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ON THE DISPERSION STRUCTURE OF THE DIFFERENTIAL EQUATION $y'' = q(t)\,y$ HAVING IN THE GENERALIZED FLOQUET THEORY THE SAME CHARACTERISTIC MULTIPLJERS

SVATOSLAV STANĚK

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Dedicated to Academician O. Borůvka on his 80th birthday

1. Introduction

If X is a dispersion of the both-sided oscillatory equation (q) : $y'' = q(t)\,y$, $q \in C^{\circ}(\mathbf{R})$, $\mathbf{R} = (-\infty, \infty)$ and u is an arbitrary solution of (q), then $\frac{uX(t)}{\sqrt{|X'(t)|}}$ represents again a solution of this equation on $\mathbf{R}$ (see [1, 2]). On this basis generalized M. Laitoch in [3] the classical Floquet theory even for the case when the coefficient q is not a periodic function in (q). To every equation (q) and a dispersion X of (q) may be (uniquely) associated with an algebraic quadratic equation, whose roots are called the characteristic multipliers of (q) relative to the dispersion X (see [4]). The values of the characteristic multipliers of (q) relative to the dispersion X are dependent on the choice of the dispersion X of (q). The object of the present paper is to investigate the structure of dispersions of (q) under which this equation has the same characteristic multipliers.

2. Basic concepts, properties and notation

We shall be concerned with equations of the type

$$y'' = q(t)\,y, \qquad q \in C^{\circ}(\mathbf{R}), \tag{q}$$

which are both-sided oscillatory on $\mathbf{R}$.

We say that a function $\alpha : \mathbf{R} \to \mathbf{R}$, $\alpha \in C^{\circ}(\mathbf{R})$ is a (first) phase of (q) if there exist independent solutions u, v of (q) with

$$\operatorname{tg} \alpha(t) = \frac{u(t)}{v(t)} \qquad \text{on } \mathbf{R} - \{t \in \mathbf{R}; v(t) = 0\}.$$

We use the symbol $\mathfrak{E}$ to denote the set of all phases of the equation $y'' = -y$. This set constitutes a group relative to the composition of functions.

Let n be an integer and let α be a phase of (q). The function $\varphi_n(t) := \alpha^{-1}[\alpha(t) + n\pi \,.\, \operatorname{sign} \alpha']$, $t \in \mathbf{R}$, is then independent of the choice of the phase α and is called the central dispersion (of the 1st kind) of (q) with the index n.

A function X, $X \in C^3(\mathbf{R})$, $X' \neq 0$ representing a solution (on $\mathbf{R}$) of the nonlinear equation

$$\sqrt{|X'|}\left(\frac{1}{\sqrt{|X'|}}\right)'' + X'^2 \,.\, q(X) = q(t),$$

is called the dispersion (of the 1st kind) of (q). If α is a phase of (q), then $\alpha^{-1}\mathfrak{E}\alpha := \{\alpha^{-1}\varepsilon\alpha; \varepsilon \in \mathfrak{E}\}$ represents the set of all dispersions of (q). If X is a dispersion of (q) and u is its solution, then $\dfrac{uX(t)}{\sqrt{|X'(t)|}}$ is again a solution of (q).

All the definitions and properties cited above were established and proved in [1].

Let X be a dispersion of (q). Following the generalized Floquet theory ([3] – [5]), there exist independent solutions u, v of (q) satisfying either

$$\frac{uX(t)}{\sqrt{|X'(t)|}} = \varrho_{-1} \,.\, u(t), \qquad \frac{vX(t)}{\sqrt{|X'(t)|}} = \varrho_1 \,.\, v(t), \qquad \varrho_{-1} \,.\, \varrho_1 = \operatorname{sign} X', \tag{1}$$

or

$$\frac{uX(t)}{\sqrt{|X'(t)|}} = \varrho_{-1} \,.\, u(t), \qquad \frac{vX(t)}{\sqrt{|X'(t)|}} = u(t) + \varrho_1 \,.\, v(t), \qquad \varrho_{-1} = \varrho_1 = \pm 1. \tag{2}$$

(Generally complex) numbers ϱ_{-1}, ϱ_1 are called the *characteristic multipliers of* (q) *relative to the dispersion* X ([4]).

Let X and φ_n be a dispersion and the central dispersion with the index n, respectively, both of (q). It has been proved in [4]:

a) If $\operatorname{sign} X' = 1$ and if for an $x_0 \in \mathbf{R}$ and for an integer n $X(x_0) = \varphi_n(x_0)$, then $(-1)^n \sqrt{\dfrac{\varphi_n'(x_0)}{X'(x_0)}}$ and $(-1)^n \sqrt{\dfrac{X'(x_0)}{\varphi_n'(x_0)}}$ are (real) characteristic multipliers of (q) relative to the dispersion X. In such a case $(1, n)$ is said to be the category of (q) relative to the dispersion X.

b) The characteristic multipliers of (q) relative to the dispersion X are complex and equal to $e^{\pm a\pi i}(0 < a < 1)$ precisely if for a phase α of (q) and for an integer n:

$$\alpha X(t) = \alpha(t) + (a + 2n)\pi.$$

In such a case $(2, n)$ is said to be the category of (q) relative to the dispersion X.

c) If $\operatorname{sign} X' = -1$ and $X(x_0) = x_0$, then $-\sqrt{|X'(x_0)|}$ and $\frac{1}{\sqrt{|X'(x_0)}}$ are the characteristic multipliers of (q) relative to the dispersion X. In such a case (3,0) said to be the category of (q) relative to the dispersion X.

Remark 1. The definitions of categories (ι, n), $\iota = 1, 2, 3$ of (q) relative to the dispersion X were introduced first in [5].

3. Main results

Theorem 1. *Let α be a phase of* (q), *let ϱ_{-1}, ϱ_1 be real numbers, $\varrho_{-1}, \varrho_1 = 1$, $\varrho_{-1} \neq \varrho_1$ and n be an integer. Denote by $\mathfrak{E}_1$ the set of all $\varepsilon \in \mathfrak{E}$ such that* $\operatorname{sign} \varepsilon' = 1$ *and there exists $\tau = \tau(\varepsilon) \in \mathbf{R}$: $\varepsilon(\tau) = \tau + n\pi \,.\, \operatorname{sign} \alpha'$ and $(-1)^n \sqrt{\varepsilon'(\tau)}$ is equal to one of the numbers ϱ_{-1}, ϱ_1. Then $\alpha^{-1}\mathfrak{E}_1\alpha$ represents the set of all dispersions of* (q), *where* (q) *relative to the dispersion $X \in \alpha^{-1}\mathfrak{E}_1\alpha$ has the characteristic multipliers ϱ_{-1} and ϱ_1 and the category* $(1, n)$.

Proof. Suppose that all assumptions of Theorem 1 are satisfied. Let $X \in \alpha^{-1}\mathfrak{E}_1\alpha$, thus $X = \alpha^{-1}\varepsilon\alpha$ with $\varepsilon \in \mathfrak{E}_1$. Then $\operatorname{sign} X' = \operatorname{sign} \varepsilon' = 1$, there exists $\tau = \tau(\varepsilon) \in \mathbf{R}$: $\varepsilon(\tau) = \tau + n\pi \,.\, \operatorname{sign} \alpha'$ and $(-1)^n \sqrt{\varepsilon'(\tau)}$ equals to one of the numbers ϱ_{-1}, ϱ_1. Let $x_0 := \alpha^{-1}(\tau)$. Then $X(x_0) = \alpha^{-1}\varepsilon\alpha(x_0) = \alpha^{-1}\varepsilon(\tau) = \alpha^{-1}[\tau + n\pi \,.\, \operatorname{sign} \alpha'] = \alpha^{-1}[\alpha(x_0) + n\pi \,.\, \operatorname{sign} \alpha'] = \varphi_n(x_0)$. Consequently $(1, n)$ is the category of (q) relative to the dispersion X and $(-1)^n \sqrt{\frac{X'(x_0)}{\varphi_n'(x_0)}} = (-1)^n \sqrt{\frac{\alpha^{-1'}\varepsilon(\tau) \,.\, \varepsilon'(\tau) \,.\, \alpha'(x_0)}{\alpha^{-1'}(\tau + n\pi \,.\, \operatorname{sign} \alpha') \,.\, \alpha'(x_0)}} = (-1)^n \sqrt{\varepsilon'(\tau)}$ is the value of one of the characteristic multipliers of (q) relative to the dispersion X. Thus ϱ_{-1}, ϱ_1 are the characteristic multipliers of (q) relative to the dispersion X.

Let X be a dispersion of (q) and let (q) relative to the dispersion X have the category $(1, n)$ with ϱ_{-1}, ϱ_1 being its characteristic multipliers. Let $X = \alpha^{-1}\varepsilon\alpha$, $\varepsilon \in \mathfrak{E}$. Then $\operatorname{sign} \varepsilon' = \operatorname{sign} X' = 1$ and there exists $x \in \mathbf{R}$: $X(x) = \varphi_n(x)$. Thus $\alpha^{-1}\varepsilon\alpha(x) = \alpha^{-1}[\alpha(x) + n\pi \,.\, \operatorname{sign} \alpha']$. Consequently $\varepsilon(\tau) = \tau + n\pi \,.\, \operatorname{sign} \alpha'$ for $\tau := \alpha(x)$. Since $(-1)^n \sqrt{\frac{X'(x)}{\varphi_n'(x)}} = (-1)^n \sqrt{\frac{\alpha^{-1'}\varepsilon(\tau) \,.\, \varepsilon'(\tau) \,.\, \alpha'(x)}{\alpha^{-1'}[\alpha(x) + n\pi \,.\, \operatorname{sign} \alpha'] \,.\, \alpha'(x)}} = (-1)^n \sqrt{\varepsilon'(\tau)}$ is one of the characteristic multipliers of (q) relative to the dispersion X, we obtain $\varepsilon \in \mathfrak{E}_1$.

Theorem 2. *Let n be an integer. Then there exists exactly one dispersion $X(= \varphi_n)$ of* (q) *under which* (q) *has the category* $(1, n)$ *and possesses two equal characteristic multipliers (equal to* $(-1)^n$*). There exist independent solutions u, v of* (q) *satisfying* (1).

The proof immediately follows from Lemma 5 [4] and from Lemma 1 [5].

Theorem 3. *Let α be a phase of* (q) *and let n be an integer. Let $\mathfrak{E}_2$ be the set of all $\varepsilon \in \mathfrak{E}$ such that* $\operatorname{sign} \varepsilon' = 1$, $\varepsilon(t) \not\equiv t + n\pi \,.\, \operatorname{sign} \alpha'$ *and there exists* $\tau = \tau(\varepsilon) \in \mathbf{R}$: $\varepsilon(\tau) = \tau + n\pi \,.\, \operatorname{sign} \alpha'$ *with* $\varepsilon'(\tau) = 1$. *Then $\alpha^{-1}\mathfrak{E}_2\alpha$ is the set of those dispersions of* (q), *where* (q) *relative to the dispersion $X \in \alpha^{-1}\mathfrak{E}_2\alpha$ possesses two equal characteristic multipliers (equal to* $(-1)^n$*) and the category* $(1, n)$ *and there exist independent solutions u, v of* (q) *satisfying* (2).

Proof. Suppose that the assumptions of Theorem 3 are satisfied. Let $X \in \alpha^{-1}\mathfrak{E}_2\alpha$. Thus $X = \alpha^{-1}\varepsilon\alpha$ for an $\varepsilon \in \mathfrak{E}_2$. Then $\operatorname{sign} X' = \operatorname{sign} \varepsilon' = 1$ and there exists $\tau = \tau(\varepsilon) \in \mathbf{R}$: $\varepsilon(\tau) = \tau + n\pi \,.\, \operatorname{sign} \alpha'$ with $\varepsilon'(\tau) = 1$ and $\varepsilon(t) \not\equiv t + n\pi \,.\, \operatorname{sign} \alpha'$. For $x_0 := \alpha^{-1}(\tau)$ we obtain $X(x_0) = \alpha^{-1}\varepsilon\alpha(x_0) = \alpha^{-1}\varepsilon(\tau) = \alpha^{-1}[\alpha(x_0) + n\pi \,.\, \operatorname{sign} \alpha'] = \varphi_n(x_0)$. Hence $(1, n)$ is the category of (q) relative to the dispersion X. Next $(-1)^n \sqrt{\dfrac{X'(x_0)}{\varphi_n'(x_0)}} = (-1)^n \sqrt{\varepsilon'(\tau)} = (-1)^n$, therefore (q) relative to the dispersion X possesses two equal characteristic multipliers (equal to $(-1)^n$). Since $X = \alpha^{-1}\varepsilon\alpha \not\equiv \alpha^{-1}[\alpha + n\pi \,.\, \operatorname{sign} \alpha'] = \varphi_n$, it follows from Lemma 7 [5] the existence of solutions u, v of (q) for which (2) applies.

Let X be a dispersion of (q) and let (q) relative to the dispersion possess two equal characteristic multipliers and the category $(1, n)$. Besides, let there exist independent solutions u, v of (q) for which (2) applies. Let $X = \alpha^{-1}\varepsilon\alpha$, $\varepsilon \in \mathfrak{E}$. Then $\operatorname{sign} \varepsilon' = \operatorname{sign} X' = 1$ and there exists $x \in \mathbf{R}$: $X(x) = \varphi_n(x)$. Consequently $\alpha^{-1}\varepsilon\alpha(x) = \alpha^{-1}[\alpha(x) + n\pi \,.\, \operatorname{sign} \alpha']$. For $\tau := \alpha(x)$ we have $\varepsilon(\tau) = \tau + n\pi \,.\, \operatorname{sign} \alpha'$. Since $(-1)^n \sqrt{\dfrac{X'(x)}{\varphi_n'(x)}} = (-1)^n \sqrt{\varepsilon'(\tau)} = (-1)^n$, we have $\varepsilon'(\tau) = 1$ and it follows from $X \not\equiv \varphi_n$ that $\varepsilon(t) \not\equiv t + n\pi \,.\, \operatorname{sign} \alpha'$. Therefore $\varepsilon \in \mathfrak{E}_2$.

Theorem 4. *Let α be a phase of* (q), *let n be an integer and* $0 < a < 1$. *Let* $\mathfrak{E}_3 := \{\varepsilon^{-1}[\varepsilon(t) + (a + 2n)\pi];\ \varepsilon \in \mathfrak{E}\}$. *Then $\alpha^{-1}\mathfrak{E}_3\alpha$ are all dispersions* of (q) *where* (q) *relative to the dispersion $X \in \alpha^{-1}\mathfrak{E}_3\alpha$ possesses the category* $(2, n)$ *and* $e^{\pm a\pi i}$ *are its characteristic multipliers.*

Proof. Suppose the assumptions of Theorem 4 to be satisfied. Let $X \in \alpha^{-1}\mathfrak{E}_3\alpha$. Then $X = \alpha^{-1}\varepsilon_1\alpha$ with $\varepsilon_1(t) = \varepsilon^{-1}[\varepsilon(t) + (a + 2n)\pi]$ for an $\varepsilon \in \mathfrak{E}$. Further $\varepsilon\alpha$ is a phase of (q) and $\varepsilon\alpha X = \varepsilon\alpha\alpha^{-1}\varepsilon^{-1}[\varepsilon\alpha + (a + 2n)\pi] = \varepsilon\alpha + (a + 2n)\pi$. Consequently (q) relative to the dispersion X possesses the category $(2, n)$ and $e^{\pm a\pi i}$ are its characteristic multipliers.

Let X be a dispersion of (q), let $(2, n)$ be the category and $e^{\pm a\pi i}$ the characteristic multipliers of (q) relative to the dispersion X. Then there exists a phase α_1 of (q):

$\alpha_1 X = \alpha_1 + (a + 2n)\pi$. Since $X = \alpha^{-1}\varepsilon_1\alpha$ and $\alpha_1 = \varepsilon\alpha$ for an $\varepsilon \in \mathfrak{E}$ and $\varepsilon_1 \in \mathfrak{E}$, we have $\alpha_1 X = \varepsilon\alpha\alpha^{-1}\varepsilon_1\alpha = \varepsilon\varepsilon_1\alpha = \varepsilon\alpha + (a + 2n)\pi$. Thus $\varepsilon\varepsilon_1 = \varepsilon + (a + 2n)\pi$, $\varepsilon_1 = \varepsilon^{-1}[\varepsilon + (a + 2n)\pi]$ and $\varepsilon_1 \in \mathfrak{E}_3$.

Theorem 5. *Let α be a phase of* (q), *let ϱ_{-1}, ϱ_1 be real numbers, $\varrho_{-1} \cdot \varrho_1 = -1$ and let n be a positive integer. Denote by $\mathfrak{E}_4$ the set of all $\varepsilon \in \mathfrak{E}$ for which* sign $\varepsilon' = -1$ *and $-\sqrt{|\varepsilon'(x_0)|}$ is equal to one of the numbers ϱ_{-1}, ϱ_1, where $x_0 = \varepsilon(x_0)$. Then $\alpha^{-1}\mathfrak{E}_4\alpha$ are all dispersions of* (q), *where* (q) *relative to the dispersion $X \in \alpha^{-1}\mathfrak{E}_4\alpha$ possesses the category* (3, 0) *with ϱ_{-1}, ϱ_1 being its characteristic multipliers.*

Proof. Suppose the assumptions of Theorem 5 to be satisfied. Let $X \in \alpha^{-1}\mathfrak{E}_4\alpha$, thus $X = \alpha^{-1}\varepsilon\alpha$ with $\varepsilon \in \mathfrak{E}_4$. Then sign $\varepsilon' = -1$. Let $x_0 = \varepsilon(x_0)$. By Corollary 6 [4], equation (q) relative to the dispersion X possesses the characteristic multipliers $-\sqrt{-\varepsilon'(x_0)}$ and $\dfrac{1}{\sqrt{-\varepsilon'(x_0)}}$. Consequently (q) relative to the dispersion X has the category (3, 0) with the characteristic multipliers ϱ_{-1}, ϱ_1.

Let X be a dispersion of (q) and let (q) relative to the dispersion X have the category (3, 0) with ϱ_{-1}, ϱ_1 being its characteristic multipliers. Let $X = \alpha^{-1}\varepsilon\alpha$, $\varepsilon \in \mathfrak{E}$. Then sign $\varepsilon' =$ sign $X' = -1$. There exists further (exactly one) $x \in \mathbf{R}$: $X(x) = x$. From this $\varepsilon(x_0) = x_0$ for $x_0 := \alpha(x)$ and $-\sqrt{-X'(x)} = -\sqrt{-\alpha^{-1\prime}\varepsilon(x_0) \, . \, \varepsilon'(x_0) \, . \, \alpha(x)} = -\sqrt{-\dfrac{\alpha'(x) \, . \, \varepsilon'(x_0)}{\alpha'\alpha^{-1}(x_0)}} = -\sqrt{-\varepsilon'(x_0)}$. Therefore $-\sqrt{-\varepsilon'(x_0)}$ is one of the characteristic multipliers of (q) relative to the dispersion X. Hence $\varepsilon \in \mathfrak{E}_4$.

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Souhrn

STRUKTURA DISPERSÍ DIFERENCIÁLNÍ ROVNICE $y'' = q(t)\,y$ PŘI NICHŽ V ZOBECNĚNÉ FLOQUETOVĚ TEORII MÁ TATO ROVNICE STEJNÉ CHARAKTERISTICKÉ KOŘENY

SVATOSLAV STANĚK

Nechť diferenciální rovnice

$$y'' = q(t)\,y, \qquad q \in C^{\circ}(\mathbf{R}) \tag{q}$$

je oboustranně oscilatorická na **R** a nechť X je řešení (na **R**) nelineární diferenciální rovnice

$$\sqrt{|X'|}\left(\frac{1}{\sqrt{|X'|}}\right)'' + X'^2 \,.\, q(X) = q(t).$$

Řešení X se nazývá disperse (1. druhu) rovnice (q). V zobecněné Floquetově teorii k rovnici (q) a dispersi X rovnice (q) lze (jednoznačně) přiřadit jistou kvadratickou algebraickou rovnici, jejíž kořeny se nazývají charakteristické kořeny rovnice (q) při dispersi X. Hodnoty charakteristických kořenů rovnice (q) při dispersi X závisí na výběru disperse X. V práci jsou vyšetřeny všechny disperse X při nichž rovnice (q) má stejné charakteristické kořeny.

Резюме

СТРУКТУРА ДИСПЕРСИЙ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ $y'' = q(t)\,y$ ОТНОСИТЕЛЬНО КОТОРЫХ ЭТО УРАВНЕНИЕ ИМЕЕТ ОДИНАКОВЫЕ КОРНИ В ОБОБЩЕННОЙ ТЕОРИИ ФЛОКЕ

СВАТОСЛАВ СТАНЕК

Пусть дифференциальное уравнение

$$y'' = q(t)\,y, \quad q \in C^{\circ}(\mathbf{R}) \tag{q}$$

колеблющееся на **R**. Если X решение на **R** нелинейного дифференциального уравнения

$$\sqrt{|X'|}\left(\frac{1}{\sqrt{|X'|}}\right)'' + X'^2 \,.\, q(X) = q(t),$$

то X называется дисперсией (1-го рода) уравнения (q). В обобщенной теории Флоке к уравнению (q) и дисперсии X уравнения (q) присоединяется квадратическое алгебраическое уравнение. Корни этого уравнения называются характеристические корни уравнения (q) относительно дисперсии X. Значения характеристических корней уравнения (q) относительно дисперсии X зависят от выбора дисперсии X. В этой статье исследуются все дисперсии уравнения (q) относительно которых это уравнение имеет одинаковые характеристические корни.