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ON SOME PROPERTIES OF FUNCTIONS RELATIVE TO THE CLASSES OF CERTAIN FACTOR SPACES

JITKA KOJECKÁ

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Dedicated to Academician O. Borůvka on his 80th birthday

M. Laitoch investigated in [1] decompositions of knots of integrals of the non-homogeneous second order linear differential equations and decompositions of knots of the first derivatives relative to these integrals. The present paper generalizes the above results for the functions relative to the classes of the factor space C^0/S of the space C^0 relative to all continuous functions modulo S, where S is a two-dimensional space of continuous functions. This article topically forms a close continuation of [3], [4] and [5].

1. Let $C^0(\iota)$ be a set of all continuous functions defined on an open interval $\iota \subset E_1(= (-\infty, +\infty))$. It is clear that $C^0(\iota)$ is a linear space and the two-dimensional space S with the definition interval ι is its linear subspace. The factor space C^0/S is then a set of all classes of elements of C^0 equivalent to modulo S, where the equivalence is defined as follows: two elements $x_1, x_2 \in C^0$ are equivalent modulo S if $x_1 - x_2 \in S$. By this equivalence the space C^0 is decomposed into mutually disjoint classes of equivalent elements. (Cf. the proof in [6]).

We assume the space S to be a regular one of a certain type with a monotone phase α, i.e. the zeros of each two independent functions of the space S are separated. In what follows our consideration will be done in a class of the factor space C^0/S denoted as X, $X \in C^0/S$.

Lemma 1.1. Let $x_0 \in X$ be an arbitrary function. Then any function $x \in X$ is of the form $x = y + x_0$, where $y \in S$ is an appropriate function.

Proof: The statement is immediate from the definition of the factor space.

Lemma 1.2. To every two numbers $t_0 \in \iota$ and $k_0 \in E_1$ there exists at least one function $x \in X$ so that $x(t_0) = k_0$.

Proof: Let $x_0 \in X$ and $y \in S$. Denoting $x_0(t_0) = k$ and $y(t_0) = k_1$ and putting $k_2 = k - k_0$ implies for the function $x = -(k_2/k_1)\, y + x_0$ that $x \in X$ and $x(t_0) = -(k_2/k_1) \,.\, k_1 + k = k_0$.

Theorem 1.1. Let $t_1, t_2 \in \imath$ be not conjugate numbers in S and let $k_1, k_2 \in E_1$ be arbitrary numbers. Then there exists exactly one function $x \in X$ for which $x(t_j) = k_j$ $(j = 1, 2)$ is valid.

Proof: Let (u, v) be a basis of the space S and $x_0 \in X$. The function $x \in X$, $x = c_1 u + c_2 v + x_0$ satisfies our requirement exactly if

$$c_1 u(t_1) + c_2 v(t_1) + x_0(t_1) = k_1$$
$$c_1 u(t_2) + c_2 v(t_2) + x_0(t_2) = k_2 \,.$$

Under the assumption that t_1, t_2 are not conjugate points in S the determinant of the system $D = u(t_1)\, v(t_2) - u(t_2)\, v(t_1)$ is different from zero and this system of equations possesses exactly one solution c_1, c_2 uniquely determining the function $x \in X$.

Remark 1.1. It is obvious from the independence of the functions of the basis relative to S that if $x_1 \equiv x_2$ holds for the two functions $x_1, x_2 \in X$ on $(a, b) \subset \imath$, then the above identity is valid on the whole $\imath$. Such functions—and so also the function $y \equiv 0$ in S—will be excluded from now on.

Theorem 1.2. Let $t_0 \in \imath$ and $k_0 \in E_1$ be arbitrary numbers. Let $x_1(t_0) = x_2(t_0) = k_0$ hold for the functions $x_1, x_2 \in X$. Let $t_1 \in \imath$, $t_1 \neq t_0$. Then $x_1(t_1) = x_2(t_1)$ exactly if t_0 and t_1 are conjugate points in S.

Proof: Let $x_1(t_1) = x_2(t_1)$. Then we have for the function $y = x_1 - x_2$ that $y \in S$, $y(t_0) = 0 = y(t_1)$, that is t_0 and t_1 are conjugate in S. If contrarywise t_0 and t_1 are conjugate points in S, then, by Lemma 1.1, x_1 may be written in the form $x_1 = y + x_2$, where $y \in S$ is an appropriate function. Because of the assumption $x_1(t_0) = x_2(t_0)$ we have $y(t_0) = 0$ and since t_0 and t_1 are conjugate numbers in S, we have also $y(t_1) = 0$ and thus $x_1(t_1) = x_2(t_1)$.

Corollary 1.1. Let $t_0 \in \imath$ and $k_0 \in E_1$ be arbitrary numbers. Let next $\{t_j\}$ be a complete system of points conjugate to t_0 in S, $j = 0, \pm 1, \pm 2, \ldots$. Then it holds for all j: $x_1(t_j) = x_2(t_j)$ and $x_1(t) \neq x_2(t)$ for $t \in (\imath - \bigcup_j \{t_j\})$ for each two functions $x_1, x_2 \in X$ satisfying the condition $x_1(t_0) = x_2(t_0) = k_0$.

Definition 1.1. Let $t_0 \in \imath$ and $k_0 \in E_1$ be arbitrary numbers. Next let $\{t_j\}$ be a complete system of points conjugate to t_0 in S $(j = 0, \pm 1, \pm 2, \ldots)$, with $x_0 \in X$ being a function for which $x_0(t_0) = k_0$ holds. The set of all points $[t_j, x_0(t_j)]$ will be called a complete system of knots relative to the initial condition $[t_0, k_0]$ written as $\mathcal{U}(t_0, k_0)$. The neighbouring knots are called points $[t_j, x_0(t_j)]$ and $[t_{j+1}, x_0(t_{j+1})]$, where t_j and t_{j+1} are the neighbouring conjugate points in S.

The set of all functions $x \in X$, for which the initial condition $x(t_0) = k_0$ holds, will be called the bundle of functions relative to the initial condition $[t_0, k_0]$ written as $\mathscr{S}(t_0, k_0)$.

Theorem 1.3. There exists exactly one system $\mathscr{U}(t_0, k_0)$ at every point $[t_0, k_0] \in \in \iota \times E_1$.

Proof: In view of Lemma 1 [4] there obviously exists at every t_0 exactly one complete system of points $\{t_j\}$ conjugated to t_0 in S. The statement follows from Theorem 1.2.

Corollary 1.2. The bundle of functions $\mathscr{S}(t_0, k_0)$ is uniquely determined by an arbitrary knot relative to $\mathscr{U}(t_0, k_0)$.

Corollary 1.3. All functions $x \in \mathscr{S}(t_0, k_0)$ have exactly the knots relative to the system $\mathscr{U}(t_0, k_0)$ in common.

Corollary 1.4. Let $[t_0, k_0] \in \iota \times E_1$ be an arbitrary point with $\bar{x} \in X$ being a function not passing through this point. Then the function $\bar{x}$ is not passing through any point of the system $\mathscr{U}(t_0, k_0)$.

Theorem 1.4. Let $[t_0, k_0] \in \iota \times E_1$ be an arbitrary point and $x \in X$, $x \in \mathscr{S}(t_0, k_0)$. Let further $[t_j, x(t_j)]$, $[t_{j+1}, x(t_{j+1})] \in \mathscr{U}(t_0, k_0)$ be neighbouring knots and $\bar{x} \in X$ be a function not passing through these knots. Then there exists exactly one point $\tau \in (t_j, t_{j+1})$ such that $[\tau, x(\tau)] = [\tau, \bar{x}(\tau)]$.

Proof: Let $x_0 \in X$ and $x_0 \in \mathscr{S}(t_0, k_0)$. By Lemma 1.1 the functions x and $\bar{x}$ may be written in the form $x = y + x_0$ and $\bar{x} = \bar{y} + x_0$, where $y, \bar{y} \in S$ are appropriate functions. Evidently $y(t_0) = 0$, hence $y(t_j) = 0$, $y(t_{j+1}) = 0$ and therefore $\bar{y}(t_j) \neq 0$. This implies with respect to the regularity of the space S that the functions y and $\bar{y}$ are independent. Now it holds for the function $z = y - \bar{y} : z \in S$; z and y are independent and due to the separation of zeros of each two independent functions in S, there lies exactly one zero of the function z in the interval (t_j, t_{j+1}) denoted as τ. The relation $z(\tau) = 0$ yields then $y(\tau) = \bar{y}(\tau)$ and therefore $x(\tau) = \bar{x}(\tau)$.

Theorem 1.5. Let $[t_0, k_0]$, $[\tau_0, \varkappa_0] \in \iota \times E_1$ be two different points. Let further $x_0, x_1 \in X$ and $x_0 \in \mathscr{S}(t_0, k_0)$ and $x_1 \in \mathscr{S}(\tau_0, \varkappa_0)$. Let $[t_j, x_0(t_j)]$, $[t_{j+1}, x_0(t_{j+1})] \in \in \mathscr{U}(t_0, k_0)$ or $[\tau_j, x_1(\tau_j)]$, $[\tau_{j+1}, x_1(\tau_{j+1})] \in \mathscr{U}(\tau_0, \varkappa_0)$ be neighbouring knots. Then there lies exactly one knot in the zone $\langle t_j, t_{j+1}) \times E_1$ or $\langle \tau_j, \tau_{j+1}) \times E_1$ relating to the system $\mathscr{U}(\tau_0, \varkappa_0)$ or $\mathscr{U}(t_0, k_0)$, respectively.

Proof: With respect to Theorem 1.3 and to its Corollaries, the statement follows immediately from the separation of points in two different systems of conjugate points in S.

Remark 1.2. It follows from Theorem 15 [7] that the space of integrals relative to the nonhomogeneous 2nd order linear differential equation

$$y'' - Q(t)\, y = r(t), \tag{xx}$$

where Q and r are continuous functions on ι, is a class of the factor space C^2/S, where S is a two-dimensional space of the solution relative to the equation

$$y'' - Q(t)\,y = 0\,. \tag{x}$$

Under the assumptions of [1] the space containing integrals of the differential equation (x) is a regular space of a certain type with a monotone first phase α, so that the Theorems proved for the functions of the class X relative to the factor space C^0/S are valid even in the space of integrals of the differential equation (xx). The same results were obtained in the space containing integrals of the differential equation (xx) from the properties of the solution of the equations (xx) and (x), which was the main purpose of paper [1].

2. Let $C^1(\iota)$ be a space of all functions defined in an open interval $\iota \subset E_1$, possessing a continuous first derivative on ι. Further let S^1 be a two-dimensional space of the functions defined on ι possessing a continuous first derivative on ι. We suppose the space S^1 in the sense that the set of derivatives S' of all functions of S^1 constitutes a two-dimensional space of continuous functions and the spaces S^1 and S' being regular, of a certain type, with monotone phases.

By the same equivalence as in part 1 we constitute the factor spaces C^1/S^1 and C'/S', where C' is the set of derivatives of all functions of the space $C^1(\iota)$, whereby $S' \subset C'$.

We are going to make further consideration in a class of the factor space C^1/S^1, which we denote as X^1, $X^1 \in C^1/S^1$.

Lemma 2.1. Let $x_0 \in X^1$ and $X' \in C'/S'$ be that class in which x_0' is lying. Then X' is a set of derivatives of all functions relative to X^1.

Proof: Let $x_0 \in X^1$. Then $X^1 = \{x_0 + y; y \in S\}$. Let $x_0' \in X' \in C'/S'$. Then $X' = \{x_0' + y'; y' \in S'\}$, hence X' is a set of derivatives of all functions relative to X^1.

Lemma 2.2. Let $x_0 \in X^1$ and $x_0' \in X'$ be its derivative. Then x_0 is the only primitive function to the function x_0' lying in X^1.

Proof: Letting $x_0 \in X^1$ and $\bar{x}_0 \in X^1$ be two primitive functions to x_0', we get for them $x_0 - \bar{x}_0 = c$ and at the same time $x_0 - \bar{x}_0 \in S^1$. By Theorem 1.2 [5] this is possible only for $c = 0$, so that $x_0 = \bar{x}_0$.

Theorem 2.1. Let X' be a set of derivatives of all functions relative to X^1. The mapping X^1 onto X' is defined by the operator $D = \dfrac{\mathrm{d}}{\mathrm{d}t}$ $(=')$ is an isomorphism of X^1 onto X'.

Proof: By Lemma 2.1 $DX^1 = X'$ and by Lemma 2.2 this mapping is schlicht. The remaining part of the statement follows from the rules for differenciating the sum of functions and the constant multiple of the function.

Theorem 2.2. Let $t_0 \in \imath$ and $k_0, k'_0 \in E_1$ be arbitrary numbers. Then there exists exactly one function $x \in X^1$ so that $x(t_0) = k_0$ and $x'(t_0) = k'_0$.

Proof: Let (u, v) be a basis of the space S^1 and $x_0 \in X^1$. The function $x \in X^1$, $x = c_1u + c_2v + x_0$ satisfies our condition exactly if

$$c_1u(t_0) + c_2v(t_0) + x_0(t_0) = k_0$$
$$c_1u'(t_0) + c_2v'(t_0) + x'_0(t_0) = k'_0.$$

On the understanding that S^1 is a space with a monotone phase α, the determinant of the system – the Wronskian of the functions of the basis (u, v) – is different from zero (See Theorem 1.11 [5]), which implies that this system has exactly one solution c_1, c_2. This pair of numbers then uniquely determines the function $x \in X^1$ for which $x(t_0) = k_0$ and $x'(t_0) = k'_0$.

Remark 2.1. In view of the fact $C^1(\imath) \subset C^0(\imath)$ and $C'(\imath) = C^0(\imath)$ all statements from part 1 are valid for the systems of the knots of functions and bundles of functions of the class relative to the factor spaces C^1/S^1 and C'/S'.

On the assumptions of [1] the space containing the derivatives of integrals of the differential equation (x) is a regular space of a certain type with a monotone phase. Consequently the Theorems of part 1 are valid in the space containing the derivatives of integrals of the differential equation (xx), which has been also obtained from the integral properties in the equations (x) and (xx) (See [1]).

REFERENCES

[1] M. Laitoch: *A modification of the Sturm's theorem on separating zeros of solutions of a linear differential equation of the 2nd order.* Acta UPO, T. 57, 1978.

[2] O. Borůvka: *Linear Differential Transformations of the Second Order.* The English Universities Press Ltd., London 1971.

[3] K. Stach: *Die allgemeinen Eigenschaften der Kummerschen Transformationen zweidimensionaler Räume von stetigen Funktionen.* "Publ. Fac. Sci. Univ. J. E. Purkyně" Brno, No 478, 1966, 389—410.

[4] S. Trávníček, J. Kojecká: *Einige Bemerkungen über die Nullstellen im zweidimensionalen Raum stetiger Funktionen.* Acta UPO, T. 37, 1972, 45—50.

[5] J. Kojecká: *Lireare zweidimensionale Räume von stetigen Funktionen mit stetigen ersten Ableitungen.* Acta UPO, T. 45, 1974, 17—30.

[6] A. E. Taylor: *Úvod do funkcionální analýzy.* Praha 1973.

[7] S. Trávníček: *Faktorové prostory.* Pokroky matematiky, fyziky a astronomie, r. XIV, č. 6, 254—265.

Shrnutí

NĚKTERÉ VLASTNOSTI FUNKCÍ ZE TŘÍD JISTÝCH FAKTOROVÝCH PROSTORŮ

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Nechť S je dvojrozměrný prostor spojitých funkcí definovaných na otevřeném intervalu $\imath \subset E_1$ a $C^0(\imath)$ je lineární prostor všech spojitých funkcí na $\imath$. Za předpokladů, že S je regulární prostor určitého typu s monotonní fází α, plyne pro rozložení uzlů funkcí libovolné třídy X faktorového prostoru C^0/S následující závěr:

Věta 1.4. Nechť $[t_0, k_0] \in \imath \times E_1$ je libovolný bod. Buďte dále $x \in X$, $x \in \mathcal{S}(t_0, k_0)$ a $[t_j, x(t_j)]$, $[t_{j+1}, x(t_{j+1})] \in \mathcal{U}(t_0, k_0)$ sousední uzly. Jestliže $\bar{x} \in X$ je funkce, která těmito uzly neprochází, pak existuje právě jeden bod $\tau \in (t_j, t_{j+1})$ takový, že $[\tau, x(\tau)] = [\tau, \bar{x}(\tau)]$.

Výsledky této práce obsahují ve speciálním případě modifikaci Sturmovy věty pro nehomogenní lineární diferenciální rovnici 2. řádu, která je dokázána v [1.]

Резюме

НЕКОТОРЫЕ СВОЙСТВА ФУНКЦИЙ ПРИНАДЛЕЖАЩИХ КЛАССАМ СПЕЦИАЛЬНЫХ ФАКТОР-ПРОСТРАНСТВ

ЙИТКА КОЙЕЦКА

Пусть S – двумерное пространство непрерывных функций определенных в открытом промежутке $i \subset E_1$ и $C^\circ(i)$ – линейное пространство всех непрерывных в i функций. Если S регулярное пространство определенного типа с монотонной фазой α, то для распределения узлов функций любого класса X фактор-пространства C^0/S получается следующий результат:

Теорема 1.4. Пусть $[t_0, k_0] \in i \times E_1$ – любая точка. Пусть дальше $x \in X$, $x \in \mathcal{S}(t_0, k_0)$ и $[t_j, x(t_j)]$, $[t_{j+1}, x(t_{j+1})] \in \mathcal{U}(t_0, k_0)$ – последовательные узлы. Если $\bar{x} \in X$ функция непроходящая через эти узлы, тогда существует только одна точка $\tau \in (t_j, t_{j+1})$ такая, что $[\tau, x(\tau)] = [\tau, \bar{x}(\tau)]$.

Полученные в статье результаты содержают как специальный случай доказанное в [1] видоизменение теоремы Штурма для неоднородного линейного дифференциального уравнения 2-ого порядка.