbf{L})$. We have seen that $\vartheta(\mathbf{V}, \mathbf{L})$ is a strong congruence relation on $(\mathbf{V}, \mathbf{L})$. Therefore, the following theorem holds true.

1.5. Theorem. *Let* $(\mathbf{V}, \mathbf{L})$ *be a language. Let* $\vartheta(\mathbf{V}, \mathbf{L})$ *be greatest strong congruence relation on* $(\mathbf{V}, \mathbf{L})$. *Then for every* $x, y \in \mathbf{V}$ *the conditions* $(x, y) \in \vartheta(\mathbf{V}, \mathbf{L})$ *and* $\sigma(x) = \sigma(y)$ *are equivalent.*

2. Operation I and its properties

Let $(\mathbf{V}, \mathbf{L})$ *be a language,* $\alpha, \beta \in \mathbf{E}(\mathbf{V})$. *For* $x, y \in \mathbf{V}$ *we put* $x(\alpha \mathbf{I} \beta)\, y$ *if the following conditions are satisfied:*

1° *If* $(u, v) \in \Sigma_\beta(x)$ *is arbitrary, then* $\alpha_* \circ \{(u, v)\} \circ \alpha_* \cap \Sigma_\beta(y) \neq \emptyset$,

2° *If* $(u, v) \in \Sigma_\beta(y)$ *is arbitrary, then* $\alpha_* \circ \{(u, v)\} \circ \alpha^* \cap \Sigma_\beta(x) \neq \emptyset$.

Let $\alpha, \beta \in \mathbf{E}(\mathbf{V})$. *For* $x, y \in \mathbf{V}$ *we put* $\beta[x] \underset{\alpha,\beta}{\geqq} \beta[y]$ *if the following condition is satisfied: for every* $s \in \beta[x]$, $(u, v) \in \sigma(s)$, *there exist* $t \in \beta[y]$, $(u', v') \in \sigma(t)$ *such that* $(u, u') \in \alpha_*$ *and* $(v, v') \in \alpha_*$. *We put* $\underset{\alpha,\beta}{\equiv} = \underset{\alpha,\beta}{\geqq} \cap (\underset{\alpha,\beta}{\geqq})^{-1}$.

2.1. Theorem. *Let* $(\mathbf{V}, \mathbf{L})$ *be a language,* $\alpha, \beta \in \mathbf{E}(\mathbf{V})$. *Then the relation* $\underset{\alpha,\beta}{\geqq}$ *is reflexive and translative on* $\mathbf{V}/\beta$.

Proof: From the definition of $\underset{\alpha,\beta}{\geqq}$, it follows that $\underset{\alpha,\beta}{\geqq}$ holds for $x, y, s, t \in \mathbf{V}$ such that $x = y$, $s = t$. Then $\underset{\alpha,\beta}{\geqq}$ is reflexive.

Let $x, y, z \in \mathbf{V}$ be such that $\beta[x] \underset{\alpha,\beta}{\geqq} \beta[y]$ and $\beta[y] \underset{\alpha,\beta}{\geqq} \beta[z]$ hold. For every $r \in \beta[x]$, $(u, v) \in \sigma(r)$, there exist $s \in \beta[y]$, $(u', v') \in \sigma(s)$ such that $(u, u') \in \alpha_*$, $(v, v') \in \alpha_*$ by definition of $\underset{\alpha,\beta}{\geqq}$. As $\beta[y] \underset{\alpha,\beta}{\geqq} \beta[z]$, there exist $t \in \beta[z]$, $(u'', v'') \in \sigma(t)$ such that $(u', u'') \in \alpha_*$, $(v', v'') \in \alpha_*$. Further, we have $(u, u'') \in \alpha_* \circ \alpha_* = \alpha_*$, $(v, v'') \in \alpha_* \circ \alpha_* = \alpha_*$. Thus, we obtain: for every $r \in \beta[x]$, $(u, v) \in \sigma(r)$, there exist $t \in \beta[z]$, $(u'', v'') \in \sigma(t)$ such that $(u, u'') \in \alpha_*$, $(v, v'') \in \alpha_*$. So $\beta[x] \underset{\alpha,\beta}{\geqq} \beta[z]$ holds and $\underset{\alpha,\beta}{\geqq}$ is transitive.

By definition of $\underset{\alpha,\beta}{\equiv}$ and by 2.1, it is clear that the relation $\underset{\alpha,\beta}{\equiv}$ is an equivalence relation on $\mathbf{V}/\beta$.

Let $\alpha, \beta \in \mathbf{E}(\mathbf{V})$, $x, y \in \mathbf{V}$. Easy to see that the following assertions are equivalent:

(I) *If* $(u, v) \in \Sigma_\beta(x)$ *is arbitrary, then* $\alpha_* \circ (u, v) \circ \alpha_* \cap \Sigma_\beta(y) \neq \emptyset$.

(II) *If* $(u, v) \in \Sigma_\beta(x)$ *is arbitrary, then there exists* $(u', v'') \in \Sigma_\beta(y)$ *such that* $(u, u') \in \alpha_*$, $(v, v') \in \alpha_*$.

(III) *For every* $(u, v) \in \mathbf{V}^* \times \mathbf{V}^*$ *such that there exists* $s \in \beta[x]$, *with the property* $(u, v) \in \sigma(s)$, *there exists* $t \in \beta[y]$ *and* $(u', v') \in \sigma(t)$ *such that* $(u, u') \in \alpha_*$, $(v, v') \in \alpha_*$.

(IV) *For every* $(u, v) \in \mathbf{V}^* \times \mathbf{V}^*$, $s \in \beta[x]$ *such that* $(u, v) \in \sigma(s)$, *there exist* $t \in \beta[y]$, $(u', v') \in \sigma(t)$ *such that* $(u, u') \in \alpha_*$ *and* $(v, v') \in \alpha_*$.

(V) $\beta[x] \underset{\alpha,\beta}{\geqq} \beta[y]$.

Let us denote by **(I′)**$\div$**(V′)** the statements obtained from **(I)**$\div$**(V)** by exchanging elements x and y.

By definition **I**, by definition of $\underset{\alpha,\beta}{\equiv}$, it is easy to see that the condition $\beta[x] \underset{\alpha,\beta}{\equiv} \beta[y]$

is equivalent with the condition $x(\alpha\mathbf{I}\beta)\,y$ for any $x, y \in \mathbf{V}$. As $\underset{\alpha,\beta}{\equiv}$ is an equivalence relation on $\mathbf{V}$, $\alpha\mathbf{I}\beta$ is an equivalence relation on $\mathbf{V}$, and $\mathbf{I}$ is an *operation* on $\mathbf{E}(\mathbf{V})$.

Let $(\mathbf{V}, \mathbf{L})$ be a language, $\alpha, \beta \in \mathbf{E}(\mathbf{V})$, $x, y \in \mathbf{V}$. We put particularly $\alpha = \beta$ in **(I)** $\div$ **(V)** and **(I′)** $\div$ **(V′)**. Then $(\mathbf{V}_1)$ $\alpha[x] \underset{\alpha,\alpha}{\geqq} \alpha[y]$ is equivalent with the following condition:

$(\mathbf{III}_1)$ *For every* $(u, v) \in \mathbf{V}^* \times \mathbf{V}^*$ *such that there exists* $s \in \alpha[x]$ *with the property* $(u, v) \in \sigma(s)$, *there exist* $t \in \alpha[y]$ *and* $(u', v') \in \sigma(t)$ *such that* $(u, u') \in \alpha_*$, $(v, v') \in \alpha^*$.

Similarly $(\mathbf{V}_1')$ is equivalent with $(\mathbf{III}_1')$.

Let $(\mathbf{V}/\alpha, \mathbf{L}(\mathbf{V}/\alpha))$ be a factor language on $\mathbf{V}/\alpha$. Then $(\mathbf{III}_1)$ is clearly equivalent with the following condition:

$(\mathbf{III}_2)$ For every $(\mathscr{U}, \mathscr{V}) \in (\mathbf{V}/\alpha)^* \times (\mathbf{V}/\alpha)^*$ such that $(\mathscr{U}, \mathscr{V}) \in \sigma(\alpha[x])$ the condition $(\mathscr{U}, \mathscr{V}) \in \sigma(\alpha[y])$ is satisfied.

Similarly the condition $(\mathbf{III}_1')$ is equivalent with the condition $(\mathbf{III}_2')$. From the definition of $\underset{\alpha,\beta}{\equiv}$ and from $(\mathbf{III}_2)$ being equivalent with $(\mathbf{V}_1)$ and $(\mathbf{III}_2')$ with $(\mathbf{V}_1')$ the following corollary holds true:

2.2. Corollary. *Let* $(\mathbf{V}, \mathbf{L})$ *be a language*, $\alpha \in \mathbf{E}(\mathbf{V})$, $(\mathbf{V}/\alpha, \mathbf{L}(\mathbf{V}/\alpha))$ *the factor language on* $\mathbf{V}/\alpha$. *Then the following assertions are equivalent.*

(1) $\alpha[x] \underset{\alpha,\alpha}{\equiv} \alpha[y]$,

(2) $\sigma(\alpha[x]) = \sigma(\alpha[y])$.

From 1.5 and 2.2 $\underset{\alpha,\alpha}{\equiv}$ is the greatest strong congruence on $(\mathbf{V}/\alpha, \mathbf{L}(\mathbf{V}/\alpha))$.

2.3. Theorem. *Let* $(\mathbf{V}, \mathbf{L})$ *be a language*, $\alpha, \beta \in \mathbf{E}(\mathbf{V})$ *arbitrary equivalence relations. Then* $\beta \subseteq \alpha\mathbf{I}\beta$ *holds.*

Proof: Let $x, y \in \mathbf{V}$ be such that $x \in \beta[y]$. Then $\beta[x] = \beta[y]$ and $\Sigma_\beta(x) = \Sigma_\beta(y)$. Let $(u, v) \in \Sigma_\beta(x)$ be arbitrary. Clearly $(u, v) \in \alpha_*$ $\cdot$ $(u, v) \circ \alpha_* \cap \Sigma_\beta(y)$ therefore condition 1° from the definition of the operation $\mathbf{I}$ is satisfied for the pair α, β. Similarly, condition 2° from the definition of the operation $\mathbf{I}$ is satisfied.

We have proved the following: If $x \in \beta[y]$, then $x(\alpha\mathbf{I}\beta)\,y$. Thus, $\beta \subseteq \alpha\mathbf{I}\beta$.

2.4. Lemma. *Let* $(\mathbf{V}, \mathbf{L})$ *be a language*, $\alpha, \beta \in \mathbf{E}(\mathbf{V})$ *arbitrary equivalence relations*, $x, y, x', y' \in \mathbf{V}$. *If* $(x, x') \in \beta$, $(y, y') \in \beta$, $(x, y) \in (\alpha\mathbf{I}\beta)$, *then* $(x', y') \in (\alpha\mathbf{I}\beta)$.

Proof: $(x', y') \in \beta \circ (\alpha\mathbf{I}\beta) \circ \beta \subseteq (\alpha\mathbf{I}\beta) \circ (\alpha\mathbf{I}\beta) \circ (\alpha\mathbf{I}\beta) \subseteq \alpha\mathbf{I}\beta$ holds by 2.3 and by transitivity of the relation $\alpha\mathbf{I}\beta$.

Let $(\mathbf{V}, \mathbf{L})$ be a language, $\alpha, \beta \in \mathbf{E}(\mathbf{V})$. We denote by $\mathbf{S}_\alpha = \{x_i;\ i = 1, 2, \ldots, n\}$ a set of elements $x_i \in \mathbf{V}$ with the following properties:

(1) $\alpha[x_i] \cap \alpha[x_j] = \emptyset$ for $i, j = 1, 2, \ldots, n, i \neq j$.

(2) $\mathbf{V} = \bigcup_{i=1}^{n} \alpha[x_i]$.

The set $\mathbf{S}_\alpha$ is said to be a *system of representatives* of the equivalence α.

By 2.3 and 2.4, the construction of equivalence relation $\alpha\mathbf{I}\beta$ can be simplified as follows:

We determine the set $\mathbf{S}_\beta$. Now for an arbitrary pair of elements $x, y \in \mathbf{S}_\beta$ we check whether they are in the equivalence relation $\alpha\mathbf{I}\beta$ or not. If such a pair does not exist, then $\beta = \alpha\mathbf{I}\beta$. If there exists at least one pair of representatives such that they are in $\alpha\mathbf{I}\beta$, then $\beta \subset \alpha\mathbf{I}\beta$ and each maximal subset of $\mathbf{S}_\beta$ in which any two different elements are not in $\alpha\mathbf{I}\beta$ defines a system of representatives $\mathbf{S}_{\alpha\mathbf{I}\beta}$ of the equivalence relation $\alpha\mathbf{I}\beta$.

2.5. Example. *We prove that the operation* **I** *need not be commutative. Let* $(\mathbf{V}, \mathbf{L})$ *be a language where* $\mathbf{V} = \{a, b\}$, $\mathbf{L} = \{ab\}$. *Let* $\alpha, \beta \in \mathbf{E}(\mathbf{V})$ *be such that* $\mathbf{V}/\alpha = \{a, b\} = \mathbf{V}$, $\mathbf{V}/\beta = \{\{a\}, \{b\}\} = id_{\mathbf{V}}$.

Proof: For the language $(\mathbf{V}, \mathbf{L})$, $\alpha \in \mathbf{E}(\mathbf{V})$ is the greatest equivalence relation on $\mathbf{V}$. From 2.3 and from **I** being an operation on $\mathbf{E}(\mathbf{V})$, we obtain $\beta\mathbf{I}\alpha = \alpha$.

We construct $\alpha\mathbf{I}\beta$. The set $\mathbf{S}_\beta = \{a, \beta\}$. We prove that $a(\alpha\mathbf{I}\beta)\, b$ does not hold. We have $\Sigma_\beta(a) = \{(\Lambda, b)\}$, $\Sigma_\beta(b) = \{(a, \Lambda)\}$. Then $\alpha_* \circ \{(\Lambda, b)\} \circ \alpha_* \cap \Sigma_\beta(b) = \{(\Lambda, b), (\Lambda, a)\} \cap \Sigma_\beta(b) = \emptyset$ holds. Thus, we obtain: $(a, b) \in (\alpha\mathbf{I}\beta)$ and $\alpha\mathbf{I}\beta = \beta$.

We have proved the following: $\alpha\mathbf{I}\beta = \beta \neq \alpha = \beta\mathbf{I}\alpha$, therefore the operation **I** is not commutative.

2.6. Example. *We prove that the operation* **I** *need not be associative. Let* $(\mathbf{V}, \mathbf{L})$ *be a language where* $\mathbf{V} = \{a, b, c, d, e\}$, $\mathbf{L} = \{abc, edc\}$. *Let* $\alpha, \beta, \gamma \in \mathbf{E}(\mathbf{V})$ *be such that* $\mathbf{V}/\alpha = \{\{a, e\}, \{b\}, \{c, d\}\}$, $\mathbf{V}/\beta = \{\{a\}, \{b, d\}, \{c, e\}\}$, $\mathbf{V}/\gamma = \{\{a, c\}, \{b\}, \{e\}, \{d\}\}$.

By a detailed analysis it can be proved that $(\alpha\mathbf{I}\beta)\,\mathbf{I}\gamma = \beta\mathbf{I}\gamma = \gamma$ and $\alpha\mathbf{I}(\beta\mathbf{I}\gamma) = \alpha\mathbf{I}\gamma = \delta$ where $\mathbf{V}/\delta = \{\{a, c\}, \{b, d\}, \{e\}\}$. Then $(\alpha\mathbf{I}\beta)\,\mathbf{I}\gamma = \gamma \neq \delta = \alpha\mathbf{I}(\beta\mathbf{I}\gamma)$ and the operation is not associative.

2.7. Theorem. *Let* $(\mathbf{V}, \mathbf{L})$ *be a language,* $\alpha, \beta, \gamma \in \mathbf{E}(\mathbf{V})$. *If* $\alpha \subseteq \gamma$, *then* $\alpha\mathbf{I}\beta \subsetneq \gamma\mathbf{I}\beta$.

Proof: Let $x, y \in \mathbf{V}$ be such that $x(\alpha\mathbf{I}\beta)\, y$. Let $(u, v) \in \Sigma_\beta(x)$ be arbitrary. Then $\gamma_* \circ \{(u, v)\} \circ \gamma_* \cap \Sigma_\beta(y) \supseteq \alpha_* \circ \{(u, v)\} \circ \alpha_* \cap \Sigma_\beta(y) \neq \emptyset$ holds and, therefore, we obtain $x(\gamma\mathbf{I}\beta)\, y$. Thus $\alpha\mathbf{I}\beta \subseteq \gamma\mathbf{I}\beta$.

3. Kulagina's operator

Let $(\mathbf{V}, \mathbf{L})$ *be a language,* $\alpha \in \mathbf{E}(\mathbf{V})$ *an arbitrary equivalence relation. We put* $\alpha' = \alpha\mathbf{I}\alpha$. *Operator* $'$ *is called Kulagina's operator.*

If $(\mathbf{V}, \mathbf{L})$ is a fragment of the Czech language, α the equivalence relation on $\mathbf{V}$ where blocks are paradigms, then α' provides blocks approximating parts of speech.

3.1. Example. Let u_1 = *stojí*, u_2 = *bojím se*, u_3 = *klaním se*, u_4 = *vidím*, u_5 = *ozvi se*, u_6 = *mluvím o*, u_7 = *pohrdám*, v_1 = *had*, v_2 = *hada*, v_3 = *hadu*, v_4 = *hade*, v_5 = *hadem*, w_1 = *hrad*, w_2 = *hradu*, w_3 = *hrade*, w_4 = *hradem*. Let $\mathbf{V} = \{u_i, v_j, w_k;\ i = 1, 2, \ldots, 7;\ j = 1, 2, \ldots, 5;\ k = 1, 2, 3, 4\}$, the set $\mathbf{L}$ of all grammatically correct sentences consists of some strings xy, where $x \in \{u_i;\ i = 1, 2, \ldots, 7\}$, $y \in \{v_j, w_k;\ j = 1, 2, \ldots, 5;\ k = 1, 2, 3, 4\}$ and is defined as follows: $\mathbf{L} = \{u_1v_1, u_2v_2, u_3v_3, u_4v_2, u_5v_4, u_6v_3, u_7v_5, u_1w_1, u_2w_2, u_3w_3, u_4w_1, u_5w_3, u_6w_2, u_7w_4\}$. Let α be the equivalence relation on $\mathbf{V}$ whose blocks are paradigms. Then $\mathbf{V}/\alpha = \{\{u_1\}, \{u_2\}, \{u_3\}, \{u_4\}, \{u_5\}\ \{u_6\}, \{u_7\}, \{v_j; j = 1, 2, \ldots, 5\}, \{w_k; k = 1, 2, 3, 4\}\}$.

By a detailed analysis it can be proved that $\mathbf{V}/\alpha' = \{\{u_i;\ i = 1, 2, \ldots, 7\}, \{v_j, w_k;\ j = 1, 2, \ldots, 5;\ k = 1, 2, 3, 4\}\}$. Then the blocks of the equivalence relation α' define parts of speech.

3.2. Theorem. *Let* $(\mathbf{V}, \mathbf{L})$ *be a language. Then* $id'_{\mathbf{V}}$ *is the greatest strong congruence on* $(\mathbf{V}, \mathbf{L})$.

Proof: For arbitrary $x \in \mathbf{V}$, we have $\Sigma_{id_{\mathbf{V}}}(x) = \sigma(x)$. Let $u, u' \in \mathbf{V}^*$; then $(u, u') \in id_{\mathbf{V}^*}$ iff $u = u'$. By definition of the operation $\mathbf{I}$, the condition $(x, y) \in id_{\mathbf{V}}\mathbf{I}id_{\mathbf{V}} = id'_{\mathbf{V}}$ is satisfied iff $\sigma(x) = \sigma(y)$ where x and y are arbitrary elements in $\mathbf{V}$. Hence, by 1.5, $id'_{\mathbf{V}}$ is the greatest strong congruence on $(\mathbf{V}, \mathbf{L})$.

3.3. Corollary. *Let* $(\mathbf{V}, \mathbf{L})$ *be a language. Then* $\alpha \subseteq id'_{\mathbf{V}}$ *holds for arbitrary* $\alpha \in \mathbf{S}(\mathbf{V}, \mathbf{L})$.

Proof: This is a consequence of 1.4 and 3.2.

3.4. Theorem. *Let* $(\mathbf{V}, \mathbf{L})$ *be a language. Then* $\alpha \subseteq \alpha'$ *holds for arbitrary* $\alpha \in \mathbf{E}(\mathbf{V})$.

Proof: $\alpha \subseteq \alpha\mathbf{I}\alpha = \alpha'$ holds by 2.3.

3.5. Theorem. *Let* $(\mathbf{V}, \mathbf{L})$ *be a language. Then* $\alpha\mathbf{I}(\alpha\mathbf{I}\alpha) = \alpha\mathbf{I}\alpha$ *holds for arbitrary* $\alpha \in \mathbf{E}(\mathbf{V})$.

Proof: $\alpha\mathbf{I}\alpha = \alpha' \subseteq \alpha\mathbf{I}\alpha' \subseteq \alpha\mathbf{I}(\alpha\mathbf{I}\alpha)$ holds by 2.3.

Let $x, y \in \mathbf{V}$ be such that $x(\alpha\mathbf{I}\alpha')\,y$ and $(u, v) \in \Sigma_\alpha(x)$ be arbitrary. Since $\alpha \subseteq \alpha'$, we have $\alpha[x] \subseteq \alpha'[x]$ and therefore $\Sigma_\alpha(x) \subseteq \Sigma_{\alpha'}(x)$. As $x(\alpha\mathbf{I}\alpha')\,y$, we obtain $\alpha_* \circ$

$\circ \{(u, v)\} \circ \alpha_* \cap \Sigma_{\alpha'}(y) \neq \emptyset$, i.e., there exists $y' \in \alpha'[y]$ such that $\alpha_* \circ \{(u, v)\} \circ \alpha_* \cap \cap \sigma(y') \neq \emptyset$. Then there exist $u', v' \in \mathbf{V}^*$ such that $(u, u') \in \alpha_*$, $(v, v') \in \alpha_*$, and $(u', v') \in \in \sigma(y')$. Further, we have $(y', y) \in \alpha' = \alpha\mathbf{I}\alpha$. Thus, there exist $y'' \in \alpha[y]$ and $(u'', v'') \in \in \sigma(y'')$ such that $(u', u'') \in \alpha_*$, $(v', v'') \in \alpha_*$. We conclude that $(u, u'') \in \alpha_* \circ \alpha_* = \alpha_*$, $(v, v'') \in \alpha_* \circ \alpha_* = \alpha_*$ and $y'' \in \alpha[y]$, i.e., $\alpha_* \circ \{(u, v)\} \circ \alpha_* \cap \Sigma_\alpha(y) \neq \emptyset$. Thus, condition 1° from the definition of the operation $\mathbf{I}$ is satisfied for the pair (α, α). Similarly condition 2° from the definition of the operation $\mathbf{I}$ is satisfied. Thus by 2.3, we obtain $\alpha' \subseteq \alpha\mathbf{I}\alpha' \subseteq \alpha'$.

We have proved that: $\alpha\mathbf{I}\alpha = \alpha' = \alpha\mathbf{I}\alpha' = \alpha\mathbf{I}(\alpha\mathbf{I}\alpha)$.

3.6. Theorem. *Let* $(\mathbf{V}, \mathbf{L})$ *be a language,* $\alpha \in \mathbf{E}(\mathbf{V})$ *be arbitrary. Then* $\alpha' \mathbf{I}\alpha = \alpha'$ *holds.*

Proof: $\alpha \subseteq \alpha\mathbf{I}\alpha = \alpha'$ holds by 2.3. By 2.7, we have $\alpha' = \alpha\mathbf{I}\alpha \subseteq \alpha'\mathbf{I}\alpha$.

It remains to prove that $\alpha'\mathbf{I}\alpha \subseteq \alpha' = \alpha\mathbf{I}\alpha$. Let $x, y \in \mathbf{V}$ be such that $x(\alpha'\mathbf{I}\alpha)\, y$. Then for every $(u, v) \in \Sigma_\alpha(x)$, $u, v \in \mathbf{V}^*$, the condition $\alpha'_* \circ \{(u, v)\} \circ \alpha'_* \cap \Sigma_\alpha(y) \neq \emptyset$ holds, i.e., there exists $y^0 \in \alpha[y]$ such that $\alpha'_* \circ \{(u, v)\} \circ \alpha'_* \cap \sigma(y^0) \neq \emptyset$.

Let us denote by n the natural number such that $u = x_1 x_2 \ldots x_m$, $v = x_{m+1} \ldots x_n$ where $x_i \in \mathbf{V}$ for $i = 1, 2, \ldots, n$. If $1 \leqq k \leqq n$, we denote by $\mathbf{V}(k)$ the following statement: There exist $x_i^k \in \mathbf{V}$ for $i = 1, 2, \ldots, n$, $y^k \in \mathbf{V}$ such that $(x_i^k, x_i) \in \alpha$ for $i \leqq k$; $(x_i^k, x_i) \in \alpha'$ for $k < i \leqq n$; $(y^k, y) \in \alpha$ and $x_1^k \ldots x_m^k y^k x_{m+1}^k \ldots x_n^k \in \mathbf{L}$.

We prove the validity of $\mathbf{V}(k)$ for $1 \leqq k \leqq n$ by induction.

(A) $\alpha'_* \circ \{(u, v)\} \circ \alpha'_* \cap \sigma(y^0) \neq \emptyset$ implies that there exists $x_i^0 \in \mathbf{V}$ for $i = 1, 2, \ldots, n$ such that $(x_i^0, x_i) \in \alpha'$ for $i = 1, 2, \ldots, n$ and $x_1^0 \ldots x_m^0 y x^{00}_{m+1} \ldots x_n^0 \in \mathbf{L}$. Hence, $\mathbf{V}(0)$ holds.

(B) Let us suppose that $\mathbf{V}(k)$ holds. Assume $k < m$. Then $(x_{k+1}^k, x_{k+1}) \in \alpha' = \alpha\mathbf{I}\alpha$ and $(x_1^k \ldots x_k^k, x_{k+2}^k \ldots x_m^k y^k x_{m+1}^k \ldots x_n^k) \in \sigma(x_{k+1}^k)$. This means that there exists $x_{k+1}^{k+1} \in \in \alpha[x_{k+1}]$ such that $\alpha_* \circ \{(x_1^k \ldots x_k^k, x_{k+2}^k \ldots x_m^k y^k x_{m+1}^k \ldots x_n^k)\} \circ \alpha_* \cap \sigma(x_{k+1}) \neq \emptyset$, i.e., there exist $x_i^{k+1} \in \mathbf{V}$, $(x_i^{k+1}, x_i^k) \in \alpha$ for $i = 1, 2, \ldots, n$, $i \neq k + 1$, $y^{k+1} \in \mathbf{V}$ such that $(y^{k+1}, y^k) \in \alpha$ and $(x_1^{k+1} \ldots x_k^{k+1}, x_{k+2}^{k+1} \ldots x_m^{k+1} y^{k+1} x_{m+1}^{k+1} \ldots x_n^{k+1}) \in \sigma(x_{k+1})$. Thus $x_1^{k+1} \ldots x_m^{k+1} y^{k+1} x_{m+1}^{k+1} \ldots x_n^{k+1} \in \mathbf{L}$. By hypothesis, we have $(x_i^{k+1}, x_i) \in \alpha \circ \alpha = \alpha$ for $i = 1, 2, \ldots, k$; $(x_{k+1}^{k+1}, x_{k+1}) \in \alpha$, $(x_i^{k+1}, x_i) \in \alpha \circ \alpha' = \alpha'$ for $i = k + 2, \ldots, n$; and $(y^{k+1}, y) \in \alpha \circ \alpha = \alpha$. For $m \leqq k \leqq n$, we proceed similarly. Hence, $\mathbf{V}(k + 1)$ is implied by $\mathbf{V}(k)$.

The validity of $\mathbf{V}(n)$ follows by induction. Thus there exist $x_1^n, \ldots, x_m^n, y^n, x_{m+1}^n, \ldots, \ldots, x_n^n \in \mathbf{V}$ such that $(x_i, x_i^n) \in \alpha$ for $i = 1, 2, \ldots, n$; $(y^n, y) \in \alpha$, and $x_1^n \ldots x_m^n y^n x_{m+1}^n \ldots \ldots x_n^n \in \mathbf{L}$. We put $u' = x_1^n \ldots x_m^n$, $v' = x_{m+1}^n \ldots x_n^n$, $y' = y^n$. Then $u'y'v' \in \mathbf{L}$; this means $(u', v') \in \sigma(y')$. Clearly $(u, u') \in \alpha_*$, $(v, v') \in \alpha_*$, and hence $(u', v') \in \alpha_* \circ \circ \{(u, v)\} \circ \alpha_* \cap \sigma(y')$. Since $y' \in \alpha[y]$, we have $\alpha_* \circ \{(u, v)\} \circ \alpha_* \cap \Sigma_\alpha(y) \neq \emptyset$.

Hence, condition 1° from the definition of the operation $\mathbf{I}$ for the pair (α, α) is satisfied. Similarly condition 2° from the definition of the operation $\mathbf{I}$ is satisfied. Thus $(x, y) \in \alpha\mathbf{I}\alpha = \alpha'$ and we have $\alpha'\mathbf{I}\alpha = \alpha'$.

3.7. Theorem. *Let* $(\mathbf{V}, \mathbf{L})$ *be a language,* $\alpha \in \mathbf{E}(\mathbf{V})$. *Then* $\alpha'' = \alpha'$.

Proof: The condition $\alpha' \subseteq \alpha'\mathbf{I}\alpha' = \alpha''$ holds by 2.3.

It remains to prove that $\alpha'' = \alpha'\mathbf{I}\alpha' \subseteq \alpha'$. Let $x, y \in \mathbf{V}$ be such that $x(\alpha'\mathbf{I}\alpha')\, y$ and $(u, v) \in \Sigma_\alpha(x)$ be arbitrary. Since $\alpha \subseteq \alpha'$ we have $\alpha[x] \subseteq \alpha'[x]$ and, therefore $\Sigma_\alpha(x) \subseteq \subseteq \Sigma_{\alpha'}(x)$. The condition $x(\alpha'\mathbf{I}\alpha')\, y$ implies that $\alpha'_* \circ \{(u, v)\} \circ \alpha'_* \cap \Sigma_{\alpha'}(y) \neq \emptyset$, i.e., there exists $y' \in \alpha'[y]$ such that $\alpha'_* \circ \{(u, v)\} \circ \alpha'_* \cap \sigma(y') \neq \emptyset$. Then there exist $u', v' \in \mathbf{V}^*$ such that $(u, u') \in \alpha'_*$, $(v, v') \in \alpha'_*$ and $(u', v') \in \sigma(y')$. Now, $(y', y) \in \alpha' = = \alpha'\mathbf{I}\alpha$ holds by 3.6. Thus there exist $y'' \in \alpha[y]$ and $(u'', v'') \in \sigma(y'')$ such that $(u', u'') \in \in \alpha'_*$, $(v', v'') \in \alpha'_*$. We have proved that $(u, u'') \in \alpha'_* \circ \alpha'_* = \alpha'_*$, $(v, v'') \in \alpha'_* \circ \alpha'_* = \alpha'_*$, and $y'' \in \alpha[y]$, i.e., $\alpha'_* \circ \{(u, v)\}\, \alpha'_* \cap \Sigma_\alpha(y) \neq \emptyset$.

Hence condition 1° from the definition of the operation $\mathbf{I}$ is satisfied for the pair (α', α). Similarly the condition 2° from the definition of the operation $\mathbf{I}$ is satisfied as well. We conclude that $\alpha'' = \alpha'\mathbf{I}\alpha' \subseteq \alpha'\mathbf{I}\alpha = \alpha'$.

We have proved that $\alpha' = \alpha''$.

3.8. Theorem. *Let* $(\mathbf{V}, \mathbf{L})$ *be a language,* $\alpha, \beta \in \mathbf{E}(\mathbf{V})$. *Then the following assertions are equivalent·*

(A) $\alpha' = \beta'$.

(B) *there exists* $\gamma \in \mathbf{E}(\mathbf{V})$ *with the property* $\alpha \subseteq \gamma \subseteq \alpha'$, $\beta \subseteq \gamma \subseteq \beta'$.

(C) $\alpha \vee \beta \subseteq \alpha'$, $\alpha \vee \beta \subseteq \beta'$.

Proof: 1. Let (A) hold. We put $\alpha' = \gamma = \beta'$. Then $\alpha \subseteq \gamma \subseteq \alpha'$, $\beta \subseteq \gamma \subseteq \beta'$ and (B) holds as well.

2. Let (B) hold. Then $\alpha \vee \beta \subseteq \gamma \subseteq \alpha'$, $\alpha \vee \beta \subseteq \gamma \subseteq \beta'$ and (C) also holds.

3. Let (C) hold. We put $\gamma = \alpha \vee \beta$; hence, $\alpha \subseteq \gamma \subseteq \alpha'$. Let $x, y \in \mathbf{V}$ be such that $(x, y) \in \alpha'\mathbf{I}\gamma$ and $(u, v) \in \Sigma_\alpha(x)$ be arbitrary. Since $x(\alpha'\mathbf{I}\gamma)\, y$ holds, we have $\alpha'_* \circ \circ \{(u, v)\} \circ \alpha'_* \cap \Sigma_\gamma(y) \neq \emptyset$, i.e., there exists $y' \in \gamma[y]$ such that $\alpha'_* \circ \{(u, v)\} \circ \alpha'_* \cap \cap\, \sigma(y') \neq \emptyset$. Then there exist $u', v' \in \mathbf{V}^*$ such that, $(u, u') \in \alpha'_*$, $(v, v') \in \alpha'_*$ and $(u', v') \in \sigma(y')$. Now $(y', y) \in \gamma \subseteq \alpha' = \alpha'\mathbf{I}\alpha$ by 3.6. Thus there exist $y'' \in \alpha[y]$, $(u'', v'') \in \in \sigma(y'')$ such that $(u', u'') \in \alpha'_*$, $(v', v'') \in \alpha'_*$. We have proved that $(u, u'') \in \alpha'_* \circ \alpha'_* = = \alpha'_*$, $(v, v'') \in \alpha'_* \circ \alpha'_* = \alpha_*$ and $y'' \in \alpha[y]$, i.e., $\alpha'_* \circ \{(u, v)\} \circ \alpha'_* \cap \Sigma_\alpha(y) \neq \emptyset$.

Hence condition 1° from the definition of the operation $\mathbf{I}$ is satisfied for the pair (α', α). Similarly condition 2^0 from the definition of the operation $\mathbf{I}$ is satisfied as well. We have proved $\alpha'\mathbf{I}\gamma \subseteq \alpha'\mathbf{I}\alpha$. Then we obtain $\gamma' = \gamma\mathbf{I}\gamma \subseteq \alpha'\mathbf{I}\gamma \subseteq \alpha'\mathbf{I}\alpha = \alpha'$ by 2.7 and 3.6.

Now let us suppose that $(x, y) \in \gamma\mathbf{I}\alpha'$, holds for $x, y \in \mathbf{V}$. Let $(u, v) \in \Sigma\gamma(x)$ be arbitrary. Since $\gamma \subseteq \alpha'$, we have $\gamma[x] \subseteq \alpha'[x]$ and therefore $\Sigma_\gamma(x) \subseteq \Sigma_{\alpha'}(x)$. Since $x(\gamma\mathbf{I}\alpha')\, y$, we obtain $\gamma_* \circ \{(u, v)\} \circ \gamma_* \cap \Sigma_{\alpha'}(y) = \emptyset$, i.e., there exists $y' \in \alpha'[y]$ such that $\gamma_* \circ \{(u, v)\} \circ \gamma_* \cap \sigma(y') \neq \emptyset$. Then there exist $u', v' \in \mathbf{V}^*$ such that $(u, u') \in \gamma_*$, $(v, v') \in \gamma_*$, and $(u', v') \in \sigma(y')$. Now we have $y' \in \alpha'[y]$. Since $\alpha' = \alpha\mathbf{I}\alpha \subseteq \gamma\mathbf{I}\alpha$ by 2.7, hence $y'(\gamma\mathbf{I}\alpha)\, y$ holds, i.e., there exists $y'' \in \alpha[y]$, $u'', v'' \in \mathbf{V}^*$ such that $(u'', v'') \in \sigma(y'')$ and $(u', u'') \in \gamma_*$, $(v', v'') \in \gamma_*$. Further $\alpha \subseteq \gamma$ holds and, therefore, $\alpha[y] \subseteq \gamma[y]$.

We have proved that $(u, u'') \in \gamma_* \circ \gamma_* = \gamma_*$, $(v, v'') \in \gamma_* \circ \gamma_* = \gamma_*$, and $y'' \in \alpha[y] \subseteq \subseteq \gamma[y]$. Thus, $\gamma_* \circ \{(u, v)\} \circ \gamma_* \cap \Sigma_\gamma(y) \neq \emptyset$.

Hence condition 1^0 from the definition of the operation **I** is satisfied for the pair (γ, γ). Similarly condition 2^0 from the definition of the operation **I** is satisfied as well. Hence $\alpha' = \alpha\mathbf{I}\alpha = \alpha\mathbf{I}\alpha' \subseteq \gamma\mathbf{I}\alpha' \subseteq \gamma\mathbf{I}\gamma = \gamma'$ by 2.7 and 3.5.

We have proved $\alpha' = \gamma'$. Similarly we prove $\beta' = \gamma'$. Hence $\alpha' = \gamma' = \beta'$ which is **(A)**.

3.9. Corollary. *Let* $(\mathbf{V}, \mathbf{L})$ *be a language,* $\alpha, \beta \in \mathbf{E}(\mathbf{V})$. *If* $\alpha \subseteq \beta \subseteq \alpha'$, *then* $\alpha' = \beta'$.

Proof: $\alpha \vee \beta = \beta \subseteq \alpha'$ holds. Further $\beta \subseteq \beta'$, hence **(C)** from 3.8 holds. Thus $\alpha' = \beta'$.

3.10. Theorem. *Let* $(\mathbf{V}, \mathbf{L})$ *be a language,* $\alpha, \beta \in \mathbf{E}(\mathbf{V})$ *be such that* $\alpha \subseteq \beta \subseteq \alpha'$. *Then the following assertions hold:*

(A) $\beta\mathbf{I}\alpha = \alpha'$.

(B) $\alpha\mathbf{I}\beta = \alpha'$.

Proof: 1. Clearly $\alpha \subseteq \beta \subseteq \alpha'$ implies $\alpha' = \alpha\mathbf{I}\alpha \subseteq \beta\mathbf{I}\alpha \subseteq (\alpha\mathbf{I}\alpha)\,\mathbf{I}\alpha = \alpha'$ by 2.7 and 3.6. We have **(A)**.

2. The condition $\alpha\mathbf{I}\beta \subseteq \beta\mathbf{I}\beta = \beta' = \alpha'$ is satisfied by 2.7 and 3.9. Let $x, y \in \mathbf{V}$ be such that $x(\alpha\mathbf{I}\beta')\, y$ and $(u, v) \in \Sigma_{\beta'}(x)$ be arbitrary; thus $\alpha_* \circ \{(u, v)\} \circ \alpha_* \cap \Sigma_{\beta'}(y) \neq \neq \emptyset$. This means that there exists $y' \in \beta'[y]$ such that $\alpha_* \circ \{(u, v)\} \circ \alpha_* \cap \sigma(y') \neq \emptyset$. Hence there exist $u', v' \in \mathbf{V}^*$ such that $(u, u') \in \alpha_*$, $(v, v') \in \alpha_*$ and $(u', v') \in \sigma(y')$. Now we have $(y', y) \in \beta' = \alpha' = \alpha\mathbf{I}\alpha$ by 3.9. Then there exist $y'' \in \alpha[y]$ and $(u'', v'') \in \in \sigma(y'')$ such that $(u', u'') \in \alpha_*$, $(v', v'') \in \alpha_*$. Since $\alpha \subseteq \beta$, we have $\alpha[y] \subseteq \beta[y]$ and therefore $y'' \in \beta[y]$. Thus we obtain $(u, u'') \in \alpha_*$, $(v, v'') \in \alpha_*$, $(u'', v'') \in \sigma(y'')$, and $y'' \in \in \beta[y]$, i.e., $\alpha_* \circ \{(u, v)\} \circ \alpha_* \cap \Sigma_\beta(y) \neq \emptyset$.

Hence condition 1^0 from the definition of the operation **I** is satisfied for the pair (α, β). Similarly condition 2^0 from the definition of the operation **I** is satisfied as well. Thus, $x(\alpha\mathbf{I}\beta')\, y$ implies $x(\alpha\mathbf{I}\beta)\, y$. Therefore, $\alpha' = \alpha\mathbf{I}\alpha = \alpha\mathbf{I}\alpha' = \alpha\mathbf{I}\beta'$ by 3.5 and 3.9.

We have proved that $\alpha' = \alpha\mathbf{I}\beta$.

3.11. Theorem. *Let* $(\mathbf{V}, \mathbf{L})$ *be a language,* $\alpha \in \mathbf{E}(\mathbf{V})$ *such that* $\alpha \subseteq id'_\mathbf{V}$. *Then* $\alpha' = = id'_\mathbf{V}$.

Proof: The condition $id_\mathbf{V} \subseteq \alpha \subseteq id'_\mathbf{V}$ holds. By 3.9, we obtain $\alpha' = id'_\mathbf{V}$.

4. Trybulec's operator

Let $(\mathbf{V}, \mathbf{L})$ *be a language,* $\alpha \in \mathbf{E}(\mathbf{V})$ *an arbitrary equivalence relation. We put* $\alpha^{\mathbf{T}} = \alpha \mathbf{I} id_{\mathbf{V}}$. *Operator* $\mathbf{T}$ *is called Trybulec's operator.*

Let $(\mathbf{V}, \mathbf{L})$ be a fragment of Czech language, α an equivalence relation on $\mathbf{V}$, whose blocks are paradigms. Let $x, y \in \mathbf{V}$. Then any word-forms x, y are in the same block of $a^{\mathbf{T}}$, iff the following conditions hold:

1. In each correct sentence containing x, each word-form different from x can be replaced by a word-form from the same paradigm and x can be replaced by y in such a way that a correct sentence is obtained after these replacements.

2. In each correct sentence containing y, each word-form different from y can be replaced by a word-form from the same paradigm and y can be replaced by x in such a way that a correct sentence is obtained after these replacements.

4.1. Example. Let us have $u_1 =$ *hodný*, $u_2 =$ *hodná*, $v_1 =$ *chlapec*, $v_2 =$ *dívka*, $w_1 =$ *zpíval*, $w_2 =$ *zpívala*, $\mathbf{V} = \{u_1, u_2, v_1, v_2, w_1, w_2\}$. Then the set $\mathbf{L}$ of all grammatically correct sentences consists of some strings xyz where $x \in \{u_1, u_2\}$, $y \in \{v_1, v_2\}$, $z \in \{w_1, w_2\}$ and is defined as follows $\mathbf{L} = \{u_1 v_1 w_1, u_2 v_2 w_2\}$. Let α be an equivalence relation whose blocks are paradigms. Then $\mathbf{V}/\alpha = \{\{u_1, u_2\}, \{v_1\}, \{v_2\}, \{w_1, w_2\}\}$.

We construct $\alpha^{\mathbf{T}} = \alpha \mathbf{I} id_{\mathbf{V}}$. By a detailed analysis it can be proved that $\mathbf{V}/\alpha^{\mathbf{T}} = \{\{u_1\}, \{u_2\}, \{v_1, v_2\}, \{w_1\}, \{w_2\}\}$.

4.2. Theorem. *Let* $(\mathbf{V}, \mathbf{L})$ *be a language,* $\alpha, \beta \in \mathbf{E}(\mathbf{V})$. If $\alpha \subseteq \beta$ *holds then* $\alpha^{\mathbf{T}} \subseteq \beta^{\mathbf{T}}$.

Proof: Clearly $\alpha^{\mathbf{T}} = \alpha \mathbf{I} id_{\mathbf{V}} \subseteq \beta \mathbf{I} id_{\mathbf{V}} = \beta^{\mathbf{T}}$ holds by 2.8.

4.3. Theorem. *Let* $(\mathbf{V}, \mathbf{L})$ *be a language,* $\alpha \in \mathbf{E}(\mathbf{V})$ *an arbitrary equivalence relation. Then* $id'_{\mathbf{V}} \subseteq \alpha^{\mathbf{T}}$ *holds.*

Proof: We have $id'_{\mathbf{V}} = id_{\mathbf{V}} \mathbf{I} id_{\mathbf{V}} \subseteq \alpha \mathbf{I} id_{\mathbf{V}} = \alpha^{\mathbf{T}}$ by 2.7.

4.4. Theorem. *Let* $(\mathbf{V}, \mathbf{L})$ *be a language,* $\alpha \in \mathbf{E}(\mathbf{V})$. *If* $\alpha \subseteq id_{\mathbf{V}}$, *then* $\alpha^{\mathbf{T}} = id'_{\mathbf{V}}$.

Proof: By 4.3, we have $id'_{\mathbf{V}} \subseteq \alpha^{\mathbf{T}}$. Since $\alpha \subseteq id'_{\mathbf{V}}$, we obtain $\alpha^{\mathbf{T}} = \alpha \mathbf{I} id'_{\mathbf{V}} \subseteq id'_{\mathbf{V}} \mathbf{I} id_{\mathbf{V}} = id_{\mathbf{V}}$ proved that by 3.6, and 2.7. We have $\alpha^{\mathbf{T}} = id'_{\mathbf{V}}$.

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Souhrn

O JEDNÉ DEFINICI GRAMATICKÝCH KATEGORIÍ

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V práci je zavedena binární operace na množině všech ekvivalencí slovníku jazyka, která uspořádané dvojici ekvivalencí přiřazuje jednoznačně další ekvivalenci. Na příkladech se ukazuje, že zavedená operace není ani asociativní ani komutativní. Má několik jednoduchých vlastností, které jsou popsány v kapitole 2.

V kapitole 3 a 4 jsou pomocí zavedené operace definovány operátory Kulaginové a Trybulcův. Ukazuje se, že operátor Kulaginové dostaneme, volíme-li dvojici ekvivalencí tak, že jsou si rovny. Konstrukce Trybulcova je zahrnut také jako zvláštní případ, volí-li se druhá ekvivalence rovna identitě.

Na příkladech je ukázán způsob, jakým lze pomocí zavedených konstrukcí generovat „velké" gramatické kategorie na jazyku s paradigmatickou strukturou, které přibližně odpovídají slovním druhům.

Резюме

ОБ ОДНОМ ОПРЕДЕЛЕНИИ ГРАММАТИЧЕСКИХ КАТЕГОРИЙ

ФРАНТИШЕК ЗЕДНИК

В работе приводится бинарная операция на множестве всех эквивалентностей словаря языка, которая к упорядоченной паре эквивалентностей присоединяет однозначную дальнейшую эквивалентность. На отдельных примерах показано, что введеная операция ни ассоциативна, ни коммутативна. Она отличается несколькими свойствами, описанными в главе 2.

В главе 3 и 4 при помощи введенной операции определены операторы Кулагиной и Трыбульца. Оказывается, что оператор Кулагиной мы получаем при избрании пары равных эквивалентностей. Построение Трыбульца также включается в качестве особого случая, когда вторая эквиваленция в паре избирается равной тождеству.

В статье на примерах показан способ, которым при помощи введенных строений можно образовать „большие" грамматические категории на языке с парадигматической структурой которые приблизительно соответствуют частям речи.