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SEMI-ORDERED GROUPS

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From the point of view of the relation theory, the notion of a lattice is based on order relations. Likewise the notion of weakly associative lattices is based on semi-order relations, that is on reflexive and antisymmetric binary relations. (For the basic properties of weakly associative lattices see [2] and [5].)

In this paper we show some properties of semi-ordered groups, whereby a semi-ordered group is a group with a semi-order relation such that the group binary operation satisfies the monotony law. In particular, there are studied some properties of *wal*-groups, i.e. of such semi-ordered groups $(G, +, \leqq)$ where the semi-ordered set $(G, \leqq)$ is a weakly associative lattice.

1. Basic definitions and examples

A *semi-order* of a set A is any reflexive and antisymmetric binary relation on A. If $\leqq$ is a semi-order of A, then the pair $(A, \leqq)$ will be called a *semi-ordered set* (*so-set*). Let $(G, +)$ be a group and let $(G, \leqq)$ be a *so*-set. Then the triple $(G, +, \leqq)$ is called a *semi-ordered group* (*so-group*) if $a \leqq b$ implies $c + a \leqq c + b$ and $a + d \leqq \leqq b + d$ for all $a, b, c, d \in G$. A *so*-group is *directed* if for each $a, b \in G$ there exists $c \in G$ such that $a, b \leqq c$. Then, evidently, for each $a, b \in G$ there exists $d \in G$ such that $d \leqq a, b$. If $(G, +, \leqq)$ is a *so*-group such that $(G, \leqq)$ is a *wa*-lattice (see [2], [5]), then $(G, +, \leqq)$ is called a *weakly associative lattice-group* (*wal-group*). A *so*-group $(G, +, \leqq)$ is called a *tournament-group* (*to-group*) if $(G, \leqq)$ is a tournament.

If a semi-order of a group $(G, +)$ is transitive, i.e. if $(G, \leqq)$ is an ordered set (*po*-set), then $(G, +, \leqq)$ is an *ordered group* (*po-group*). It is evident that if a *wal*-group $(G, +, \leqq)$ is also a *po*-group, then $(G, \leqq)$ is a lattice and so $(G, +, \leqq)$ is a *lattice-ordered group* (*l-group*). Analogously, if a *to*-group $(G, +, \leqq)$ is also a *po*-group, then $(G, \leqq)$ is a *linearly ordered set* and $(G, +, \leqq)$ is a *linearly ordered group* (*o-group*).

Let $(G, +)$ be a group and let $(G, \leqq)$ be a *so*-set (a *po*-set) such that $a \leqq b$ implies $a + c \leqq b + c$ for all $a, b, c \in G$. Then $(G, +, \leqq)$ is called a *right so-group* (*a right po-group*).

Example 1. Let us consider a group $(G, +)$, where $G = \{0, a, b, c, d, e\}$ and addition is defined as:

$+$	0	a	b	c	d	e
0	0	a	b	c	d	e
a	a	0	d	e	b	c
b	b	c	0	a	e	d
c	c	b	e	d	0	a
d	d	e	a	0	c	b
e	e	d	c	b	a	0

We define the binary relation $\leqq$ on G: $0 \leqq c$, $a \leqq b$, $b \leqq e$, $c \leqq d$, $d \leqq 0$, $e \leqq a$ and $x \leqq x$ for each $x \in G$. Let us show that $(G, +, \leqq)$ is a right *so*-group which is neither a *so*-group nor a right *po*-group. It holds:

$$0 \leqq c \text{ and } a = 0 + a \leqq c + a = b,\ b = 0 + b \leqq c + b = e,\ c = 0 + c \leqq c + c = d,\ d = 0 + d \leqq c + d = 0,\ e = 0 + e \leqq c + e = a;$$

$$a \leqq b \text{ and } 0 = a + a \leqq b + a = c,\ d = a + b \leqq b + b = 0,\ e = a + c \leqq b + c = a,\ b = a + d \leqq b + d = e,\ c = a + e \leqq b + e = d;$$

$$b \leqq e \text{ and } c = b + a \leqq e + a = d,\ 0 = b + b \leqq e + b = c,\ a = b + c \leqq e + c = b,\ e = b + d \leqq e + d = a,\ d = b + e \leqq e + e = 0;$$

$$c \leqq d \text{ and } b = c + a \leqq d + a = e,\ e = c + b \leqq d + b = a,\ d = c + c \leqq d + c = 0,\ 0 = c + d \leqq d + d = c,\ a = c + e \leqq d + e = b;$$

$$d \leqq 0 \text{ and } e = d + a \leqq 0 + a = a,\ a = d + b \leqq 0 + b = b,\ 0 = d + c \leqq 0 + c = c,\ c = d + d \leqq 0 + d = d,\ b = d + e \leqq 0 + e = e;$$

$$e \leqq a \text{ and } d = e + a \leqq a + a = 0,\ c = e + b \leqq a + b = d,\ b = e + c \leqq a + c = e,\ a = e + d \leqq a + d = b,\ 0 = e + e \leqq a + e = c.$$

Therefore G is a right *so*-group. But G is not a *so*-group, since $a = a + 0 \nleqq a + c = e$. Moreover, G is not a right *po*-group since $0 \leqq c$, $c \leqq d$ and $0 \nleqq d$.

Example 2. As is known from the theory of *po*-groups, a group admitting a linear order (i.e. if it is an 0-group) is torsion free. In the case of abelian groups, this condition is also sufficient. (See [4].) E. Fried proved (in [3]) that the class of all groups admitting tournament semi-orders is essentially larger than the class of all 0-groups. For example, a torsion group admits a tournament semi-order if and only if it contains no element of order 2. We shall show a concrete construction of a tournament semi-order for the cyclic group of order n, where n is an arbitrary odd positive integer.

Let $n > 1$ be an odd number and let $(G, \oplus)$ be the cyclic group of the numbers $0, 1, \ldots, n - 1$ with addition $\oplus$ mod n. ($+$ and $-$ means addition and subtraction of integers, $\leqq$ and $<$ denote the relations "to be less than or equal to" and "to be strictly less than" in the natural ordering of integers, and $\langle x, y\rangle = \{z \in Z;\ x \leqq z \leqq y\}$ for $x, y \in Z$.)

We define $\prec$ on G as:

(I)
$$0 \prec y \Leftrightarrow y \in \left\langle 1, \frac{n-1}{2} \right\rangle \text{ for all } y \in G$$
$$z \prec 0 \Leftrightarrow z \in \left\langle \frac{n+1}{2}, n - 1 \right\rangle \text{ for all } z \in G$$

(II) Let $0 < x \leqq \frac{n-1}{2}$. Then
$$x \prec y \Leftrightarrow y \in \left\langle x + 1, \frac{n-1}{2} + x \right\rangle \text{ for all } y \in G$$
$$z \prec x \Leftrightarrow z \in \left\langle \frac{n+1}{2} + x, n - 1 \right\rangle \cup \langle 0, x - 1\rangle$$

(III) Let $\frac{n-1}{2} < x < n - 1$. Then
$$x \prec y \Leftrightarrow y \in \langle x + 1, n - 1\rangle \cup \left\langle 0, x - \frac{n+1}{2} \right\rangle \text{ for all } y \in G$$
$$z \prec x \Leftrightarrow z \in \left\langle x - \frac{n-1}{2}, x - 1 \right\rangle \text{ for all } z \in G$$

(IV)
$$n - 1 \prec y \Leftrightarrow y \in \left\langle 0, \frac{n-1}{2} - 1 \right\rangle \text{ for all } y \in G$$
$$z \prec n - 1 \Leftrightarrow z \in \left\langle \frac{n-1}{2}, n - 2 \right\rangle \text{ for all } z \in G$$

Prove that the relation $\preceq\ = (\prec \cup =)$ is a tournament semi-order of the group G. The reflexivity is trivial. We prove the antisymmetry.

1. Since $n \geqq 3$, we have $1 \leqq \frac{n-1}{2} < \frac{n+1}{2} \leqq n - 1$, thus $\left\langle 1, \frac{n-1}{2} \right\rangle \cap \cap \left\langle \frac{n+1}{2}, n - 1 \right\rangle = \emptyset$.

2. Let $0 < x \leqq \frac{n-1}{2}$. Then $x + 1 \leqq \frac{n-1}{2} + x < \frac{n+1}{2} + x \leqq n - 1$, therefore $\left\langle x + 1, \frac{n-1}{2} + x \right\rangle \cap \left\langle \frac{n+1}{2} + x, n - 1 \right\rangle = \emptyset$. Simultaneously $\left\langle x + 1, \frac{n-1}{2} + x \right\rangle \cap \langle 0, x - 1 \rangle = \emptyset$.

3. Let $\frac{n-1}{2} < x < n - 1$. Then $0 < x - \frac{n-1}{2} \leqq x - 1 < x + 1 \leqq n - 1$, hence $\langle x + 1, n - 1 \rangle \cap \left\langle x - \frac{n-1}{2}, x - 1 \right\rangle = \emptyset$. Moreover $0 \leqq x - \frac{n+1}{2} < x - \frac{n-1}{2} \leqq x - 1$, therefore $\left\langle 0, x - \frac{n+1}{2} \right\rangle \cap \left\langle x - \frac{n-1}{2}, x - 1 \right\rangle = \emptyset$.

4. If $0 \leqq \frac{n-1}{2} - 1 < \frac{n-1}{2} \leqq n - 2$, then $\left\langle 0, \frac{n-1}{2} - 1 \right\rangle \cap \left\langle \frac{n-1}{2}, n - 2 \right\rangle = \emptyset$. By the definition of $\prec$ we obtain the antisymmetry.

Next we show that $\preceq$ is a tournament semi-order.

1. It is clear that $\left\langle 1, \frac{n-1}{2} \right\rangle \cup \left\langle \frac{n+1}{2}, n - 1 \right\rangle \cup \{0\} = \{1, \ldots, n - 1\}$.

2. In the case $0 < x \leqq \frac{n-1}{2}$, we have $\left\langle x + 1, \frac{n-1}{2} + x \right\rangle \cup \left\langle \frac{n+1}{2} + x, n - 1 \right\rangle \cup \langle 0, x - 1 \rangle \cup \{x\} = \{1, \ldots, n - 1\}$.

3. For $\frac{n-1}{2} < x < n - 1$ there is $\langle x + 1, n - 1 \rangle \cup \left\langle 0, x - \frac{n+1}{2} \right\rangle \cup \left\langle x - \frac{n-1}{2}, x - 1 \right\rangle \cup \{x\} = \{1, \ldots, n - 1\}$.

4. It holds $\left\langle 0, \frac{n-1}{2} - 1 \right\rangle \cup \left\langle \frac{n-1}{2}, n - 2 \right\rangle \cup \{n - 1\} = \{1, \ldots, n - 1\}$.

Finally we prove that $a \preceq b \Rightarrow a \oplus c \preceq b \oplus c$ for all $a, b, c \in G$. Let $a, b, c \in G$, $a \prec b$.

1. Let $a = 0$. Then $1 \leqq b \leqq \frac{n-1}{2}$.

1α) Suppose $0 < c \leqq \frac{n-1}{2}$. Then $a + c = c$, $c + 1 \leqq b + c \leqq \frac{n-1}{2} + c$. But this means (by (II)) that $a \oplus c \prec b \oplus c$.

1β) Let $\frac{n-1}{2} < c < n - 1$. Then $a + c = c$, $c + 1 < b + c$. Hereby either $b + c \leqq n - 1$ or $n \leqq b + c < \frac{n-1}{2} + c$. In the second case it is $b \oplus c < \frac{n-1}{2} + c - n$, i.e. $b \oplus c < c - \frac{n+1}{2}$. Hence by (III) $a \oplus c \prec b \oplus c$.

1γ) Let $c = n - 1$. Then $a + c = c$, $n \leqq b + c \leqq \frac{n-1}{2} + n - 1$. This means that $0 \leqq b \oplus c \leqq \frac{n-1}{2} - 1$, therefore by (IV) $a \oplus c \prec b \oplus c$.

2. Let $b = 0$. Then $\frac{n+1}{2} \leqq a \leqq n - 1$.

2α) Suppose $0 < c \leqq n - 1$. Then $b + c = c$, $\frac{n+1}{2} + c < a + c \leqq n - 1 + c$. Let $n \leqq n - 1 + c$. Then $a \oplus c \leqq c - 1$, and so by (II) $a \oplus c \prec b \oplus c$.

2β) Let $\frac{n-1}{2} < c < n - 1$. Then $b + c = c$, $n \leqq \frac{n+1}{2} + c \leqq a + c < n - 1 + c$, i.e. $c - \frac{n-1}{2} \leqq a \oplus c < c - 1$. Hence by (III) $a \oplus c \prec b \oplus c$.

2γ) Let $c = n - 1$. Then $b + c = c$, $\frac{n+1}{2} + n - 1 \leqq a + c \leqq 2n - 2$, therefore $\frac{n-1}{2} \leqq a \oplus c \leqq n - 2$ and this means $a \oplus c \prec b \oplus c$.

3. From $0 < a \leqq \frac{n-1}{2}$ and from $0 < b \leqq \frac{n-1}{2}$ it follows $a + 1 \leqq b \leqq \frac{n-1}{2} + a$.

3α) Suppose $0 < c \leqq \frac{n-1}{2}$. Then evidently $0 < a + c < b + c \leqq n - 1$.

3αa) Let $a + c \leqq \frac{n-1}{2}$. Then $a + c + 1 \leqq b + c \leqq \frac{n-1}{2} + a + c$ and thereby (II) $a \oplus c \prec b \oplus c$.

3αb) Let $\frac{n-1}{2} < a + c < n - 1$. Then $a + c + 1 \leqq b + c \leqq n - 1$. From this and from (III) it follows $a \oplus c \prec b \oplus c$.

3β) Let $\frac{n-1}{2} < c < n - 1$.

3βa) Suppose $\frac{n-1}{2} < a + c < n - 1$. Hence $a + c + 1 < b + c$. Indeed, let $n \leqq b + c \leqq \frac{n-1}{2} + a + c$, i.e. $0 \leqq b \oplus c \leqq a + c - \frac{n+1}{2}$. Therefore by (III) $a \oplus c \prec b \oplus c$.

3βb) Let $n \leqq a + c$. Then $n < b + c < \frac{n-1}{2} + n - 1$, therefore $0 < b \oplus c < \frac{n-1}{2} - 1$. This means $0 \leqq a \oplus c < b \oplus c < \frac{n-1}{2} - 1$. Thus by (I) and (II) $a \oplus c \prec b \oplus c$.

3γ) Let $c = n - 1$. Then $n \leqq a + c < b + c \leqq \frac{n-1}{2} + n - 1$, and so $0 \leqq$ $\leqq a \oplus c < b \oplus c \leqq \frac{n-1}{2} - 1$. Hence by (I) and (II) $a \oplus c \prec b \oplus c$.

4. Suppose $0 < a \leqq \frac{n-1}{2}$, $\frac{n-1}{2} < b < n - 1$. Thus $a + 1 \leqq b \leqq \frac{n-1}{2} +$ $+ a$.

4α) Let $0 < c \leqq \frac{n-1}{2}$. Then $0 < a + c \leqq n - 1$.

4αa) If $a + c \leqq \frac{n-1}{2}$, then $a + c + 1 \leqq b + c \leqq \frac{n-1}{2} + a + c$, therefore by (II) $a \oplus c \prec b \oplus c$.

4αb) Let $\frac{n-1}{2} < a + c < n - 1$. Then $a + c + 1 < b + c \leqq \frac{n-1}{2} + a +$ $+ c$; therefore if $n \leqq b + c$, then $0 \leqq b \oplus c \leqq a + c - \frac{n+1}{2}$. But by (III) $a \oplus c \prec b \oplus c$. ;

4αc) Let $a + c = n - 1$. Then $n \leqq b + c \leqq \frac{n-1}{2} + a + \frac{n-1}{2} = n - 1 + a$, i.e. $0 \leqq b \oplus c \leqq a - 1 \leqq \frac{n-1}{2} - 1$. Thus by (IV) $a \oplus c \prec b \oplus c$.

4β) Let $\frac{n-1}{2} < c < n - 1$. Then $\frac{n-1}{2} < a + c < \frac{n-1}{2} + n - 1$.

4βa) Suppose $a + c < n - 1$. Then $a + c + 1 \leqq b + c$. If $n \leqq b + c$, then $b + c \leqq \frac{n-1}{2} + a + c$, hence $0 \leqq b \oplus c \leqq a + c - \frac{n+1}{2}$. Therefore by (III) $a \oplus c \prec b \oplus c$.

4βb) Let $a + c = n - 1$. Then $n \leqq b + c \leqq \frac{n-1}{2} + a + c = \frac{n-1}{2} + n -$ $- 1$. Therefore $0 \leqq b \oplus c \leqq \frac{n-1}{2} - 1$, i.e. by (IV) $a \oplus c \prec b \oplus c$.

4βc) Let $n \leqq a + c$. Then $a + c < \frac{n-1}{2} + n - 1$, which means $0 \leqq a \oplus c <$ $< \frac{n-1}{2} - 1$. Simultaneously $n < b + c \leqq \frac{n-1}{2} + a + c$. This means $0 <$ $< b \oplus c \leqq \frac{n-1}{2} + (a \oplus c)$. Hence by (I) and (II) $a \oplus c \prec b \oplus c$.

4γ) Suppose $c = n - 1$. Then $n \leqq a + c \leqq \frac{n-1}{2} + n - 1$, and so $0 \leqq a \oplus$ $\oplus c \leqq \frac{n-1}{2} - 1$. In addition there is $n \leqq a + c < b + c \leqq \frac{n-1}{2} + a + n - 1$, therefore $0 \leqq a \oplus c < b \oplus c \leqq \frac{n-1}{2} + a - 1$, hence $a \oplus c \prec b \oplus c$.

5. Suppose $\frac{n-1}{2} < a < n-1$, $0 < b \leqq \frac{n-1}{2}$. Then $0 < b \leqq a - \frac{n+1}{2}$.

5α) Let $0 < c \leqq \frac{n-1}{2}$. Then $0 < b + c \leqq n - 1$.

5α*a*) Let $\frac{n-1}{2} < a + c < n - 1$. Then $0 < b + c \leqq a + c - \frac{n+1}{2}$. Hence by (III) $a \oplus c \prec b \oplus c$.

5α*b*) Let $a + c = n - 1$. Then $0 < b + c \leqq a - \frac{n+1}{2} + c = (n-1) - \frac{n+1}{2} = \frac{n-1}{2} - 1$. Thus by (IV) $a \oplus c \prec b \oplus c$.

5α*c*) Let $n \leqq a + c$, i.e. $n \leqq a + c < n - 1 + \frac{n-1}{2}$, and so $0 \leqq a \oplus c < \frac{n-1}{2} - 1$. Then $0 < b + c \leqq a - \frac{n+1}{2} + c = (a \oplus c) + n - \frac{n+1}{2} = \frac{n-1}{2} + (a \oplus c)$. Thus by (I) and (II) $a \oplus c \prec b \oplus c$.

5β) Suppose $\frac{n-1}{2} < c < n - 1$. Hence $n \leqq a + c < 2n - 2$.

5β*a*) Let $0 \leqq a \oplus c \leqq \frac{n-1}{2}$. Then $b + c \leqq a - \frac{n+1}{2} + c = (a \oplus c) + n - \frac{n+1}{2} = \frac{n-1}{2} + (a \oplus c)$. In addition $\frac{n-1}{2} + (a \oplus c) \leqq n - 1$, which means $\frac{n-1}{2} < b + c \leqq n - 1$, thus $a \oplus c < b \oplus c$. Therefore by (I) and (II) $a \oplus c \prec b \oplus c$.

5β*b*) Let $\frac{n-1}{2} < a \oplus c < n - 2$. Hence, if $n \leqq b + c$, then $n \leqq b + c \leqq a + c - \frac{n+1}{2} = (a \oplus c) + n - \frac{n+1}{2}$ and so $0 \leqq b \oplus c \leqq (a \oplus c) - \frac{n+1}{2}$. If $b + c \leqq n - 1$, then $n \leqq a + c < b + c + n$ implies $0 \leqq a \oplus c < b + c$. Thus by (III) $a \oplus c \prec b \oplus c$.

5γ) Let $c = n - 1$. Then $\frac{n-1}{2} + n - 1 < a + c < 2n - 2$, hence $\frac{n-1}{2} \leqq a \oplus c < n - 2$. In addition $n < b + c \leqq a + c - \frac{n+1}{2}$, therefore $0 < b \oplus c \leqq (a \oplus c) - \frac{n+1}{2}$. Thus by (III) $a \oplus c \prec b \oplus c$.

6. Let $\frac{n-1}{2} < a < n-1$, $\frac{n-1}{2} < b < n-1$. Thus $b - \frac{n-1}{2} \leqq a \leqq \leqq b - 1$. Thus $b - \frac{n-1}{2} \leqq a \leqq b - 1$.

6α) Suppose $0 < c \leqq \frac{n-1}{2}$.

6αa) Let $\frac{n-1}{2} < b + c < n - 1$. Then $b + c - 1 \leqq a + c \leqq b + c - 1$, therefore by (III) $a \oplus c \prec b \oplus c$.

6αb) Let $b + c = n - 1$. Then $\frac{n-1}{2} < a + c \leqq n - 2$ and this implies by (IV) $a \oplus c \prec b \oplus c$.

6αc) Let $n \leqq b + c < \frac{n-1}{2} + n - 1$, i.e. $0 \leqq b \oplus c < \frac{n-1}{2} - 1$. If $a + c \leqq \leqq n - 1$, then $(b \oplus c) + n - \frac{n-1}{2} \leqq a + c$, which means $(b \oplus c) + \frac{n+1}{2} \leqq \leqq a + c$. Hence by (II) $a \oplus c \prec b \oplus c$.

6β) Let $\frac{n-1}{2} < c \leqq n - 1$. Then $n \leqq a + c < b + c < 2n - 2$, i.e. $0 \leqq \leqq a \oplus c < b \oplus c < n - 2$. Moreover, $n < b + c \leqq a + \frac{n-1}{2} + c = (a \oplus c) + + n + \frac{n-1}{2}$, thus $0 < b \oplus c \leqq (a \oplus c) + \frac{n-1}{2}$. Therefore by (II) $a \oplus c \prec \prec b \oplus c$.

7. Suppose $a = n - 1$. Hence $0 < b \leqq \frac{n-1}{2} - 1$. Furthermore $0 < c$ implies $a + c = n - 1 + c$, i.e. $a \oplus c = c - 1$.

7α) Let $0 < c \leqq \frac{n-1}{2}$. Then $a \oplus c < \frac{n-1}{2}$ and $0 < b + c \leqq \frac{n-1}{2} - 1 + + c = \frac{n-1}{2} + (a \oplus c)$. Thus by (II) $a \oplus c \prec b \oplus c$.

7β) Let $\frac{n-1}{2} < c < n - 1$. Hence $\frac{n-1}{2} \leqq a \oplus c < n - 2$.

7βa) Let $a \oplus c = \frac{n-1}{2}$, i.e. $c - 1 = \frac{n-1}{2}$. Then $c = \frac{n+1}{2}$ and $b + c = = b + \frac{n+1}{2} \leqq \frac{n-1}{2} - 1 + \frac{n+1}{2} = (a \oplus c) + \frac{n-1}{2}$. Moreover $\frac{n+1}{2} < < b + c$. Thus by (II) $a \oplus c \prec b \oplus c$.

7βb) Let $\frac{n-1}{2} < a \oplus c < n-2$. If $n \leqq b + c$, then $n \leqq b + c \leqq \frac{n-1}{2} - 1 + c = (a \oplus c) + \frac{n-1}{2}$, i.e. $0 \leqq b \oplus c \leqq (a \oplus c) - n + \frac{n-1}{2}$ and so $0 \leqq b \oplus c \leqq (a \oplus c) - \frac{n+1}{2}$. Let $b + c \leqq n - 1$. Then $a \oplus c = c - 1 < b + c$. In both cases we obtain by (III) $a \oplus c \prec b \oplus c$.

7γ) Let $c = n - 1$. Then $a + c = 2n - 2$, $n \leqq b + c \leqq \frac{n-1}{2} + n - 2$. Thus $a \oplus c = n - 2$ and $0 \leqq b \oplus c \leqq \frac{n-1}{2} - 2$. $n \geqq 5$ must hold, therefore $\frac{n-1}{2} < n - 2$. Furthermore, $b \oplus c \leqq \frac{n-1}{2} - 2 = (n-2) - \frac{n+1}{2} = (a \oplus c) - \frac{n+1}{2}$. Thus by (III) $a \oplus c \prec b \oplus c$.

8. Suppose $b = n - 1$. Then $\frac{n-1}{2} \leqq a \leqq n - 2$. In addition, $0 < c$ implies $b + c = n - 1 + c$, i.e. $b \oplus c = c - 1$.

8α) Let $0 < c \leqq \frac{n-1}{2}$. Then $n \leqq b + c \leqq \frac{n-1}{2} + n - 1$, and so $0 \leqq b \oplus c \leqq \frac{n-1}{2} - 1$.

8αa) If $b \oplus c = 0$, then $c = 1$ and $\frac{n-1}{2} + 1 \leqq a + c \leqq n - 1$, this means $\frac{n+1}{2} \leqq a + c \leqq n - 1$. Thus by (I) $a \oplus c \prec b \oplus c$.

8αb) Let $0 < b \oplus c \leqq \frac{n-1}{2} - 1$. If $a + c \leqq n - 1$, then $\frac{n-1}{2} + c \leqq a + c$. By this $\frac{n+1}{2} + c - 1 \leqq a + c$, therefore $\frac{n+1}{2} + (b \oplus c) \leqq a + c$. Thus by (II) $a \oplus c \prec b \oplus c$. Let $n \leqq a + c \leqq n - 2 + c$. Then $0 \leqq a \oplus c \leqq c - 2 = (c-1) - 1 = (b \oplus c) - 1$. Hence also by (II) $a \oplus c \prec b \oplus c$.

8β) Let $\frac{n-1}{2} < c < n - 1$. Then $n \leqq a + c < 2n - 3$, therefore $0 \leqq a \oplus c < n - 3$. Moreover, $\frac{n-1}{2} \leqq b \oplus c < n - 2$.

8βa) Let $a \oplus c = 0$. Then $a + c = n$, and so $2 \leqq c \leqq \frac{n-1}{2}$. This means $1 \leqq c - 1 \leqq \frac{n-1}{2} - 1$, thus by (I) $a \oplus c \prec b \oplus c$.

$8\beta b)$ Let $0 < a \oplus c \leqq \frac{n-1}{2}$. Then $(a \oplus c) + \frac{n-1}{2} = (a + c) - n + + \frac{n-1}{2} \geqq \frac{n-1}{2} + c - \frac{n+1}{2} = c > c - 1$ and thus $(a \oplus c) + \frac{n-1}{2} > > b \oplus c$. Furthermore $(a \oplus c) + 1 = (a + c) - n + 1 \leqq (n - 2) + c - n + 1 = = c - 1 = b \oplus c$. Therefore by (II) $a \oplus c \prec b \oplus c$.

$8\beta c)$ Let $\frac{n-1}{2} < a \oplus c < n - 3$. We know that $\frac{n-1}{2} \leqq b \oplus c$. But $\frac{n-1}{2} = = b \oplus c$ cannot hold. Namely, in the other case $a + c = a + \frac{n+1}{2}$ and thus $a + c = a + \frac{n+1}{2}$, and $(a \oplus c) - \frac{n+1}{2} = a \oplus c$, which implies $n = -1$. Therefore $\frac{n-1}{2} < b \oplus c$ holds always. In addition $(a \oplus c) + \frac{n-1}{2} = (a + c) - n + + \frac{n-1}{2} = a + c - \frac{n+1}{2} \geqq \frac{n-1}{2} + c - \frac{n+1}{2} = c - 1 = b \oplus c$, thus $a \oplus \oplus c \geqq (b \oplus c) - \frac{n-1}{2}$. And since $a \leqq n - 2$ and $n \leqq a + c < b + c < 2n - 2$, $a \oplus c < b \oplus c$. Hence by (III) we obtain $a + c \prec b \oplus c$.

$8\gamma)$ Let $c = n - 1$. Then $\frac{n-1}{2} + n - 1 \leqq a + c \leqq 2n - 3$, hence $\frac{n-1}{2} - - 1 \leqq a \oplus c \leqq n - 3$. If $n = 3$, then $a \oplus c = 0$ and $b \oplus c = 1$, thus $a \oplus c \prec \prec b \oplus c$. If $n > 3$, then $\frac{n-1}{2} < b \oplus c = n - 2$. Since $\frac{n-1}{2} - 1 \leqq a \oplus c$, it is $(n - 2) - \frac{n-1}{2} \leqq a \oplus c$, thus $(b \oplus c) - \frac{n-1}{2} \leqq a \oplus c$. And since $a \oplus c \leqq \leqq n - 3 = (b \oplus c) - 1$, by (III) $a \oplus c \prec b \oplus c$.

Example 3. Let $(Z, +)$ be the additive group of integers, "$\leqq$" the relation "to be less than or equal to" in the ordinary sense. Let us define a relation "$\preceq$" on Z as:
$a \preceq b \Leftrightarrow_{\mathrm{df}} a \leqq b$ and $b - a \neq 2$. Then $(Z, +, \preceq)$ is a *wal*-group which is neither a *to*-group nor an *l*-group.

2. Semi-ordered groups

Let G be a *so*-group, $G^+ = \{x \in G;\ 0 \leqq x\}$.

Theorem 1. a) *If* $(G, +, \leqq)$ *is a so-group, then* G^+ *is an invariant subset with* 0 *in* G *such that* $a \in G^+$ *and* $-a \in G^+$ *imply* $a = 0$ *for each* $a \in G$.

b) *If* $(G, +)$ *is a group,* P *an invariant subset with* 0 *in* G *containing no non-zero element with its opposite element, then* $(G, +, \leqq)$, *where* $a \leqq b$ *iff* $b - a \in P$ *for all* a, $b \in G$, *is a so-group and* $G^+ = P$.

Theorem 2. *A so-group* $(G, +, \leqq)$ *is a po-group if and only if* G^+ *is a subsemigroup of* $(G, +)$.

Proof. Let G^+ be a subsemigroup of $(G, +)$, $a \leqq b$, $b \leqq c$. Then $b - a$, $c - b \in G^+$ and $c - a = (c - b) + (b - a) \in G^+$, therefore $a \leqq c$.

Theorem 3. *Let* $G = (G, +, \leqq)$ *be a so-group. Then the following conditions are equivalent:*

(1) *G is directed.*

(2) $G = \{y - z; a \leqq y, a \leqq z\}$ *for each* $a \in G$.

(3) $G = \{y - z; y, z \in G^+\}$, *i.e.* $G = G^+ - G^+$.

(4) *For each* $x \in G$ *there exists* $y \in G^+$ *such that* $x \leqq y$.

Proof. (1) $\Rightarrow$ (2), (4) $\Rightarrow$ (2): Let $a, b \in G$ and let $c \in G^+$ such that $b \leqq c$. We denote $y = c + a$, $z = -b + c + a$. Then $y - z = c + a - (-b + c + a) = b$ and $y = c + a \geqq a$, $z = -b + c + a \geqq a$.

(2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (4): Let $x \in G$ and y, $z \in G^+$ such that $x = y - z$. Then $y = x + z \geqq x$.

(4) $\Rightarrow$ (1): Let $a, b \in G$, $d \in G^+$ such that $a - b \leqq d$. Then $a \leqq d + b$, $b \leqq d + b$, thus G is directed.

Let $G = (G, +, \leqq)$ be a *so*-group, $\emptyset \neq A \subseteq G$. Then we say that A is a *convex subset* of G if $a \leqq x$, $x \leqq b$ imply $x \in A$ for all $a, b \in A$, $x \in G$. A subgroup A of G is called a *convex subgroup* of G if A is a convex subset of G.

Theorem 4. *L et* $G = (G, +, \leqq)$ *be a so-group, A a subgroup of G. Then A is convex if and only if* $0 \leqq x$, $x \leqq a$ *imply* $x \in A$ *for each* $a \in A$, $x \in G$.

Proof. Let $a, b \in A$, $x \in G$, $a \leqq x$, $x \leqq b$. Then $0 \leqq -a + x$, $-a + x \leqq -a + b$, thus $-a + x \in A$ and so $x \in A$.

Let $(G, +, \leqq)$ and $(G', +, \leqq)$ be *so*-groups. A mapping φ: $G \to G'$ will be called a *so-homomorphism* $(G, +, \leqq) \to (G', +, \leqq)$ if φ is a homomorphism $(G, +) \to (G', +)$ and simultaneously φ is a homomorphism $(G, \leqq) \to (G', \leqq)$ (i.e. $a \leqq b$ implies $a\varphi \leqq b\varphi$ for all $a, b \in G$).

Theorem 5. *Let* $G = (G, +, \leqq)$ *be a so-group. Then a normal subgroup A of G is the kernel of a so-homomorphism if and only if A is convex.*

Proof. a) Let φ: $G \to G'$ be a *so*-homomorphism, $0'$ the zero-element in G'. Let us denote $A = \operatorname{Ker} \varphi$. Suppose $a \in A$, $x \in G$, $0 \leqq x$, $x \leqq a$. Then $0\varphi \leqq x\varphi$, $x\varphi \leqq a\varphi$, i.e. $0' \leqq x\varphi$, $x\varphi \leqq 0'$, and thus $x\varphi \in A$.

b) Let A be a normal convex subgroup of G, $\bar{G} = G/A$. Let us consider the relation "$\leqq$" on $\bar{G}$ defined as:

$x + A \leqq y + A \Leftrightarrow_{\mathrm{df}}$ there exists $a \in A$ such that $x + a \leqq y$. Let us show that this definition is correct. Suppose that $x, x_1, y, y_1 \in G$ and that $x_1 + A = x + A$, $y_1 + A = y + A$. Then there exist $b, c \in A$ such that $x_1 + b = x$, $y_1 + c = y$, i.e. $x_1 + b + a \leqq y_1 + c$. Therefore $x_1 + (b + a - c) \leqq y_1$ and thus $x_1 + A \leqq y_1 + A$.

The reflexivity of $\leqq$ is evident. Let us show that $\leqq$ is antisymmetric. Let $x, y \in G$, $x + A \leqq y + A$, $y + A \leqq x + A$. Then there exist $a, b \in A$ such that $x + a \leqq y$, $y + b \leqq x$. By this $y + b + a \leqq x + a$, $x + a \leqq y$, thus $b + a \leqq -y + x + a$, $-y + x + a \leqq 0$. Since A is convex, $-y + x + a \in A$. Therefore $-y + x \in A$, and so $x + A = y + A$.

Now, let $x, y, z \in A$, $x + A \leqq y + A$. Then there exists $a \in A$ such that $x + a \leqq$ $\leqq y$. Thus $x + a + z \leqq y + z$ and since A is normal, $x + z + a_1 \leqq y + z$ for $a_1 \in A$ satisfying $a + z = z + a_1$. Therefore $(x + A) + (z + A) \leqq (y + A) + (z + A)$. Similarly $(z + A) + (x + A) \leqq (z + A) + (y + A)$.

Finally, it is evident that the natural mapping $v: G \to G/A$ is a *so*-homomorphism.

Note. The semi-order $\leqq$ of the factor group G/A defined in the proof of Theorem 5 is called an *induced semi-order.*

3. Weakly associative lattice-groups

Now, we shall show some properties of *wal*-groups. Let $G = (G, +, \leqq)$ be a *wal*-group. If $a, b \in G$, then $a \vee b$ denotes the element $c \in G$ such that $a \leqq c$, $b \leqq c$ and $c \leqq c'$ for all $c' \in G$ satisfying $a \leqq c'$, $b \leqq c'$. By the duality we define $a \wedge b$.

Theorem 6. *If G is a wal-group,* $a, b, c \in G$, *then*

1. $a + (b \vee c) = (a + b) \vee (a + c)$;
2. $a + (b \wedge c) = (a + b) \wedge (a + c)$;
3. $a \wedge b = -(-a \vee -b)$.

Proof. 1. From $b, c \leqq b \vee c$ it holds $a + b$, $a + c \leqq a + (b \vee c)$ and thus $(a + b) \vee (a + c) \leqq a + (b \vee c)$. Let $x \in G$ such that $a + b$, $a + c \leqq x$, Then $b \leqq -a + x$, $c \leqq -a + x$, thus $b \vee c \leqq -a + x$ and this implies $a + (b \vee c) \leqq x$.

2. Dually.

3. Since $-a, -b \leqq -a \vee -b$, $-(-a \vee -b) \leqq a, b$. Let $x \in G$ such that $x \leqq$ $\leqq a, b$. Then $-a, -b \leqq -x$, therefore $-a \vee -b \leqq -x$, and so $x \leqq -(-a \vee -b)$.

Note. Now, the wal-groups evidently form a variety of algebras of the type $\langle 2, 0, 1, 2 \rangle$ with two binary operations $+$ and $\vee$, with one nullary operation 0 and with one unary operation $-$ (.).

Theorem 7. *If* $(G, +, \leqq)$ *is a so-group, then the following conditions are equivalent*:

(1) *G is a wal-group.*
(2) *For each* $g \in G$ *there exists* $g \vee 0$.

Proof. (1) $\Rightarrow$ (2): Trivial.
(2) $\Rightarrow$ (1): Let $a, b \in G$. Then $[(a - b) \vee 0] + b = (a - b + b) \vee b = a \vee b$.

Let G be a *wal*-group, $x \in G$. Let us denote $|x| = x \vee -x$. It is clear that $-|x| \leqq x$, $x \leqq |x|$.

Theorem 8. *If A is a convex subgroup of a wal-group G, $a \in A$, $x \in G$ and if* $0 \leqq |x|$, $|x| \leqq |a|$ *or* $|x| \leqq 0$, $|a| \leqq |x|$, *then* $x \in A$.

Proof. Let $a \in A$, $x \in G$, $0 \leqq |x|$, $|x| \leqq |a|$. But then $|x| \in A$. And since $-|x| \leqq x$ and $x \leqq |x|$, $x \in A$. Similarly we can prove the case $|x| \leqq 0$, $|a| \leqq |x|$.

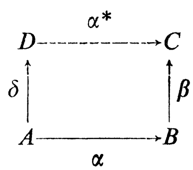
Let $G = (G, +, \leqq)$ be a *wal*-group, A a subgroup of G. Then A is called a *wal-subgroup* of G, if A is a *wa*-sublattice of $(G, \leqq)$. A *wal-ideal* of G is any normal convex *wal*-subgroup A of G which satisfies the following condition: For all $a, b \in A$, $x, y \in G$ such that $x \leqq a$, $y \leqq b$ there exists $c \in A$ such that $x \vee y \leqq c$. (It is clear that if G is an *l*-group, A a normal subgroup of G, then A is a *wal*-ideal of G if and only if A is an *l*-ideal of G.)

Let $(G, +, \leqq)$, $(G', +, \leqq)$ be *wal*-groups. A mapping φ: $G \to G'$ is called a *wal*-homomorphism $(G, +, \leqq) \to (G', +, \leqq)$ if simultaneously φ is a group homomorphism $(G, +) \to (G', +)$ and a *wa*-lattice homomorphism $(G, \leqq) \to (G', \leqq)$. It is evident that each *wal*-homomorphism is a *so*-homomorphism.

Theorem 9. *If G, G' are wal-groups, $\varphi : G \to G'$ a wal-homomorphism, then* Ker φ *is a wal-ideal of G.*

Proof. Let $\varphi : G \to G'$ be a *wal*-homomorphism and let $0'$ be the zero-element in G'. Let $A = \operatorname{Ker} \varphi$. By Theorem 5 A is convex. Let $a, b \in A$. Then $(a \vee b) \varphi = a\varphi \vee b\varphi = 0' \vee 0' = 0'$, thus $a \vee b \in A$. Let $x, y \in G$, $a, b \in A$, $x \leqq a$, $y \leqq b$. Then $x\varphi \leqq a\varphi = 0'$, $y\varphi \leqq b\varphi = 0'$, therefore $(x \vee y) \varphi = x\varphi \vee y\varphi \leqq 0'$ and so $(x \vee y) \varphi \vee 0' = 0'$. Let $d \in A$. Then $[(x \vee y) \vee d] \varphi = (x \vee y) \varphi \vee d\varphi = (x \vee y) \varphi \vee 0' = 0'$, thus $(x \vee y) \vee d \in A$. This implies the existence of $c \in A$ such that $(x \vee y) \vee d = c$ and therefore $x \vee y \leqq c$.

Theorem 10. *Let A, B, C, D be wal-groups and let $\alpha : A \to B$, $\beta : B \to C$, $\delta : A \to D$ be wal-homomorphisms such that δ is surjective and* $(\operatorname{Ker} \delta) \alpha \subseteq \operatorname{Ker} \beta$. *Then there exists exactly one wal-homomorphism $\alpha^* : D \to C$ such that the diagram*



commutes.

Proof. The existence of the unique group homomorphism α^* is known. Let $d \in D$ and let $a \in A$ such that $a\delta = d$. Then $(d \vee 0_D) \alpha^* = (a\delta \vee 0_A\delta) \alpha^* = (a \vee 0_A) \delta\alpha^* = (a \vee 0_A) \alpha\beta = a\alpha\beta \vee 0_A\alpha\beta = a\delta\alpha^* \vee 0_C = d\alpha^* \vee 0_D\alpha^*$. ($0_A$, 0_C, 0_D is the zero-element in A, C, D, respectively.) Then α^* is a *wal*-homomorphism.

Theorem 11. *If A is a wal-ideal of a wal-group G, then A is the kernel of a wal-homomorphism. Moreover, if $\varphi : G \to G'$ is a wal-homomorphism with the kernel A, then the mapping $\psi : G/A \to G'$, defined by $(x + A)\,\psi = x\varphi$ for all $x \in G$, is a wal-isomorphism.*

Proof. By the proof of Theorem 5, G/A is a *so*-group with respect to the induced semi-order. Let $x, y \in G$. Then $x + A$, $y + A \leqq (x \vee y) + A$. Let $z \in G$ such that $x + A, y + A \leqq z + A$. Then there exist $a, b \in A$ for which $x + a \leqq z$, $y + b \leqq z$, i.e. $-z + x \leqq -a$, $-z + y \leqq -b$. Since A is a *wal*-ideal, there exists $c \in A$ such that $(-z + x) \vee (-z + y) \leqq -c$. This implies $-z + (x \vee y) \leqq -c$, hence $(x \vee y) + + c \leqq z$ and thus $(x \vee y) + A \leqq z + A$. But this means that $(x + A) \vee (y + A) = = (x \vee y) + A$, and so G/A is a *wal*-group and the natural homomorphism $\nu : (G, +) \to \to (G/A, +)$ is a *wal*-homomorphism.

Now, let $\varphi : G \to G'$ be a *wal*-homomorphism. Then by Theorem 10, the diagram

$$\begin{array}{ccc} G/\mathrm{Ker}\,\varphi & \xrightarrow{\varphi^* = \psi} & G' \\ \nu \uparrow & & \uparrow 1_{G'} \\ G & \xrightarrow[\varphi]{} & G' \end{array}$$

commutes, ψ is a *wal*-homomorphism and $(x + \mathrm{Ker}\,\varphi)\,\psi = x\nu\psi = x\varphi\,1_{G'} = x\varphi$ for each $x \in G$.

Let G be a *wal*-group. We denote the set of all *wal*-ideals of G by $\mathscr{L}(G)$.

Theorem 12. *Let G be a wal-group, $A, B \in \mathscr{L}(G)$, $A \subseteq B$. Then $B/A \in \mathscr{L}(G/A)$ and the natural group isomorphism $\nu : G/B \to (G/A)/(B/A)$ is a wal-isomorphism.*

Proof. By Theorem 10, the diagram

$$\begin{array}{ccc} G/B & \xrightarrow{\nu_A^* = \nu} & (G/A)/(B/A) \\ \nu_B \uparrow & & \uparrow \nu_{B/A} \\ G & \xrightarrow[\nu_A]{} & G/A \end{array}$$

where ν_B, ν_A, $\nu_{B/A}$ are the natural homomorphisms, commutes and ν is a *wal*-isomorphism.

Let G be a group, $\emptyset \neq A \subseteq G$. Then $[A]$ denotes the subgroup of G generated by A.

Theorem 13. *Let G be a wal-group, H a wal-subgroup of G and C a convex wal-subgroup of G which is a wal-ideal of $[H \cup C]$. Then $H \cap C \in \mathscr{L}(H)$, $H + C$ is a wal-subgroup of G and the natural isomorphism $\nu : H/(H \cap C) \to (H + C)/C$ is a wal-isomorphism.*

Proof. Since C is a normal subgroup of $[H \cup C]$, $[H \cup C] = H + C$. Let $x = h + c \in H + C$. Then $x + C = h + C$, hence $(x \vee 0) + C = (x + C) \vee C = (h + C) \vee C = (h \vee 0) + C$ and this means $x \vee 0 = (h \vee 0) + d$, where $d \in C$, therefore $x \vee 0 \in H + C$. Thus $H + C$ is a *wal*-subgroup of G.

Let $h \in H$, $c \in H \cap C$, $0 \leqq h$, $h \leqq c$. Since C is convex in G, $h \in C$ and $H \cap C$ is convex in H. Then it is evident that $H \cap C \in \mathscr{L}(H)$.

Let us consider the diagram

$$\begin{array}{ccc} H/(H \cap C) & \xrightarrow{\nu = \alpha^*} & (H + C)/C \\ \uparrow {\scriptstyle \nu_{H \cap C}} & & \uparrow {\scriptstyle \nu_C} \\ H & \xrightarrow[\alpha = 1_{H,H+C}]{} & H + C \end{array}$$

where $\nu_{H \cap C}$, ν_C are the natural homomorphisms. Since $(\operatorname{Ker} \nu_{H \cap C})\, 1_{H,H+C} = (H \cap C)\, 1_{H,H+C} = H \cap C \subseteq C = \operatorname{Ker} \nu_C$, the diagram (by Theorem 10) commutes and the group isomorphism ν is a *wal*-isomorphism.

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Souhrn

SEMIUSPOŘÁDANÉ GRUPY

JIŘÍ RACHŮNEK

Semiuspořádanou grupou se rozumí grupa s relací semiuspořádání, tj. s reflexivní a antisymetrickou binární relací, taková, že grupová binární operace splňuje zákon monotonie. V článku jsou ukázány některé vlastnosti semiuspořádaných grup, speciálně pak *wal*-grup, tzn. semiuspořádaných grup $(G, +, \leqq)$ takových, že $(G, \leqq)$ je slabě asociativní svaz.

Резюме

ПОЛУУПОРЯДОЧЕННЫЕ ГРУППЫ

ЙИРЖИ РАХУНЕК

Полуупорядоченная группа — это группа с отношением полупорядка, т. е. с рефлексивным и антисимметрическим бинарным отношением, такая, что групповая бинарная операция выполняет закон монотонии.

В статье показаны некоторые свойства полуупорядоченных групп, именно *wal*-групп, т. е. таких полуупорядоченных групп $(G, + \leqq)$, что $(G, \leqq)$ — слабо ассоциативная решётка.