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Sborník prací Přírodovědecké fakulty University Palackého v Olomouci. Matematika, Vol. 18 (1979), No. 1, 59--71

Persistent URL: `http://dml.cz/dmlcz/120083`

1979 — ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM — TOM 61

*Katedra matematické analýzy a numerické matematiky
přírodovědecké fakulty Univerzity Palackého v Olomouci
Vedoucí katedry: prof. RNDr. Miroslav Laitoch, CSc.*

TWO-POINT BOUNDARY PROBLEM IN A SECOND ORDER NONHOMOGENEOUS LINEAR DIFFERENTIAL EQUATION

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(Received September 19, 1977)

Dedicated to Academician O. Borůvka on his 80th birthday

1. In this paper we investigate an equation

$$y'' - q(t, \lambda)\, y = r(t, \mu), \qquad q \in C^0(D), \qquad r \in C^0(D), \tag{1}$$

with $D = j \times \mathbf{R}$, $j = (a, b)$ $(-\infty \leqq a < b \leqq \infty)$, $\mathbf{R} = (-\infty, \infty)$, comprising two parameters λ and μ. Let (x_0, y_0) and (x_1, y_1) be arbitrary points of D, $x_0 < x_1$. The object of this article is:

a) to find sufficient conditions for the existence of the solution y of (1) where $y(x_0) = y_0$, $y(x_1) = y_1$ and for the relative homogeneous equation

$$y'' = q(t, \lambda)\, y, \qquad q \in C^0(D), \tag{2}$$

to have a nontrivial solution v such that $v(x_0) = v(x_1) = 0$ and $v(t) \neq 0$ for $t \in (x_0, x_1)$,

b) to find satisfactory conditions for the solution of the above problem, where instead of the solution y of (1) and of the solution v of (2) we consider the derivative of these solutions.

Besides, there is investigated the uniqueness of the solutions of both problems.

2. Basic definitions, relations and notation.

Let $x \in j$ and u be a nontrivial solution of

$$y'' = p(t)\, y, \qquad p \in C^0(j), \tag{p}$$

$u(x) = 0$. Denote by $\varphi(x)$ the first zero of the solution u (as far as such exists) lying to the right of the point x. The function φ is called the fundamental dispersion of the 1st kind of (p).

Let $p(t) < 0$ for $t \in j$ and let v be a nontrivial solution of (p), $v'(x) = 0$. Denote by $\psi(x)$ the first zero of the function v' (as far as such exists) lying to the right of the point x. The function ψ is called the fundamental dispersion of the 2nd kind of (p).

Let $p \in C^2(j)$, $p(t) < 0$ for $t \in j$. We set $p_1(t) := p(t) + \sqrt{|p(t)|}\left(\frac{1}{\sqrt{|p(t)|}}\right)''$, $t \in j$. Then the 2nd kind fundamental dispersion of (p) is equal to the 1st kind fundamental dispersion of (p_1): $y'' = p_1(t)\, y$. For more details see [1, 2].

Throughout the functions $\varphi(t)$ and $\psi(t)$ ($\varphi(t, \lambda)$ and $\psi(t, \lambda)$) will denote the fundamental dispersions of the 1st and 2nd kinds of the equation (p) (the equation (2)), respectively.

If for any λ_1 and λ_2 holds that $q(t, \lambda_1) < q(t, \lambda_2)$ for $t \in j$, then we conclude from the Sturm comparison theorem that $\varphi(t, \lambda_1) < \varphi(t, \lambda_2)$ for t from the interval of definition of the function $\varphi(t, \lambda_2)$. This set may be also empty.

It follows from [5, 6]: Let $x_0 \in j$, y_0, y_0' be arbitrary numbers. Let u_1 and u_2 be two different solutions of

$$y'' - p(t)\, y = f(t), \qquad p \in C^0(j), \qquad f \in C^0(j), \qquad f(t) \not\equiv 0, \tag{3}$$

satisfying the condition $u_1(x_0) = u_2(x_0) = y_0$, and the 1st kind fundamental dispersion φ of (p) be defined at x_0. Then $u_1(t) \neq u_2(t)$ for $t \in (x_0, \varphi(x_0))$ and $u_1(\varphi(x_0)) = u_2(\varphi(x_0)) := y_1$. In this case the points (x_0, y_0) and $(\varphi(x_0), y_1)$ are called the *neighbouring knots of the 1st kind relative to* (3) *and to the initial condition* (x_0, y_0). Let $p(t) < 0$ for $t \in j$. Let v_1 and v_2 be two different solutions of (3) satisfying the condition $v_1'(x_0) = v_2'(x_0) = y_0'$ and the 2nd kind fundamental dispersion ψ of (p) be defined at x_0. Then $v_1'(t) \neq v_2'(t)$ for $t \in (x_0, \psi(x_0))$ and $v_1'(\psi(x_0)) = v_2'(\psi(x_0)) := := y_1'$. In this case the points (x_0, y_0') and $(\psi(x_0), y_1')$ are called the *neighbouring knots of the 2nd kind relative to* (3) *and to the initial condition* (x_0, y_0').

Convention. Throughout this article we use ' to denote the derivative with respect to the independent variable t to shorten the writing even in case of functions examined being of two independent variables.

In what follows we will occasionally investigate the function $q(t, \lambda)$ for which one of the following assumptions applies:

(i) $q \in C^0(D)$, $q(t, \lambda_1) < q(t, \lambda_2)$ for $\lambda_1 < \lambda_2$, $t \in j$ and

$$\lim_{\lambda \to -\infty} q(t, \lambda) = -\infty, \qquad \lim_{\lambda \to \infty} q(t, \lambda) = \infty, \qquad t \in j \tag{4}$$

(ii) $q(t, \lambda) \equiv \lambda p(t)$, $p \in C^0(j)$ and $p(t) \not\equiv 0$ on every interval ($\subset j$);

(iii) $q \in C^0(D)$, $q''(t, \lambda) \in C^0(D)$, $q(t, \lambda) < 0$ for $(t, \lambda) \in D$,

$$\begin{aligned} &\lim_{\lambda \to -\infty} \left(q(t, \lambda) + \sqrt{|q(t, \lambda)|}\left(\frac{1}{\sqrt{|q(t, \lambda)|}}\right)'' \right) = -\infty \\ &\lim_{\lambda \to \infty} \left(q(t, \lambda) + \sqrt{|q(t, \lambda)|}\left(\frac{1}{\sqrt{|q(t, \lambda)|}}\right)'' \right) = \infty, \qquad t \in j \end{aligned} \tag{5}$$

and

$$q(t, \lambda_1) + \sqrt{|q(t, \lambda_1)|}\left(\frac{1}{\sqrt{|q(t, \lambda_1)|}}\right)'' < q(t, \lambda_2) +$$

$$+ \sqrt{|q(t, \lambda_2)|}\left(\frac{1}{\sqrt{|q(t, \lambda_2)|}}\right)''$$

for $\lambda_1 < \lambda_2$, $t \in j$;

(iv) $q \in C^0(D)$, $q''(t, \lambda) \in C^0(D)$, $q(t, \lambda) < 0$ for $(t, \lambda) \in D$,

$$q(t, y) + \sqrt{|q(t, \lambda)|}\left(\frac{1}{\sqrt{|q(t, \lambda)|}}\right)'' \equiv \lambda p(t)$$

and $p(t) \not\equiv 0$ on every interval $(\subset j)$.

Lemma 1. Let $x_0 \in j$, $x_1 \in j$ *be arbitrary numbers*, $x_0 < x_1$. *If the function q satisfies the assumption* (i), *then there exists exactly one number* $\lambda_0 : \varphi(x_0, \lambda_0) = x_1$. *If the function q satisfies the assumption* (ii) *and* $\inf \{x; x \in j, x > x_0, p(x) < 0\} < x_1$ ($\inf \{x; x \in j, x > x_0, p(x) > 0\} < x_1$), *then there exists exactly one* $\lambda_1 > 0$ ($\lambda_2 < 0$) *with* $\varphi(x_0, \lambda_1) = x_1(\varphi(x_0, \lambda_2) = x_1)$.

Proof. Following Lemma 1 [3] the function $\varphi(x_0, \lambda)$ is continuous on its interval of definition. If the function q satisfies the assumption (i), then $\varphi(x_0, \lambda)$ is an increasing function mapping the interval of definition onto (x_0, b). Hence, there exists exactly one number $\lambda_0 : \varphi(x_0, \lambda_0) = x_1$. Let q satisfy the assumption (ii) and $\inf\{x; x \in j, x > x_0, p(x) < 0\} < x_1$ ($\inf\{x; x \in j, x > x_0, p(x) > 0\} < x_1$). It follows from Theorem 1 [7] and from its proof that $\varphi(x_0, \lambda)$ is for $\lambda > 0$ for which $\varphi(x_0, \lambda)$ is defined (it is for $\lambda < 0$ for which $\varphi(x_0, \lambda)$ is defined) a decreasing (an increasing) function. The rest of the statement of the Lemma follows from Corollary 5.1. [4, p. 408] and from Corollary 1 [7].

Remark 1. Let the function q satisfy the assumption (ii). Then it follows from Lemma 1 that there always exists at least one number $\lambda_0 : \varphi(x_0, \lambda_0) = x_1$.

Corollary 1. *Let* $x_0 \in j$ $x_1 \in j$ *be arbitrary numbers,* $x_0 < x_1$. *If the function q satisfies the assumption* (iii), *then there exists exactly one number* $\lambda_0 : \psi(x_0, \lambda_0) = x_1$. *If the function q satisfies the assumption* (iv) *and* $\inf\{x; x \in j, x > x_0, p(x) < 0\} <$ $< x_1$ ($\inf\{x; x \in j, x > x_0, p(x) > 0\} < x_1$) *then there exists exactly one* $\lambda_1 > 0$ ($\lambda_2 < 0$) *with* $\psi(x_0, \lambda_1) = x_1$ ($\psi(x_0, \lambda_2) = x_1$).

The proof follows from Lemma 1 and from the fact that the 2nd kind fundamental dispersion of (2) is equal to the 1st kind fundamental dispersion of $y'' -$ $- \left(q(t, \lambda) + \sqrt{|q(t, \lambda)|}\left(\frac{1}{\sqrt{|q(t, \lambda)|}}\right)''\right) y = 0.$

Lemma 2. *Let $x_0 \in j$, y_0 be arbitrary numbers, $r \in C^0(D)$, let the function $\varphi(t)$ be defined at x_0 and let*

$$\lim_{\mu \to -\infty} r(t, \mu) = -\infty, \qquad \lim_{\mu \to \infty} r(t, \mu) = \infty \tag{6}$$

uniformly on the interval $\langle x_0, \varphi(x_0) \rangle$ $(\subset j)$. Let $u(t, \mu)$ be a solution of

$$y'' - p(t)\, y = r(t, \mu), \qquad p \in C^0(j), r \in C^0(D) \tag{7}$$

satisfying the condition $u(x_0, \mu) = y_0$. Setting

$$\mathcal{M} := \{u(\varphi(x_0), \mu);\ \mu \in \mathbf{R}\}, \tag{8}$$

then

$$\mathcal{M} = \mathbf{R}$$

Proof. Let $\mathcal{M}$ be the set defined by (8). It follows from the continuous dependence of solutions on the parameter that $\mathcal{M}$ is a convex set. To prove Lemma 2 it suffices to show that $\inf \mathcal{M} = -\infty$, $\sup \mathcal{M} = \infty$. We prove the second of the given equalities (the proof of the first one proceeds similarly). Let $\sup \mathcal{M} = L < \infty$. Let y_1, y_2 be solutions of (p) satisfying the initial conditions $y_1(x_0) = y_2'(x_0) = 0$, $y_1'(x_0) = y_2(x_0) = 1$. Then $y_1(\varphi(x_0)) = 0$, $y_2(\varphi(x_0)) < 0$. We set

$$k := -\frac{L + 1 - y_0 \,.\, y_2(\varphi(x_0))}{y_2(\varphi(x_0))} \Big(\int_{x_0}^{\varphi(x_0)} y_1(t)\, \mathrm{d}t \Big)^{-1}.$$

According to the assumption there holds (6) uniformly on the interval $\langle x_0, \varphi(x_0) \rangle$ and consequently there exists $\mu_0 \in \mathbf{R}$ such that $r(t, \mu_0) > k$ for $t \in \langle x_0, \varphi(x_0) \rangle$. Let v be a solution of the equation $y'' - p(t)\, y = k$, $v(x_0) = y_0$, $v'(x_0) = u'(x_0, \mu_0) := y_0'$. Then

$$v(t) = y_0 y_2(t) + y_0' y_1(t) + k \int_{x_0}^{t} (y_1(t)\, y_2(s) - y_1(s)\, y_2(t))\, \mathrm{d}s.$$

Setting $w(t) := u(t, \mu_0) - v(t)$, $t \in j$, then $w'' - p(t)\, w = r(t, \mu_0) - k$, $w(x_0) = w'(x_0) = 0$. Hence, by Theorem 1.1 [6] and its proof, we have $w(\varphi(x_0)) > 0$ and therefore $u(\varphi(x_0), \mu_0) > v(\varphi(x_0))$. We have next

$$v(\varphi(x_0)) = y_0 y_2(\varphi(x_0)) - k y_2(\varphi(x_0)) \int_{x_0}^{\varphi(x_0)} y_1(s)\, \mathrm{d}s = L + 1\,.$$

Thus $u(\varphi(x_0), \mu_0) > v(\varphi(x_0)) = L + 1$ contrary to the assumption $u(\varphi(x_0), \mu_0) \leqq L$.

Lemma 3. *Let $x_0 \in j$, y_0 be arbitrary numbers, let the function $\varphi(t)$ be defined at x_0 and let the function $r \in C^0(D)$ satisfy (6) for $t \in \langle x_0, \varphi(x_0) \rangle$ and*

$$r(t, \mu_1) < r(t, \mu_2) \qquad \text{for } \mu_1 < \mu_2 \text{ and } t \in \langle x_0, \varphi(x_0) \rangle \tag{9}$$

Let $u(t, \mu)$ be a solution of (7) *satisfying the condition $u(x_0, \mu) = y_0$. Then the function*

$$\alpha(\mu) := u(\varphi(x_0), \mu), \qquad \mu \in \mathbf{R} \tag{10}$$

is an increasing function on $\mathbf{R}$ *and* $\alpha(\mathbf{R}) = \mathbf{R}$.

Remark 2. As stated before, it follows from [5, 6], that all solutions $u(t, \mu)$ of (7) satisfying the condition $u(x_0, \mu) = y_0$ have equal values at the point $(\varphi(x_0), \mu)$ Evidently this implies that the function α is by relation (10) correctly defined.

The proof of Lemma 3. From assumptions (6) and (9) laid on the function then follows the uniformly convergence of (6) on the interval $\langle x_0, \varphi(x_0)\rangle$ whic implies by Lemma 2 that $\alpha(\mathbf{R}) = \mathbf{R}$, where the function α is defined by (10). Let μ_1, μ be arbitrary numbers $\mu_1 < \mu_2$. Let y_1, y_2 be solutions of (p), $y_1(x_0) = y_2'(x_0) = 0$ $y_1'(x_0) = y_2(x_0) = 1$. Then

$$u(t, \mu_1) = y_0 y_2(t) + u'(x_0, \mu_1)\, y_1(t) + \int_{x_0}^{t} \big(y_1(t)\, y_2(s) - y_1(s)\, y_2(t)\big)\, r(s, \mu_1)\, ds\,,$$

$$u(t, \mu_2) = y_0 y_2(t) + u'(x_0, \mu_2)\, y_1(t) + \int_{x_0}^{t} \big(y_1(t)\, y_2(s) - y_1(s)\, y_2(t)\big)\, r(s, \mu_2)\, ds$$

and assumption (9) and the inequalities $y_1(t) > 0$ for $t \in (x_0, \varphi(x_0))$, $y_2(\varphi(x_0)) < 0$ yield

$$\alpha(\mu_2) - \alpha(\mu_1) = u(\varphi(x_0), \mu_2) - u(\varphi(x_0), \mu_1) =$$
$$= -y_2(\varphi(x_0)) \int_{x_0}^{\varphi(x_0)} y_1(s)\big(r(s, \mu_2) - r(s, \mu_1)\big)\, ds > 0.$$

Consequently the function α is increasing on $\mathbf{R}$.

Remark 3. The function $r(t, \mu) \equiv f(t) + \mu$, $f \in C^0(j)$ satisfies the assumptions of Lemma 3.

Lemma 4. *Let $x_0 \in j$, y_0 be arbitrary numbers, $f \in C^0(j)$ and let the function $\varphi(t)$ be defined at x_0. Next let*

$$\int_{x_0}^{\varphi(x_0)} f(t)\, y_1(t)\, dt \neq 0\,,$$

where y_1 is a solution of (p), $y_1(x_0) = 0$, $y_1'(x_0) = 1$. *Let $u(t, \mu)$ be a solution of*

$$y'' - p(t)\, y = \mu f(t)$$

satisfying the condition $u(x_0, \mu) = y_0$. Then the function $\alpha(\mu)$ defined by (10) *is strictly monotone on* $\mathbf{R}$, $\alpha(\mathbf{R}) = \mathbf{R}$, $\alpha' \neq 0$.

Proof. Let y_2 be a solution of (p), $y_2(x_0) = 1$, $y_2'(x_0) = 0$ and y_1, u be the function defined in Lemma 4. Then

$$u(t, \mu) = y_0 y_2(t) + u'(x_0, \mu)\, y_1(t) + \mu \int_{x_0}^{t} \big(y_1(t)\, y_2(s) - y_1(s)\, y_2(t)\big)\, f(s)\, ds$$

and from this we obtain

$$\alpha(\mu) = u(\varphi(x_0), \mu) = y_0 y_2(\varphi(x_0)) - \mu y_2(\varphi(x_0)) \int_{x_0}^{\varphi(x_0)} y_1(s) f(s)\, \mathrm{d}s\,,$$

$$\alpha'(\mu) = -y_2(\varphi(x_0)) \int_{x_0}^{\varphi(x_0)} y_1(s) f(s)\, \mathrm{d}s := k \neq 0\,,$$

$$\alpha(\mu) = k\mu + c \qquad (c = \text{a constant})$$

from which the statement of the Lemma results.

Lemma 5. *Let* $p \in C^2(j)$, $p(t) \neq 0$ *for* $t \in j$ and $f \in C^1(j)$. *Then for every solution* y *of the equation*

$$y'' - p(t)\, y = f(t) \tag{11}$$

the function $z(t) := \dfrac{y'(t)}{\sqrt{|p(t)|}}$, $t \in j$ *represents a solution of the equation*

$$z'' - \left(p(t) + \sqrt{|p(t)|}\left(\frac{1}{\sqrt{|p(t)|}}\right)''\right) z =$$

$$= 2f(t)\left(\frac{1}{\sqrt{|p(t)|}}\right)' + \frac{f'(t)}{\sqrt{|p(t)|}} \tag{12}$$

and vicer versa, for every solution z *of* (12) *the function* $z(t)\sqrt{|p(t)|}$ *is the derivative of exactly one solution of* (11).

Proof. Let y be a solution of (11) and $z(t) := \dfrac{y'(t)}{\sqrt{|p(t)|}}$, $t \in j$. It is easily verified that the function z is a solution of (12).

Let z be a solution if (12) and $v(t) := z(t)\sqrt{|p(t)|}$, $t \in j$. Assume that v is the derivative of a solution y of (11). Then the solution y and its derivative y' at $x_0 \in j$ have necessarily the following values

$$v_0 := \frac{1}{p(x_0)}(v'(x_0) - f(x_0)), \qquad v_0' ;= z(x_0)\sqrt{|p(x_0)|}.$$

Setting

$$w(t) := v_0 + \int_{x_0}^{t} z(s)\sqrt{|p(s)|}\, \mathrm{d}s, \qquad t \in j,$$

then $w'(t) = z(t)\sqrt{|p(t)|}$, $w(x_0) = v_0$, $w'(x_0) = v_0'$ and further

$$w''' = z''\sqrt{|p|} + \frac{z'p' \operatorname{sign} p}{\sqrt{|p|}} + z(\sqrt{|p|})'' =$$

$$= \left[\left(p - \frac{p''}{2p} + \frac{3}{4}\left(\frac{p'}{p}\right)^2\right) z + 2f\left(\frac{1}{\sqrt{|p|}}\right)' + \frac{f'}{\sqrt{|p|}}\right]\sqrt{|p|} +$$

$$+\frac{p'}{p}\left(w''-\frac{zp'\operatorname{sign}p}{2\sqrt{|p|}}\right)+\frac{(2pp''-p'^2)\operatorname{sign}p}{4p\sqrt{|p|}}z=$$

$$=pw'+\frac{p'}{p}w''+2f\sqrt{|p|}\left(\frac{1}{\sqrt{|p|}}\right)'+f'.$$

Thus

$$w'''-\frac{p'}{p}w''-pw'=-\frac{fp'}{p}+f'$$

From this we obtain

$$\left(\frac{w''-pw}{p}\right)'=\left(\frac{f}{p}\right)'.$$

$$w''-pw=f+kp \qquad (k=\text{a constant}).$$

The definition of the function w and $w(x_0)=v_0$, $w'(x_0)=v_0'$ yields $k=0$. This completes the proof of the Lemma.

Remark 4. Lemma 5 was proved in [1, p. 9] under the assumption $f(t)\equiv 0$.

Lemma 6. *Let $x_0\in j$, y_0' be arbitrary numbers, $l\in C^0(D)$, $l'(t,\mu)\in C^0(D)$, $k\in C^2(j)$, $k(t)<0$ for $t\in j$. Let the second order fundamental dispersion $\bar{\psi}$ of* (k): $y''=k(t)y$ *be defined $x_0\in j$ and uniformly on $\langle x_0,\bar{\psi}(x_0)\rangle$*

$$\lim_{\mu\to-\infty}\left\{2l(t,\mu)\left(\frac{1}{\sqrt{|k(t)|}}\right)'+\frac{l'(t,\mu)}{\sqrt{|k(t)|}}\right\}=-\infty,$$
$$\lim_{\mu\to\infty}\left\{2l(t,\mu)\left(\frac{1}{\sqrt{|k(t)|}}\right)'+\frac{l'(t,\mu)}{\sqrt{|k(t)|}}\right\}=\infty. \tag{13}$$

Let $v(t,\mu)$ be a solution of

$$y''-k(t)y=i(t,\mu) \tag{14}$$

satisfying the condition $v'(x_0,\mu)=y_0'$. Setting

$$\mathcal{M}_1:=\{v'(\bar{\psi}(x_0),\mu);\ \mu\in\mathbf{R}\}, \tag{15}$$

then

$$\mathcal{M}_1=\mathbf{R}.$$

Proof. Let $\mathcal{M}_1$ be the set defined by (15). Let $v(t,\mu)$ be a solution of (14) satisfying the condition $v'(x_0,\mu)=y_0'$ and let $u(t,\mu):=\dfrac{v'(t,\mu)}{\sqrt{|k(t)|}}$, $(t,\mu)\in D$. Then $u(x_0,\mu)=$ $=\dfrac{y_0'}{\sqrt{|k(x_0)|}}$ and according to Lemma 5 $u(t,\mu)$ is a solution of

$$y'' - \left(k(t) + \sqrt{|k(t)|}\left(\frac{1}{\sqrt{|k(t)|}}\right)''\right)y =$$
$$= 2l(t, \mu)\left(\frac{1}{\sqrt{|k(t)|}}\right)' + \frac{l'(t, \mu)}{\sqrt{|k(t)|}}.$$

Putting

$$p(t) := k(t) + \sqrt{|k(t)|}\left(\frac{1}{\sqrt{|k(t)|}}\right)'', \qquad t \in j,$$
$$r(t, \mu) := 2l(t, \mu)\left(\frac{1}{\sqrt{|k(t)|}}\right)' + \frac{l'(t, \mu)}{\sqrt{|k(t)|}}, \qquad (t, \mu) \in D, \tag{16}$$

then $u(t, \mu)$ is a solution of (7) where the functions p, r are defined by (16), $u(x_0, \mu) = \frac{y_0'}{\sqrt{|k(x_0)|}}$. Since $\bar{\psi}$ is the 1st kind fundamental dispersion of (p), we have with respect to Lemma 2

$$\mathbf{R} = \{u(\bar{\psi}(x_0), \mu); \mu \in \mathbf{R}\} = \left\{v'(\bar{\psi}(x_0), \mu)\frac{1}{\sqrt{|k(\bar{\psi}(x_0))|}};\ \mu \in \mathbf{R}\right\} =$$
$$= \{v'(\bar{\psi}(x_0), \mu); \mu \in \mathbf{R}\} = \mathcal{M}_1.$$

Remark 5. Let $k \in C^2(j)$, $k(t) < 0$ and $\left(\frac{1}{\sqrt{|k(t)|}}\right)' > 0$ for $t \in j$, $h \in C^1(j)$. Setting $l(t, \mu) := h(t) + \mu$, $(t, \mu) \in D$, then (13) applies uniformly on every compact subinterval of j.

Lemma 7. *Let $x_0 \in j$, y_0' be arbitrary numbers, $l \in C^0(D)$, $l'(t, \mu) \in C^0(D)$, $k \in C^2(j)$, $k(t) < 0$ for $t \in j$. Let the 2nd order fundamental dispersion $\bar{\psi}$ of* (k) *be defined at x_0, let the function l satisfy* (13) *for $t \in \langle x_0, \bar{\psi}(x_0)\rangle$ and*

$$2l(t, \mu_1)\left(\frac{1}{\sqrt{|k(t)|}}\right)' + \frac{l'(t, \mu_1)}{\sqrt{|k(t)|}} <$$
$$< 2l(t, \mu_2)\left(\frac{1}{\sqrt{|k(t)|}}\right)' + \frac{l'(t, \mu_2)}{\sqrt{|k(t)|}}$$

for $\mu_1 < \mu_2$ and $t \in \langle x_0, \bar{\psi}(x_0)\rangle$. Let next $v(t, \mu)$ be a solution of (14) *satisfying the condition $v'(x_0, \mu) = y_0'$. Then the function*

$$\beta(\mu) := v'(\bar{\psi}(x_0), \mu), \qquad \mu \in \mathbf{R}, \tag{17}$$

is an increasing one on $\mathbf{R}$ *and* $\beta(\mathbf{R}) = \mathbf{R}$.

Remark 6. It follows from [5] that all solutions $v(t, \mu)$ of (14) satisfying the condition $v'(x_0, \mu) = y_0'$ have the same values at the point $(\bar{\psi}(x_0), \mu)$. Therefore the function β is defined correctly by relation (17).

The proof of Lemma 7. Let $v(t, \mu)$ be a solution of (14), $v'(x_0, \mu) = y'_0$. We put $u(t, \mu) := \frac{v'(t, \mu)}{\sqrt{|k(t)|}}$, $(t, \mu) \in D$. Then, with respect to Lemma 5, u is a solution of (7), where the functions p, r are defined by (16), $u(x_0, \mu) = \frac{y'_0}{\sqrt{|k(x_0)|}}$. Since $\bar{\psi}$ is the 1st kind fundamental dispersion of (p), the assumptions of Lemma 3 are satisfied and $\beta(\mu) = v'(\bar{\psi}(x_0), \mu) = \sqrt{|k(\bar{\psi}(x_0)|} \cdot u(\bar{\psi}(x_0), \mu)$ and the properties of the function β under proof, immediately follow from the properties of the function α defined in Lemma 3.

3. We prove the following

Theorem 1. *Let $x_0 \in j$, $x_1 \in j$, y_0, y_1 be arbitrary numbers, $x_0 < x_1$ and $q \in C^0(D)$, $r \in C^0(D)$. Let next* (4) *and* (6) *hold uniformly on every compact subinterval of j. Then there exist numbers λ_0, μ_0 such that the points (x_0, y_0), (x_1, y_1) are the 1st kind neighbouring knots relative to equation* (1) *with $\lambda = \lambda_0$, $\mu = \mu_0$, and to the initial condition (x_0, y_0).*

Proof. The function $\varphi(x_0, \lambda)$ is continuous on its interval of definition with respect to Lemma 1 [3] and it follows from (4) holding by assumption uniformly on every compact subinterval of j that: $\lim_{\lambda \to -\infty} \varphi(x_0, \lambda) = x_0$ and there exists a number λ_1, where the function $\varphi(x_0, \lambda)$ is mapping the interval $(-\infty, \lambda_1)$ onto the interval (x_0, b). There exists therefore at least one number $\lambda_0 (\in (-\infty, \lambda_1))$: : $\varphi(x_0, \lambda_0) = x_1$. Let $u(t, \mu)$ be a solution of

$$y'' - q(t, \lambda_0)\, y = r(t, \mu)$$

$u(x_0, \mu) = y_0$. With respect to Lemma 2 then follows the existence of a number $\mu_0 : u(x_1, \mu_0) = y_1$.

Corollary 2. *Let $x_0 \in j$, $x_1 \in j$, y_0, y_1 be arbitrary numbers, $x_0 < x_1$. Let q satisfy the assumption* (i) *and let r satisfy one of the following assumptions*:

(v) *$r \in C^0(D)$ and* (6) *and* (9) *are satisfied for $t \in j$,*

(vi) *$r(t, \mu) \equiv \mu f(t)$, where $f \in C^0(j)$ and $\int_{x_0}^{x_1} f(t)\, y_1(t)\, dt \neq 0$.*

Here y_1 is a nontrivial solution of $y'' = q(t, \lambda_0)\, y$, $y_1(x_0) = 0$ and λ_0 is the number occurring in the statement of Lemma 1.

Then there exists exactly one value of the parameter λ which we write as λ_0 and exactly one value of the parameter μ written as μ_0 with the points (x_0, y_0), (x_1, y_1) being the 1st kind neighbouring knots relative to (1), *where $\lambda = \lambda_0$ and $\mu = \mu_0$, and to the initial condition (x_0, y_0).*

Proof. With respect to Lemma 1 there exists exactly one number $\lambda_0 : \varphi(x_0, \lambda_0) = x_1$. We set $p(t) := q(t, \lambda_0)$, $t \in j$. If r satisfies the assumption (v) and (vi) then—with respect to Lemmas 3 and 4 respectively—there exists exactly one value of the parameter μ written as μ_0, where the equation $y'' - q(t, \lambda_0)\, y = r(t, \mu_0)$ has a solution u for which $u(x_0) = y_0$ and $u(x_1) = y_1$.

Corollary 3. *Let $x_0 \in j$, $x_1 \in j$, y_0, y_1 be arbitrary numbers, $x_0 < x_1$. Let q satisfy the assumption* (ii) *and* $\inf\{x;\, x \in j,\, x > x_0, p(x) < 0\} < x_1$. *Let r satisfy either the assumption* (v) *or the assumption*

(vii) $r(t, \mu) \equiv \mu f(t)$, *where* $f \in C^0(j)$ *and* $\int_{x_0}^{x_1} f(t)\, y_1(t)\, \mathrm{d}t \neq 0$ *with y_1 being a nontrivial solution of* $y'' = q(t, \lambda_1)\, y$, $y_1(x_0) = 0$ *and* $\lambda_1 > 0$ *a number occurring in the statement of Lemma* 1.

Then there exists exactly one positive value of the parameter λ written as λ_1 and exactly one value of the parameter μ written as μ_0 such that the points (x_0, y_0), (x_1, y_1) are the 1st kind neighbouring knots relative to (1), *where $\lambda = \lambda_1$ and $\mu = \mu_0$, and to the initial condition (x_0, y_0).*

Proof. With respect to Lemma 1 there exists exactly one number $\lambda_1 > 0$: $\varphi(x_0, \lambda_1) = x_1$. We set $p(t) := q(t, \lambda_1)$, $t \in j$. The rest of the proof proceeds completly analogous to the proof of Corollary 2.

Corollary 4. *Let $x_0 \in j$, $x_1 \in j$, y_0, y_1 be arbitrary numbers, $x_0 < x_1$. Let q satisfy the assumption* (ii) *and* $\inf\{x;\, x \in j,\, x > x_0, p(x) > 0\} < x_1$. *Let r satisfy the assumption* (v) *or the assumption* (vii), *where instead of λ_1 we consider $\lambda_2 < 0$ occurring in the statement of Lemma* 1. *Then there exists one negative value of the parameter λ written as λ_2 and exactly one value of the parameter μ written as μ_0 such that the points (x_0, y_0), (x_1, y_1) are the 1st kind neighbouring knots relative to* (1), *where $\lambda = \lambda_2$ and $\mu = \mu_0$, and to the initial condition (x_0, y_0).*

We refrain from proving these assertions since the proof is an exact repetition of the previons one.

Theorem 2. *Let $x_0 \in j$, $x_1 \in j$, y'_0, y'_1 be arbitrary numbers, $x_0 < x_1$, $q \in C^0(D)$, $q''(t, \lambda) \in C^0(D)$, $r \in C^0(D)$, $r'(t, \mu) \in C^0(D)$ and $q(t, \lambda) < 0$ for $(t, \lambda) \in D$. Let* (5) *hold uniformly on every compact subinterval of j and let uniformly with respect to the variable t on every compact subinterval of j:*

$$\lim_{\mu \to -\infty} \left\{ 2r(t, \mu) \left(\frac{1}{\sqrt{|q(t, \lambda)|}} \right)' + \frac{r'(t, \mu)}{\sqrt{|q(t, \lambda)|}} \right\} = -\infty,$$
$$\lim_{\mu \to \infty} \left\{ 2r(t, \mu) \left(\frac{1}{\sqrt{|q(t, \lambda)|}} \right)' + \frac{r'(t, \mu)}{\sqrt{|q(t, \lambda)|}} \right\} = \infty, \qquad (18)$$
$$(\lambda \in \mathbf{R}).$$

Then there exist numbers λ_0, μ_0 where points $(x_0, y'_0), (x_1, y'_1)$ are the 2nd kind neighbouring knots relative to (1) (*with* $\lambda = \lambda_0$, $\mu = \mu_0$) *and to the initial condition* (x_0, y'_0).

Proof. We set $q_1(t, \lambda) := q(t, \lambda) + \sqrt{|q(t, \lambda)|}\left(\frac{1}{\sqrt{|q(t, \lambda|}}\right)''$, $(t, \lambda) \in D$. Then the 2nd kind fundamental dispersion $\psi(t, \lambda)$ of (2) is equal to the 1st kind fundamental dispersion of the equation $y'' = q_1(t, \lambda)\, y$. From the assumption (5) we can prove the existence of a number $\lambda_0 : \psi(x_0, \lambda_0) = x_1$ by a completly analogous method to that used in the first part of the proof of Theorem 1. We set $r_1(t, \mu) := = 2r(t, \mu)\left(\frac{1}{\sqrt{|g(t, \lambda_0)|}}\right)' + \frac{r'(t, \mu)}{\sqrt{|q(t, \lambda_0)|}}$, $(t, \mu) \in D$. Let $v(t, \mu)$ be a solution of $y'' - q_1(t, \lambda_0)\, y = r_1(t, \mu), v(x_0, \mu) = \frac{y'_0}{\sqrt{|q(t, \lambda_0)|}}$. Then, with respect to Lemma 6, the function $v(t, \mu)\sqrt{|q(t, \lambda_0)|}$ is the derivative of the exactly one solution of $y'' - - q(t, \lambda_0)\, y = r(t, \mu)$, written as $u(t, \mu)$. Evidently $u'(x_0, \mu) = y'_0$. With respect to Lemma 6 there exists μ_0, and (1) (with $\lambda = \lambda_0$, $\mu = \mu_0$) has the solution $u(t, \mu_0)$ satisfying $u'(x_0, \mu_0) = y_0, u'(x_1, \mu_0) = u'(\psi(x_0), \mu_0) = y'_1$. Thus Theorem 2 is proved.

Corollary 5. *Let $x_0 \in j$, $x_1 \in j$, y'_0, y'_1 be arbitrary numbers, $x_0 < x_1$. Let q satisfy the assumption* (iii), $r \in C^0(D)$, $r'(t, \mu) \in C^0(D)$ *and*

$$2r(t, \mu_1)\left(\frac{1}{\sqrt{|g(t, \lambda)|}}\right)' + \frac{r'(t, \mu_1)}{\sqrt{|q(t, \lambda)|}} < 2r(t, \mu_2)\left(\frac{1}{\sqrt{|q(t, \lambda)|}}\right)' + $$
$$+ \frac{r'(t, \mu_2)}{\sqrt{|q(t, \lambda)|}} \quad \textit{for } \mu_1 < \mu_2 \textit{ and } (t, \lambda) \in D. \tag{19}$$

Let next (18) *be true for every* $(t, \lambda) \in D$.

Then there exists exactly one value of the parameter λ written as λ_0 and exactly one value of the parameter μ written as μ_0 such that the points (x_0, y'_0), (x_1, y'_1) are the 2nd kind neighbouring knots relative to (1) (*with* $\lambda = \lambda_0$, $\mu = \mu_0$) *and to the initial condition* (x_0, y'_0).

Proof. Since q satisfies the assumption (iii), there exists exactly one number $\lambda_0 : \psi(x_0, \lambda_0) = x_1$. Then the statement of Corollary 5 follows from inequality (19) (where we put λ_0 in place of λ) and from Lemma 7.

Corollary 6. *Let $x_0 \in j$, $x_1 \in j$, y'_0, y'_1 be arbitrary numbers, $x_0 < x_1$. Let q satisfy the assumption* (iv), $r \in C^0(D)$, $r'(t, \mu) \in C^0(D)$ *and let* (18) *for* $(t, \lambda) \in D$ *and* (19) *hold.*

If $\inf \{x; x \in j, x > x_0, p(x) < 0\} < x_1$, *then there exists exactly one positive value of the parameter λ written as λ_1 and exactly one value of the parameter μ written as μ_0 such that the points (x_0, y'_0), (x_1, y'_1) are the 2nd kind neighbouring knots relative to* (1) (*with* $\lambda = \lambda_1$, $\mu = \mu_0$) *and to the initial condition* (x_0, y_0).

If $\inf\{x; x \in j, x > x_0, p(x) > 0\} < x_1$, *then there exists exactly one negative value of the parameter* λ *written as* λ_2 *and exactly one value of the parameter* μ *written as* μ_0 *such that the points* (x_0, y'_0), (x_1, y'_1) *are the 2nd order neighbouring knots relative to* (1) (*with* $\lambda = \lambda_2$, $\mu = \mu_0$) *and to the initial condition* (x_0, y'_0).

The proof follows from Corollary 1 and from Lemma 7.

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Souhrn

DVOUBODOVÝ OKRAJOVÝ PROBLÉM PRO NEHOMOGENNÍ LINEÁRNÍ DIFERENCIÁLNÍ ROVNICI 2. ŘÁDU

SVATOSLAV STANĚK

V práci je vyšetřována lineární diferenciální rovnice

$$y'' - q(t, \lambda) y = r(r, \mu), \tag{1}$$

kde $q \in C^0(D)$, $r \in C^0(D)$, $D = j \times \mathbf{R}$, $j = (a, b)$ $(-\infty \leqq a < b \leqq \infty)$, která závisí na dvou reálných parametrech λ, μ. Nechť (x_0, y_0), (x_0, y'_0), (x_1, y_1), (x_1, y'_1) jsou libovolné body v D, $x_0 < x_1$. Jsou uvedeny postačující podmínky k tomu, aby:

(i) existovalo řešení y rovnice (1) pro něž $y(x_0) = y_0$, $y(x_1) = y_1$ a příslušná homogenní rovnice

$$y'' = q(t, \lambda) y \tag{2}$$

měla netriviální řešení v, kde $v(x_0) = v(x_1) = 0$ a $v(t) \neq 0$ pro $t \in (x_0, x_1)$;

(ii) existovalo řešení z rovnice (1) pro něž $z'(x_0) = y'_0$, $z'(x_1) = y'_1$ a příslušná

homogenní rovnice (2) měla netriviální řešení u, kde $u'(x_0) = u'(x_1) = 0$ a $u'(t) \neq 0$ pro $t \in (x_0, x_1)$.

Rovněž je vyšetřována jednoznačnost řešení obou problémů.

Резюме

ДВУТОЧЕЧНАЯ ЗАДАЧА ДЛЯ НЕОДНОРОДНОГО ЛИНЕЙНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ВТОРОГО ПОРЯДКА

СВАТОСЛАВ СТАНЕК

В работе исследуется неоднородное линейное дифференциальное уравнение

$$y'' - q(t, \lambda)\, y = r(t, \mu), \tag{1}$$

где $q \in C^\circ(D)$, $r \in C^\circ(D)$, $D = j \times \mathbf{R}$, $j = (a, b)$ $(-\infty \leqq a - b \leqq \infty)$ которое зависит от двух действительных параметров λ, μ. Пусть (x_0, y_0), (x_0, y'_0), (x_1, y_1) и (x_1, y'_1) произвольные точки из D, $x_0 < x_1$. Приведены достаточные условия для того, чтобы

(i) существовало решение y уравнения (1), $y(x_0) = y_0$, $y(x_1) = y_1$ и одновременно соответствующее однородное уравнение

$$y'' = q(t, \lambda)\, y \tag{2}$$

имело нетривиальное решение v, где $v(x_0) = v(x_1) = 0$ и $v(t) \neq 0$ для $t \in (x_0, x_1)$;

(ii) существовало решение z уравнения (1), $z'(x_0) = y'_0$, $z'(x_1) = y'_1$ и одновременно соответствующее однородное уравнение (2) имело нетривиальное решение u, где $u'(x_0) = u'(x_1) = 0$ и $u'(t) \neq 0$ для $t \in (x_0, x_1)$.

Исследуется тоже однозначность решения обеих проблем.