Vladimír Vlček

On functions describing a distribution of zeros of solutions of an iterated linear differential equation of the fourth order

Sborník prací Přírodovědecké fakulty University Palackého v Olomouci. Matematika, Vol. 18 (1979), No. 1, 103--116

Persistent URL: http://dml.cz/dmlcz/120073

1979 — ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM — TOM 61

Katedra matematické analýzy a numerické matematiky
přírodovědecké fakulty Univerzity Palackého v Olomouci
Vedoucí katedry: prof. RNDr. Miroslav Laitoch, CSc.

ON FUNCTIONS DESCRIBING A DISTRIBUTION OF ZEROS OF SOLUTIONS OF AN ITERATED LINEAR DIFFERENTIAL EQUATION OF THE FOURTH ORDER

VLADIMÍR VLČEK

(Received March 31, 1978)

Dedicated to Academician O. Borůvka on his 80th birthday

Preliminary note

This article is topically a close continuation of [3]. Our object now is to study a distribution and to describe the properties of the so-called weakly conjugate points which, together with the strongly conjugate points (whose distribution and properties were considered in [3]), constitute a thorough description of zeros of any oscillatory solution — more precisely: of any oscillatory bundle of solutions — of an ordinary linear homogeneous differential equation of the fourth order. Its form is

$$y^{(IV)}(t) + 10[q(t)\,y'(t)]' + 3[3q^2(t) + q''(t)]\,y(t) = 0 \tag{1}$$

where the function $q(t) \in C_I^2$, $I = (-\infty, +\infty)$ and $q(t) > 0$ for all $t \in (-\infty, +\infty)$, produced by iteration of an ordinary linear homogeneous differential equation of the second order. Its form is

$$y''(t) + q(t)\,y(t) = 0 \tag{2}$$

whose basis $[u(t), v(t)]$ is formed by a pair of linearly independent functions $u(t), v(t)$ being of oscillatory type in the sense of [2] (that is, to every $t \in (-\infty, +\infty)$ there exist infinitely many zeros of an arbitrary non-trivial solution of (2) lying to the left and to the right of t) satisfying the condition

$$u(t_0) = v'(t_0) = 0 \tag{P}$$

where $t_0 \in (-\infty, +\infty)$ is an arbitrary firmly chosen point. This implies that $u'(t_0) \neq 0$, $v(t_0) \neq 0$ and therefore the point t_0 is a simple zero of the function $u(t)$. It is known of the differential equation (1), called iterated to the differential equation (2) that, if $[u(t), v(t)]$ is a basis of (2), than $u^3(t)$, $u^2(t)\,v(t)$, $u(t)\,v^2(t)$, $v^3(t)$ is a basis of (1) such that every non-trivial solution of (1) is of the form

$$y(t) = \sum_{i=1}^{4} C_i u^{4-i}(t)\, v^{i-1}(t) \tag{3}$$

where $C_i \in \mathbf{R}$, $i = 1, \ldots, 4$, $\sum_{i=1}^{4} C_i^2 > 0$.

Since we assume that (2) is oscillatory, it follows that any solution of this form is oscillatory too, and therefore the differential equation (1) is in short called the oscillatory equation.

In view of the fact that the differential equation (1) is of the fourth order, it is evident that any arbitrary zero of its non-trivial solution $y(t)$ is at most of multiplicity $\nu = 3$.

Throughout every solution both of (1) and (2) will be understood to be non-trivial, only.

Functions describing a distribution of the weakly conjugate points of a bundle of solutions of the differential equation (1)

Let $t_0 \in (-\infty, +\infty)$ be an arbitrary firmly chosen point and let us consider throughout that the system of all oscillatory solutions of the differential equation (1) which are vanishing at this point, form a three-parametric bundle

$$y(t, C_1, C_2, C_3) = u(t)\,[C_1 u^2(t) + C_2 u(t)\, v(t) + C_3 v^2(t)] \tag{4}$$

with $C_i \in \mathbf{R}$, $i = 1, 2, 3$, such that $\sum_{i=1}^{3} C_i^2 > 0$ and $[u(t), v(t)]$ being an oscillatory basis of all solutions of the differential equation (2) satisfying the condition (P) at the point t_0.

For the point t_0 to be a three- (or two- or one-) fold zero of the solution $y(t)$ of the differential equation (1), it is (by Lemma 1 in [3]) necessary and sufficient if $C_1 \neq 0$ and $C_2 = C_3 = 0$ (or $C_2 \neq 0$ and $C_3 = 0$ or $C_3 \neq 0$), i.e. the subsystem of the system (4) of all solutions of the differential equation (1) having at the point t_0

1. a triple zero, is exactly of the form

$$y(t, C_1) = C_1 u^3(t), \qquad C_1 \neq 0$$

2. a double zero, is exactly of the form

$$y(t, C_1, C_2) = u^2(t)\,[C_1 u(t) + C_2 v(t)], \qquad C_2 \neq 0$$

3. a simple zero, is exactly of the form

$$y(t, C_1, C_2, C_3) = u(t)\,[C_1u^2(t) + C_2u(t)\,v(t) + C_3v^2(t)], \qquad C_3 \neq 0.$$

Since (according to Theorem 1.1 in [3]) among all subsystems of the system (4) of all solutions of the differential equation (1) there exist exactly two such ones having all their zeros strongly conjugate, more precisely, exactly the subsystems of the form

$$y(t, C_1) = C_1u^3(t), \qquad C_1 \neq 0$$

(with all zeros being triple) and

$$y(t, C_1, C_2, C_3) = u(t)\,[C_1u^2(t) + C_2u(t)\,v(t) + C_3v^2(t)]$$
$$C_3 \neq 0, \qquad C_2^2 - 4C_1C_3 < 0$$

(with all zeros being simple) whereby – as proved – all these zeros are incident (they coincide) with all zeros of the function $u(t)$ of the basis $[u(t), v(t)]$ of the differential equation (2), it is possible (by Theorem 4.1 in [3]) to describe their distribution by means of the function $\varphi_k(t)$. Now let us consider only the remaining subsystems of the system (4) of all solutions of the differential equation (1) having besides the strongly conjugate zeros also the weakly conjugate ones. Such bundles of solutions are the subsystems of the system (4) either of the form

$$y(t, C_1, C_2) = u^2(t)\,[C_1u(t) + C_2v(t)], \qquad C_2 \neq 0$$

or

$$y(t, C_1, C_2, C_3) = u(t)\,[C_1u^2(t) + C_2u(t)\,v(t) + C_3v^2(t)]$$

$C_3 \neq 0$, where $C_2^2 - 4C_1C_3 \geqq 0$.

Function describing a distribution of the weakly conjugate points of the subsystem $y(t, C_1, C_2) = u^2(t)\,[C_1u(t) + C_2v(t)],\ C_2 \neq 0$

As known, (see Theorem 2.2 in [3]), between any two neighbouring strongly conjugate points of the subsystem

$$y(t, C_1, C_2) = u^2(t)\,[C_1u(t) + C_2v(t)], \qquad C_2 \neq 0$$

of all solutions of the differential equation (1) having a zero at the point t_0, with multiplicity $\nu = 2$, there lies exactly one weakly conjugate point of this subsystem, more precisely, with multiplicity $\mu = 1$.

While all the double strongly conjugate points of the above subsystem are coincident with all zeros of the function $u(t)$ of the basis $[u(t), v(t)]$ of the differential equation (2), all the simple weakly conjugate points of this subsystem are coincident with all zeros of the two-parametric subbundle of this subsystem:

$$y^*(t, C_1, C_2) = C_1u(t) + C_2v(t)$$

where $C_2 \neq 0$.

However, every function $y^*(t)$ belonging to this subbundle obtained in an arbitrary choice of constants $C_i \in \mathbf{R}$, i = 1, 2, $C_2 \neq 0$, is simultaneously a solution of the differential equation (2) which is linearly independent of the solution $u(t)$ of this equation on the interval $(-\infty, +\infty)$. By the STURM separation theorem this implies that all zeros of both functions $u(t)$ and $y^*(t)$ separate themselves mutually.

If we denote by T_1 the nearest zero of $u(t)$ lying to the right of the point t_0 (where $u(t_0) = 0$) and $t_1^* \in (t_0, T_1)$ is a zero of the function $y^*(t)$, then the nearest zero t_2^* of $y^*(t)$ lying to the right of the point t_1^* lies between T_1, T_2, where T_2 again is the further neighbouring zero of the point T_1 of $u(t)$ lying to the right of the point T_1, i.e. $t_2^* \in (T_1, T_2)$ and so on.

Generally: If we denote by ${}^2t_{2k}$, $k = 0, \pm 1, \pm 2, \ldots$, all the zeros of $u(t)$ that are the double strongly conjugate zeros of the subsystem $y(t, C_1, C_2)$ of the solutions of the differential equation (1), and if we denote by ${}^1t^*_{2k+1}$, $k = 0, \pm 1, \pm 2, \ldots$, all the (simple) zeros of whatever function $y^*(t)$ of the subbundle $y^*(t, C_1, C_2)$ obtained in an arbitrary firm choice of the constants $C_i \in \mathbf{R}$, i = 1, 2, $(C_2 \neq 0)$, then always

$${}^2t_{2k} < {}^1t^*_{2k+1} < {}^2t_{2k+2}$$

i.e. ${}^1t^*_{2k+1} \in ({}^2t_{2k}, {}^2t_{2k+2})$ for all k = $0, \pm 1, \pm 2, \ldots$

On using the function $\varphi_n(t)$, $n = 0, \pm 1, \pm 2, \ldots$ (cf. [2]) describing a distribution of all zeros of whatever solution of the differential equation (2) again, we observe that this function may be used to describe not only the distribution of all the strongly conjugate points ${}^2t_{2k}$ (as proved in Theorem 4.1 in [3]), but the distribution of all the weakly conjugate points ${}^1t^*_{2k+1}$, $k = 0, \pm 1, \pm 2, \ldots$, of the subsystem $y(t, C_1, C_2)$ of the solutions of the differential equation (1).

Function describing a distribution of the weakly conjugate points of the subsystem $y(t, C_1, C_2, C_3) = u(t)\,[C_1u^2(t) + C_2u(t)\,v(t) + C_3v^2(t)]$, $C_3 \neq 0$

A general survey of zeros of the tree-parametric bundle $y(t, C_1, C_2, C_3)$ of solutions of the differential equation (1) resulting from the analysis of this algebraic structure and the possibility to describe them by using the function $\varphi_k(t)$ applied even to the description of the weakly conjugate points, are given in

Theorem 4.2: Let $t_0 \in (-\infty, +\infty)$ be an arbitrary firmly chosen point. Let us consider a system

$$y(t, C_1, C_2, C_3) = u(t)\,[C_1u^2(t) + C_2u(t)\,v(t) + C_3v^2(t)] \tag{5}$$

$C_i \in \mathbf{R}$, $i = 1, 2, 3$, $C_3 \neq 0$, of all solutions of the differential equation (1), where $[u(t), v(t)]$ is a basis of an oscillatory differential equation (2) satisfying the condition (P) at t_0. Denote by T_1 the neighbouring zero of $u(t)$ lying to the right of the point t_0 (such that t_0, T_1 are the neighbouring strongly conjugate points of an arbitrary solution relating to the system (5)).

Then,

1. to the condition $C_2^2 - 4C_1C_3 = 0$ there corresponds a subsystem of all solutions of the differential equation (1) being exactly of the form (with an arbitrary non-zero multiplicative constant $C \in \mathbf{R}$)

$$y(t, C_1', C_2') = u(t)\, y_1^2(t, C_1', C_2') \tag{6}$$

where $y_1(t, C_1', C_2') = C_1'u(t) + C_2'v(t)$, $C_i' \in \mathbf{R}$, $i = 1, 2$, $C_2' \neq 0$, denotes a two-parametric system of all solutions of the differential equation (2) linearly independent with the function $u(t)$, whereby every solution relative to this system possesses exactly one zero between t_0, T_1, which is a double weakly conjugate point of the subsystem (6) of solutions of the differential equation (1)

2. to the condition $C_2^2 - 4C_1C_3 > 0$ there corresponds a subsystem of all solutions of the differential equation (1) being exactly of the form (with an arbitrary non-zero multiplicative constant $C \in \mathbf{R}$)

$$y(t, C_1', C_2', C_1'', C_2'') = u(t)\, y_1(t, C_1', C_2')\, y_2(t, C_1'', C_2'') \tag{7}$$

where $y_1(t, C_1', C_2') = C_1'u(t) + C_2'v(t)$, $y_2(t, C_1'', C_2'') = C_1''u(t) + C_2''v(t)$, $C_i', C_i'' \in \mathbf{R}$, $i = 1, 2$, $C_2'C_2'' \neq 0$, $C_1'C_2'' - C_2'C_1'' \neq 0$, denotes two double-parametric systems of all solutions of the differential equation (2) such that each two of the three functions $u(t)$, $y_1(t)$ and $y_2(t)$ are linearly independent on the interval $(-\infty, +\infty)$. Hereby either of the solutions $y_1(t)$, $y_2(t)$ selected consecutively from the systems $y_1(t, C_1', C_2')$, $y_2(t, C_1'', C_2'')$, with an arbitrary choice of constants $C_i', C_i'' \in \mathbf{R}$, $i = 1, 2$, satisfying the given conditions, possesses exactly one zero between t_0, T_1, either of which is a simple weakly conjugate point relative to the subsystem (7) of solutions of the differential equation (1)

3. to the condition $C_2^2 - 4C_1C_3 < 0$ there corresponds a subsystem of all solutions of the differential equation (1) being exactly of the form (with an arbitrary non-zero multiplicative constant $C \in \mathbf{R}$)

$$y(t, C_1', C_2', C_1'', C_2'') = u(t)\, y^*(t, C_1', C_2', C_1'', C_2'') \tag{8}$$

where $y^*(t, C_1', C_2', C_1'', C_2'')$ denotes a four-parametric system of functions – more precisely: a sum of the quadrates of two linearly independent solutions $y_1(t, C_1', C_2') = C_1'u(t) + C_2'v(t)$, $y_2(t, C_1'', C_2'') = C_1''u(t) + C_2''v(t)$, $C_i', C_i'' \in \mathbf{R}$, $i = 1, 2$, $C_1'C_2'' - C_2'C_1'' \neq 0$ (hence $C_i'^2 + C_i''^2 > 0$, $i = 1, 2$) of the differential equation (2) – which have no zeros on the interval $(-\infty, +\infty)$. It follows in this case that the subsystem (8) of solutions of the differential equation (1), having all the (simple) zeros of the function $u(t)$ as strongly conjugate points, has no weakly conjugate points.

Proof: By Lemma 1 [3] – with respect to the assumption of our Theorem – the system of all solutions of the differential equation (1) vanishing together with the function $u(t)$ at t_0, is exactly of the form

$$y(t, C_1, C_2, C_3) = u(t)\left[C_1 u^2(t) + C_2 u(t)\, v(t) + C_3 v^2(t)\right]$$

where $C_i \in \mathbf{R}$, $i = 1, 2, 3$, $C_3 \neq 0$, whose single, more precisely simple strongly conjugate points are exactly all zeros of the function $u(t)$. The weakly conjugate points of this system (so far such exist), may be only zeros of its three-parametric subbundle

$$y^*(t, C_1, C_2, C_3) = C_1 u^2(t) + C_2 u(t)\, v(t) + C_3 v^2(t), \qquad C_3 \neq 0.$$

1. Let $C_2^2 - 4C_1C_3 = 0$; we will show that there exist constants $C_i' \in \mathbf{R}$, $i = 1, 2$, $C_2' \neq 0$, such that for all $t \in (-\infty, +\infty)$

$$C_1 u^2(t) + C_2 u(t)\, v(t) + C_3 v^2(t) = k\left[C_1' u(t) + C_2' v(t)\right]^2$$

with $k \neq 0$ holds.

Since $\left[C_1' u(t) + C_2' v(t)\right]^2 = C_2'^1 u^2(t) + 2C_1'C_2' u(t)\, v(t) + C_2'^2 v^2(t)$, there must hold at the same time

$$C_1 = kC_1'^2, \qquad C_2 = 2kC_1'C_2', \qquad C_3 = kC_2'^2$$

whence it can be seen that in case of $C_1 \neq 0$, then $\operatorname{sgn} C_1 = \operatorname{sgn} C_3 = \operatorname{sgn} k$ (whereby $C_2 = 0$ exactly if $C_1 = 0$). On substituting for C_1, C_2, C_3 from the preceding relations, it is easy to verify that

$$C_2^2 - 4C_1C_3 = (2kC_1'C_2')^2 - 4k^2C_1'^2C_2'^2 = 0$$

and vice versa.

The existence of the exactly one zero t^* of an arbitrary function relative to the two-parametric system of solutions of the differential equation (2) having the form

$$y_1(t, C_1', C_2') = C_1' u(t) + C_2' v(t), \qquad C_2' \neq 0$$

between the two neighbouring simple zeros t_0, T_1 of the function $u(t)$, follows from the STURM theorem on reciprocal separation of zeros of any two linearly independent solutions of the differential equation (2), since the function $u(t)$ on an interval $(-\infty, +\infty)$ is linearly independent with an arbitrary function from the bundle $y_1(t, C_1', C_2')$, obtained in every choice of constants $C_i' \in \mathbf{R}$, $i = 1, 2$, $C_2' \neq 0$.

All zeros of any oscillatory function from the three-parametric subbundle

$$y^*(t, C_1, C_2, C_3) = C_1 u^2(t) + C_2 u(t)\, v(t) + C_3 v^2(t) = k\left[C_1' u(t) + C_2' v(t)\right]^2$$

where $k \neq 0$, $C_2' \neq 0$, are thus exactly the only, more precisely twofold – weakly conjugate points of the bundle of solutions

$$y(t, C_1, C_2, C_3) = u(t)\left[C_1 u^2(t) + C_2 u(t) v(t) + C_3 v^2(t)\right]$$

$C_3 \neq 0$, of the differential equation (1).

So, to describe them, we may use the function $\varphi_k(t)$ [2].

2. Let $C_2^2 - 4C_1C_3 > 0$; we will show that there exist the constants $C_i', C_i'' \in \mathbf{R}$, $i = 1, 2$, $C_2'C_2'' \neq 0$ and $C_1'C_2'' - C_2'C_1'' \neq 0$, such that for all $t \in (-\infty, +\infty)$ it holds (eventually up to an appropriate multiplicative constant $k \neq 0$):

$$C_1 u^2(t) + C_2 u(t) v(t) + C_3 v^2(t) = \left[C_1' u(t) + C_2' v(t)\right] . \left[C_1'' u(t) + C_2'' v(t)\right].$$

Because of

$$\begin{aligned}&\left[C_1' u(t) + C_2' v(t)\right]\left[C_1'' u(t) + C_2'' v(t)\right] = \\ &= C_1'C_1'' u^2(t) + (C_1'C_2'' + C_2'C_1'')\, u(t) v(t) + C_2'C_2'' v^2(t),\end{aligned}$$

it must hold at the same time also

$$C_1 = C_1'C_1'', \qquad C_2 = C_2'C_1'' + C_1'C_2'', \qquad C_3 = C_2'C_2''.$$

Hence we see that there must be both $C_2' \neq 0$ and $C_2'' \neq 0$ (especially if $C_2 = 0$, then for the purpose of $-4C_1C_3 > 0$, it must be $\operatorname{sgn} C_1 \neq \operatorname{sgn} C_3$ both different from zero, i.e. either $C_1 > 0$, $C_3 < 0$ or $C_1 < 0$, $C_3 > 0$). Then it holds indeed:

$$\begin{aligned}&C_2^2 - 4C_1C_3 = (C_2'C_1'' + C_1'C_2'')^2 - 4C_1'C_1''C_2'C_2'' = \\ &= (C_2'C_1'')^2 - 2C_2'C_1''C_1'C_2'' + (C_1'C_2'')^2 = (C_2'C_1'' - C_1'C_2'')^2 > 0\end{aligned}$$

exactly if $C_2'C_1'' - C_1'C_2'' \neq 0$, i.e. exactly if both two-parametric systems $y_1(t, C_1', C_2') = C_1' u(t) + C_2' v(t)$, $y_2(t, C_1'', C_2'') = C_1'' u(t) + C_2'' v(t)$ of solutions of the differential equation (2) are linearly independent and besides, (with respect to the condition $C_2'C_2'' \neq 0$), either of them is linearly independent with the function $u(t)$. This implies that all zeros of the system $y_1(t, C_1', C_2')$ are different from the zeros of the system $y_2(t, C_1'', C_2'')$ and, at the same time, all zeros of both these systems mutually separate among themselves (by the STURM separation theorem) and moreover they are separating with all zeros of the function $u(t)$, so that between any arbitrary two neighbouring zeros of $u(t)$ there always lies exactly one point of both the arbitrary function relative to the two-parametric system $y_1(t, C_1', C_2')$ and of the arbitrary function relative to the two-parametric system $y_2(t, C_1'', C_2'')$. All zeros, both the strongly and the weakly conjugate ones, of the three-parametric bundle $y(t, C_1, C_2, C_3)$ of solutions of the differential equation (1) are altogether simple in this case.

By [2], the function $\varphi_k(t)$ may be applied to the description of all the weakly conjugate points (i.e. zeros of any function of the both bundles $y_1(t, C_1', C_2')$,

$y_2(t, C_1'', C_2''))$, as well as to the description of the strongly conjugate points (i.e. zeros of the function $u(t)$) relative to the bundle $y(t, C_1, C_2, C_3)$ of solutions of the differential equation (1).

3. Let $C_2^2 - 4C_1C_3 < 0$; we will show that there exist real constants $C_i', C_i'' \in \mathbf{R}$, $i = 1, 2$, $C_1'^2 + C_1''^2 > 0$ and $C_2'^2 + C_2''^2 > 0$, $C_1'C_2'' - C_2'C_1'' \neq 0$, such that it holds (eventually up to an appropriate non-zero multiplicative constant $k \in \mathbf{R}$) for all $t \in (-\infty, +\infty)$:

$$C_1u^2(t) + C_2u(t)\,v(t) + C_3v^2(t) = \\ = [C_1'u(t) + C_2'v(t)]^2 + [C_1''u(t) + C_2''v(t)]^2$$

We will show first that there exist complex constants $D_i', D_i'' \in \mathbf{K}$ (where $\mathbf{K}$ denotes the set of all complex numbers), $i = 1, 2$, such that it holds (up to an appropriate non-zero multiplicative real constant again) for all $t \in (-\infty, +\infty)$:

$$C_1u^2(t) + C_2u(t)\,v(t) + C_3v^2(t) = \\ = [D_1'u(t) + D_2'v(t)]\,.\,[D_1''u(t) + D_2''v(t)]$$

where $D_1'D_2'' - D_2'D_1'' \neq 0$.
Since

$$[D_1'u(t) + D_2'v(t)]\,[D_1''u(t) + D_2''v(t)] = \\ = D_1'D_1'u^2(t) + (D_1'D_2'' + D_2'D_1'')\,u(t)\,v(t) + D_2'D_2''v^2(t)$$

it must simultaneously hold

$$C_1 = D_1'D_1'', \qquad C_2 = D_1'D_2'' + D_2'D_1'', \qquad C_3 = D_2'D_2''.$$

Since $C_i \in \mathbf{R}$, $i = 1, 2, 3$, $C_3 \neq 0$, then for the purpose $D_1'D_1'' \in \mathbf{R}$ and $D_2'D_2'' \in \mathbf{R} - \{0\}$, the complex numbers D_1', D_1'' as well as D_2', D_2'' must be complex conjugate, i.e. there exist numbers $C_j', C_j'' \in \mathbf{R}$, $j = 1, 2$, such that

$$D_1' = C_1' + iC_1'', \qquad D_1'' = C_1' - iC_1''$$
$$D_2' = C_2' + iC_2'', \qquad D_2'' = C_2' - iC_2''$$

(where i is the complex unit), which leads to

$$C_1 = C_1'^2 + C_1''^2, \qquad C_3 = C_2'^2 + C_2''^2.$$

Hereby (with respect to the assumption $C_2^2 - 4C_1C_3 < 0$, i.e. when there must be $\operatorname{sgn} C_1 = \operatorname{sgn} C_3 \neq 0$ for all $C_2 \in \mathbf{R}$), it holds

$$C_1'^2 + C_1''^2 > 0 \qquad \text{and} \qquad C_2'^2 + C_2''^2 > 0$$

simultaneously (in the following it appears that neither $C_1' = C_2' = 0$ nor $C_1'' = C_2'' = 0$ hold).

From this we get for C_2:

$$C_2 = D_1'D_2'' + D_2'D_1'' =$$
$$= (C_1 + iC_1'')(C_2' - iC_2'') + (C_2' + iC_2'')(C_1' - iC_1'') = 2(C_1'C_2' + C_1''C_2'') \in \mathbf{R}.$$

Since by the assumption $C_2^2 - 4C_1C_3 < 0$ and because of

$$C_2^2 - 4C_1C_3 = [2(C_1'C_2' + C_1''C_2'')]^2 - 4(C_1'^2 + C_1''^2)(C_2'^2 + C_2''^2) =$$
$$= 4(2C_1'C_2'C_1''C_2'' - C_1''^2C_2'^2 - C_1'^2C_2''^2) = -4(C_1'C_2'' - C_2'C_1'')^2$$

there must be necessary $C_1'C_2'' - C_2'C_1'' \neq 0$, and vice versa.

Consequently the three-parametric subsystem $y^*(t, C_1, C_2, C_3) = C_1u^2(t) + C_2u(t)\,v(t) + C_3v^2(t)$, $C_3 \neq 0$, relative to the bundle $y(t, C_1, C_2, C_3)$ of solutions of the differential equation (1) may be written in this case as

$$C_1u^2(t) + C_2u(t)\,v(t) + C_3v^2(t) =$$
$$= (C_1'^2 + C_1''^2)\,u^2(t) + 2(C_1'C_2' + C_1''C_2'')\,u(t)\,v(t) + (C_2'^2 + C_2''^2)\,v^2(t) =$$
$$= [C_1'u(t) + C_2'v(t)]^2 + [C_1''u(t) + C_2''v(t)]^2.$$

Since both the two-parametric systems $y_1(t, C_1', C_2') = C_1'u(t) + C_2'v(t)$, $y_2(t, C_1'', C_2'') = C_1''u(t) + C_2''v(t)$ of solutions of the differential equation (2) are on the interval $(-\infty, +\infty)$ with respect to the assumption $C_2'C_2'' - C_2'C_1'' \neq 0$ linearly independent, then no zero can exist on the interval $(-\infty, +\infty)$, where both these systems of functions would be vanishing at the same time. Hence it holds on the interval $(-\infty, +\infty)$ that

$$[C_1'u(t) + C_2'v(t)]^2 + [C_1''u(t) + C_2''v(t)]^2 > 0.$$

The three-parametric bundle $y(t, C_1, C_2, C_3)$ of solutions of the differential equation (1) in this case being (up to an arbitrary non-zero multiplicative constant $C \in \mathbf{R}$) of the form

$$y(t, C_1, C_2, C_3) = u(t)\,\{[C_1'u(t) + C_2'v(t)]^2 + [C_1''u(t) + C_2''v(t)]^2\}$$

has thus on the interval $(-\infty, +\infty)$ exactly all simple zeros coinciding with all zeros of the function $u(t)$ only, which are mutually strongly conjugate. Hence all zeros of this system may be described by means of the function $\varphi_k(t)$ [2].

SUPPLEMENT

In closing we show one more way for describing the position of the weakly conjugate points relative to the subbundles

$$y(t, C_1, C_2) = u^2(t)\,[C_1u(t) + C_2v(t)]$$

$C_i \in \mathbf{R}$, i = 1, 2, $C_2 \neq 0$, and

$$y(t, C_1, C_2, C_3) = u(t)\,[C_1u^2(t) + C_2u(t)\,v(t) + C_3v^2(t)]$$

$C_i \in \mathbf{R}$, i = 1, 2, 3, $C_3 \neq 0$, $C_2^2 - 4C_1C_3 \geqq 0$

of solutions of the differential equation (1), following on the application of the function $\eta_k(t)$ defined in § 4 in [3] for the purpose to describe the strongly conjugate points relative to the bundle $\sum_{i=1}^{3} C_i u^{4-i}(t)\, v^{i-1}(t)$ of solutions of the differential equation (1) (and – as proved in Theorem 4.1 [3] – to be coinciding with the function $\varphi_k(t)$).

In analogy with the preceding considerations in § 4 [3], let us now introduce a function describing a distribution of the weakly conjugate points relative to the bundle $y(t, C_1, C_2, C_3)$ of solutions of the differential equation (1) by following

Definition 4.2: Let us denote ${}^{\nu\mu}_{\Theta_k}\eta_k(t)$, where $\nu, \mu \in \{1, 2\}$, $\nu + \mu < 4$, $\Theta_k \in (0, 1)$, $k = \pm 1, \pm 2, \ldots$, a function which associates a k^{th} weakly conjugate point to the point $t \in (-\infty, +\infty)$ in the following way:

1. if $\nu + \mu = 3$, then ${}^{\nu\mu}_{\Theta_k}\eta_k(t)$ is such a $(2k - 1)^{\text{st}}$ conjugate point to the point t, that the proportion of its distance to the $2k^{\text{th}}$ conjugate point to the point t, to the distance of the $2(k + 1)^{\text{st}}$ and the $2k^{\text{th}}$ conjugate points to the point t is equal to Θ_k

2. if $\nu = \mu = 1$, then ${}^{11}_{\Theta_k}\eta_k(t)$ is such a

a) $(3m - 2)^{\text{nd}}$ – (in case that $k = 2m - 1$, $m = 0, \pm 1, \pm 2, \ldots$)
b) $(3m - 1)^{\text{nd}}$ – (in case that $k = 2m$, $m = \pm 1, \pm 2, \ldots$)

conjugate point to the point t that the proportion of its distance to the $3(m - 1)^{\text{st}}$ conjugate point to the point t, to the distance of the $3m^{\text{th}}$ and $3(m - 1)^{\text{st}}$ conjugate points to the point t is equal to Θ_k.

Theorem 4.3:

1. The interval $I = (-\infty, +\infty)$ is the interval of definition and the domain of the function values of all functions ${}^{\nu\nu}_{\Theta_k}\eta_k(t)$ introduced in the preceding Definition
2. For all $k = \pm 1, \pm 2, \ldots$ it holds the inequality

$$[{}^{\nu\mu}_{\Theta_k}\eta_k(t) - t] \operatorname{sgn} k > 0$$

3. $\lim\limits_{t \to -\infty} {}^{\nu\mu}_{\Theta_k}\eta_k(t) = -\infty, \quad \lim\limits_{t \to +\infty} {}^{\nu\mu}_{\Theta_k}\eta_k(t) = +\infty$
4. ${}^{\nu\mu}_{\Theta_k}\eta_k(t) \in C_I^3$ for $k = \pm 1, \pm 2, \ldots$
5. ${}^{\nu\mu}_{\Theta_k}\eta_k'(t) > 0$ for all $t \in (-\infty, +\infty)$ and for $k = \pm 1, \pm 2, \ldots$
6. For all $t \in (-\infty, +\infty)$, $k = \pm 1, \pm 2, \ldots$ and $\Theta_k \in (0, 1)$ it holds

$$\begin{aligned}
{}^{21}_{\Theta_k}\eta_k(t) &= \eta_{k-1}(t) + \Theta_k[\eta_k(t) - \eta_{k-1}(t)] \\
{}^{12}_{\Theta_k}\eta_k(t) &= \eta_{k-1}(t) + \Theta_k'[\eta_k(t) - \eta_{k-1}(t)] \\
{}^{11}_{{}_1\Theta_k}\eta_{2k-1}(t) &= \eta_{k-1}(t) + {}_1\Theta_k[\eta_k(t) - \eta_{k-1}(t)] \\
{}^{11}_{{}_2\Theta}\eta_{2k}(t) &= \eta_{k-1}(t) + {}_2\Theta_k[\eta_k(t) - \eta_{k-1}(t)]
\end{aligned}$$

whereby $0 < {}_1\Theta_k < {}_2\Theta_k < 1$, $\Theta' = \frac{1}{2}({}_1\Theta_k + {}_2\Theta_k)$.

Proof: All relations introduced in statement 6) of Theorem 4.3 follow directly from Definition 4.2. With some modification – for example – of the first relation in 6) in the form

$$\overset{21}{_{\Theta_k}}\eta_k(t) = \Theta_k\eta_k(t) + (1 - \Theta_k)\,\eta_{k-1}(t)$$

(and likewise of the others in 6)) it can be seen that on the interval $(-\infty, +\infty)$ there exists a derivative

$$\overset{21}{_{\Theta_k}}\eta_k(t) = \Theta_k\eta_k'(t) + (1 - \Theta_k)\,\eta_{k-1}'(t) > 0$$

$k = \pm 1, \pm 2, \ldots$, so that statement 5) generally is valid for the functions $\overset{\nu\mu}{_{\Theta_k}}\eta_k(t)$, $k = \pm 1, \pm 2, \ldots$ From the respective properties of the function $\eta_k(t)$ given by Theorem 4.1 [3] we get even all the remaining statements 1)–4) established by Theorem 4.3.

From statement 6) of the preceding Theorem 4.3 immediately follows the

Corollary: It holds for all $t \in (-\infty, +\infty)$ and for all $k = \pm 1, \pm 2, \ldots$:

1. if $\nu + \mu = 3$, then

$$\lim_{\Theta_k \to 0^+} \overset{\nu\mu}{_{\Theta_k}}\eta_k(t) = \eta_{k-1}(t)$$

$$\lim_{\Theta_k \to 1^-} \overset{\nu\mu}{_{\Theta_k}}\eta_k(t) = \eta_k(t),$$

where $\Theta_k \in (0, 1)$

2. if $\nu = \mu = 1$, then

$$\lim_{{}_1\Theta_k \to 0^+} \overset{11}{_{{}_1\Theta_k}}\eta_{2k-1}(t) = \lim_{{}_2\Theta_k \to 0^+} \overset{11}{_{{}_2\Theta_k}}\eta_{2k}(t) = \eta_{k-1}(t)$$

$$\lim_{{}_1\Theta_k \to 1^-} \overset{11}{_{{}_1\Theta_k}}\eta_{2k-1}(t) = \lim_{{}_2\Theta_k \to 1^-} \overset{11}{_{{}_2\Theta_k}}\eta_{2k}(t) = \eta_k(t)$$

where $0 < {}_1\Theta_k < {}_2\Theta_k < 1$.

Theorem 4.4: Let $\eta_p(t)$ or $\overset{\Theta_n}{_{\nu\mu}}\eta_n(t)$, where $\nu, \mu \in \{1, 2\}$, $p, n \in \{\pm 1, \pm 2, \ldots\}$, $\Theta_n \in (0, 1)$, denote the functions describing a distribution of the strongly or the weakly conjugate points to an arbitrary point $t \in (-\infty, +\infty)$, introduced by Definition 4.1 [3] or 4.2.

Then for all $k, m \in \{\pm 1, \pm 2, \ldots\}$ and for an arbitrary firmly chosen Θ_n (or Θ_n', ${}_1\Theta_n$, ${}_2\Theta_n$) $\in (0, 1)$, depending on the index n of the function $\overset{\nu\mu}{_{\Theta_n}}\eta_n(t)$, the following relations hold:

1. if $\nu = 2, \mu = 1$, then

$$\overset{21}{_{\Theta_k}}\eta_k(t) = \overset{21}{_{\Theta_m}}\eta_m[\eta_{k-m}(t)]$$

2. if $\nu = 1, \mu = 2$, then

$$\overset{12}{_{\Theta_k'}}\eta_k(t) = \overset{12}{_{\Theta_m'}}\eta_m[\eta_{k-m}(t)]$$

3. if $\nu = \mu = 1$, then

$$ {}^{11}_{{}_1\Theta_k}\eta_{2k-1}(t) = {}^{11}_{{}_1\Theta_m}\eta_m[\eta_{k-m}(t)] $$
$$ {}^{11}_{{}_2\Theta_k}\eta_{2k}(t) = {}^{11}_{{}_2\Theta_m}\eta_m[\eta_{k-m}(t)] $$

where $0 < {}_1\Theta_k < {}_2\Theta_k < 1$.

Proof: We will only prove the property 1) because the remaining properties can be proved completely analogous.

By Theorem 4.3, 6), it holds for every point $\tau \in (-\infty, +\infty)$, for all $\mathbf{m} = \pm 1, \pm 2, \dots$ and for $\Theta_m \in (0, 1)$:

$$ {}^{21}_{\Theta_m}\eta_m(\tau) = \eta_{m-1}(\tau) + \Theta_m[\eta_m(\tau) - \eta_{m-1}(\tau)]. $$

For $\tau = \eta_{k-m}(t)$, where $k = \pm 1, \pm 2, \dots$, and by utilizing the property of composition of the function η describing a distribution of the strongly conjugate points, we obtain

$$ {}^{21}_{\Theta_m}\eta_m[\eta_{k-m}(t)] = \eta_{m-1}[\eta_{k-m}(t)] + \Theta_m[\eta_m(\eta_{k-m}(t)) - \eta_{m-1}(\eta_{k-m}(t))] = $$
$$ = \eta_{k-1}(t) + \Theta_k[\eta_k(t) - \eta_{k-1}(t)] = {}^{21}_{\Theta_k}\eta_k(t) $$

and vice versa, q.e.d.

BIBLIOGRAPHY

[1] T. L. Sherman: Properties of solutions of N[th] order linear differential equations, Pacific J. Math., Vol. 15, No. 3, 1965.

[2] O. Borůvka: Lineare Differentialtransformationen 2. Ordnung, VEB Deutscher Verlag der Wissenschaften, Berlin, 1967.

[3] V. Vlček: On a distribution of zeros of solutions of a fourth-order iterated linear differential equation, Acta Univ. Palackianae Olomucensis (1977).

Souhrn

FUNKCE POPISUJÍCÍ ROZLOŽENÍ NULOVÝCH BODŮ ŘEŠENÍ ITEROVANÉ LINEÁRNÍ DIFERENCIÁLNÍ ROVNICE 4. ŘÁDU

VLADIMÍR VLČEK

V práci, bezprostředně navazující na [3], je studováno rozložení slabě konjugovaných bodů svazků řešení lineární diferenciální rovnice 4. řádu

$$y^{(IV)}(t) + 10[q(t)\,y'(t)]' + 3[3q^2(t) + q''(t)]\,y(t) = 0 \tag{1}$$

kde funkce $q(t) \in C_I^2$, $I = (-\infty, +\infty)$ a $q(t) > 0$ na int. $(-\infty, +\infty)$, získané iterací (oscilatorické) dif. rovnice 2. řádu tvaru

$$y''(t) + q(t)\,y(t) = 0. \tag{2}$$

Nejprve jsou vyšetřovány dva (dvou- a tříparametrické) podsvazky svazku řešení dif. rovnice (1) s výskytem jednoduchých resp. dvojnásobných slabě konjugovaných bodů.

Na závěr (s použitím výsledků rozboru algebraické struktury příslušného homogenního funkcionálního polynomu 2. stupně, obsaženého ve svazku řešení dif. rovnice (1)), jsou nalezeny i tvary odpovídajících řešení, vyjádřených pomocí svazků řešení dif. rovnice (2).

V práci je ukázáno, že rozložení všech slabě konjugovaných bodů řešení dif. rovnice (1) lze ve všech případech popsat toutéž funkcí $\varphi_k(t)$ [2], jakou bylo popsáno [3] rozložení silně konjugovaných bodů řešení dif. rovnice (1).

ФУНКЦИИ ОПИСЫВАЮЩИЕ РАЗЛОЖЕНИЕ НУЛЕВЫХ ТОЧЕК РЕШЕНИЙ ИТЕРИРОВАННОГО ЛИНЕЙНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ 4-го ПОРЯДКА

ВЛАДИМИР ВЛЧЕК

В работе, которая является непосредственным продолжением работы [3], исследуется разложение так называемых слабо сопряжённых точек пучков решений линейного дифференциального уравнения 4-го порядка

$$y^{(IV)}(t) + 10[q(t)\,y'(t)]' + 3[3q^2(t) + q''(t)]\,y(t) = 0, \tag{1}$$

с функцией $q(t) \in C_I^2$, $I = (-\infty, +\infty)$ и $q(t) > 0$ на интервале $(-\infty, +\infty)$, которое получится интерированием (колеблющегося) дифференциального уравнения 2-го порядка

$$y''(t) + q(t)\,y(t) = 0 \tag{2}$$

Прежде всего здесь изучаются два типа двух- или трёхпараметрических пучков решений диф. уравнения (1), у которых появляются только слабо сопряжённые точки.

В заключении — согласно с обнаружением всех возможных собственностей алгебраического характера, касающихся именно разложимости однородного функционального многочлена второй степени, который содержится в трёхпараметрическом пучке решений диф. уравнения (1) — построены соответствующие решения диф. уравнения (1), которые — как указывается — зависят на пучках решений самого диф. уравнения (2), при помощи которых они и прямо выражаются.

В работе показано, что разложение слабо сопряжённых точек решений диф. уравнения (1) в каждом отдельном случае можно описать совсем той же самой функцией $\varphi_k(t)$ [2], которая была уже использована [3] для описывания разложения сильно сопряжённых точек решений диф. уравнения (1).