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ALGEBRAIC PROPERTIES OF PHASES OF THE DIFFERENTIAL EQUATION $y'' = q(t)\,y$

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This paper is devoted to the description of the algebraic structure of the set of phases of the second order ordinary linear differential equation $y'' = q(t)\,y$.

Basic concepts and relations used in this paper are taken from [1], where they are defined and proved. For completeness we give below a brief summary of them.

We shall consider a both-side oscillatory differential equation

$$y'' = q(t)\,y, \tag{q}$$

where the carrier $q(t)$ is a continuous function on the interval $(-\infty, \infty)$, that is, $q(t) \in C^0$. Let $u(t)$, $v(t)$ be a base of the differential equation (q), that is, a pair of linearly independent solutions of (q). A function α, continuous on $(-\infty, \infty)$ and satisfying the relation

$$\tan \alpha(t) = u(t)/v(t) \tag{1}$$

wherever $v(t) \neq 0$, is called the first phase of (q) corresponding to the base $u(t)$, $v(t)$ (henceforth a phase of (q)). For every phase α of the differential equation (q) there holds $\alpha \in C^3$, $\alpha'(t) \neq 0$ for $t \in (-\infty, \infty)$. The converse is valid, too. Namely, the function α satisfying the property

$$\alpha \in C^3, \qquad \alpha'(t) \neq 0 \qquad \text{for } t \in (-\infty, \infty)$$

is a phase of the differential equation (q) where (q) is determined by the relation

$$q(t) = -\{\tan \alpha, t\} = -\{\alpha, t\} - (\alpha'(t))^2 =$$
$$= -(1/2)\,\alpha'''/\alpha' + (3/4)\,(\alpha''/\alpha')^2 - (\alpha')^2.$$

Let $t_0 \in (-\infty, \infty)$, and y be a nontrivial solution of (q), whereby $y(t_0) = 0$. Let $\varphi(t_0) \in (-\infty, \infty)$ be the first zero of the solution y lying on the right of t_0. Then φ is called the basic central dispersion of the 1st kind of the differential equation (q) (henceforth the basic central dispersion). Similarly, if $\varphi_n(t_0)$ $[\varphi_{-n}(t_0)]$ is the n-th zero of the solution y lying on the right [on the left] of t_0, the function $\varphi_n[\varphi_{-n}]$ is called the n-th $[-n\text{-th}]$ central dispersion of the 1st kind of (q) (henceforth n-th $[-n\text{-th}]$ central dispersion).

If α is a phase of the differential equation (q) and φ is the 1st kind basic central dispersion of the differential equation (q), then Abel's equation

$$\alpha(\varphi(t)) = \alpha(t) + \pi \operatorname{sgn} \alpha'$$

is satisfied on the whole interval $(-\infty, \infty)$. Similarly the n-th dispersion φ_n, $n = 0, \pm 1, \pm 2, \ldots$, satisfies

$$\alpha(\varphi_n(t)) = \alpha(t) + n\pi \operatorname{sgn} \alpha'.$$

Moreover, in [1], there are defined more general transformation functions than central dispersions. There are dispersions of the 1st kind (henceforth dispersions) of (q).

The set $\boldsymbol{D}_1$ of all dispersions of (q) with an operation of composition of functions forms a group. The set $\boldsymbol{C}_1 \subset \boldsymbol{D}_1$ of all central dispersions forms a cyclic subgroup and the set $\boldsymbol{S}_1 \subset \boldsymbol{C}_1$ of all central dispersions with an even index forms a cyclic subgroup which is a normal subgroup of $\boldsymbol{D}_1$. The factor group $\boldsymbol{D}_1/\boldsymbol{S}_1$ and the group $\boldsymbol{L}$ of all unimodular matrices of the 2nd order are isomorphic. (Unimodular matrices possessing determinants equal to $+1$ or -1)

1. Let us denote the space of all solutions of (q) by R, the set of all phases of (q) by Q, the set of all integers by I. In [2], there was introduced a scalar product in R as follows:

Let $\alpha(t)$ be an arbitrary fixed chosen phase of (q) and $\varphi(t)$ be the basic central dispersion of (q). Then the composition

$$(f, g) = \int_t^{\varphi(t)} [\alpha'(\tau)]^2 f(\tau)\, g(\tau)\, \mathrm{d}\tau, \tag{2}$$

where f, g are arbitrary elements of R, t an arbitrary element in $j = (-\infty, \infty)$, is a scalar product in R.

The functions

$$u(t) = \sin \alpha(t)/\sqrt{|\alpha'(t)|}, \qquad v(t) = \cos \alpha(t)/\sqrt{|\alpha'(t)|} \tag{3}$$

form an ortonormal base in R with respect to the scalar product (2).

Note. Throughout this paper $\alpha(t)$ will always denote the phase from (2) and $u(t)$, $v(t)$ the base (3).

2. Let us consider the base $u(t), v(t)$. There exists the countable phase system $\alpha_n(t) = \alpha(t) + n\pi$, $n = 0, \pm 1, \pm 2, \ldots$ belonging to u, v. Moreover the phases

$$\alpha_{2k}(t) = \alpha(t) + 2k\pi, \qquad k = 0, \pm 1, \pm 2, \ldots$$

are proper phases of u, v and the phases

$$\alpha_{2k+1}(t) = \alpha(t) + (2k + 1)\,\pi, \qquad k = 0, \pm 1, \pm 2, \ldots$$

are improper phases of u, v. The set of all proper phases of u, v will be denoted by A.

$$A = \{\alpha_{2k}(t) = \alpha(t) + 2k\pi, \qquad k \in I\}. \tag{4}$$

Now we introduce the relation $\sim$ in the set Q as follows: Let α_i, $\alpha_k \in Q$, then $\alpha_i \sim \alpha_k$ iff there exists $m \in I$ such that $\alpha_i - \alpha_k = 2m\pi$. The relation $\sim$ is an equivalence relation on Q and each class of equivalence $A_i \in Q/\sim$ consists of all proper (improper) phases of the same base of (q). Denoting this base U, V, the class A will be called a class of phases of the base U, V.

Let us denote the set of the functions $t + 2k\pi$, $k \in I$ by S.

$$S = \{t + 2k\pi, k \in I\}.$$

Then we can write for $A_i \in Q/\sim$

$$A_i = S\alpha_i, \tag{5}$$

where α_i is an arbitrary fixed phase of A_i.

In particular $A = S\alpha$.

Note. The set $S\alpha_i$ consists of all functions $f(\alpha_i)$, $f \in S$.

We shall introduce the relation $\approx$ in the set Q. Let $\alpha_i, \alpha_k \in Q$, then $\alpha_i \approx \alpha_k$ iff there exists $\varphi_{2n} \in \boldsymbol{S}_1$ such that $\alpha_i(\varphi_{2n}) = \alpha_k$, $n \in I$.

Note. We shall also write $\alpha\varphi$ instead of $\alpha(\varphi)$.

In analogy with Theorem 1.3 in [3] it can be proved that the relation $\approx$ is an equivalence relation on Q.

3. **Proposition.** *The decompositions $Q/\sim$ and $Q/\approx$ coincide.*

Proof. a) Let $\alpha_i \sim \alpha_j$, $\alpha_i, \alpha_j \in Q$. Then there exists $k \in I$ such that $\alpha_j = \alpha_i + 2k\pi$. Let $\varepsilon = \operatorname{sgn} \alpha_i'$. From Abel's equation $\alpha_i\varphi_{\varepsilon 2k} = \alpha_i + \varepsilon^2 2k\pi$ we obtain $\alpha_i\varphi_{\varepsilon 2k} = \alpha_j$ and consequently $\alpha_i \approx \alpha_j$.

b) Let $\alpha_i \approx \alpha_j$. Then there exists $\varphi_{2n} \in \boldsymbol{S}_1$ such that $\alpha_i\varphi_{2n} = \alpha_j$. Therefore $\alpha_i + \varepsilon 2n\pi = \alpha_j$ and so $\alpha_j \in S\alpha_i$. Thus $\alpha_i \sim \alpha_j$. Consequently any class $A_i \in Q/\sim$ can be writen as

$$A_i = S\alpha_i = \alpha_i\boldsymbol{S}_1, \qquad \alpha_i \in A_i \tag{6}$$

and for $A \in Q/\sim$ we have

$$A = S\alpha = \alpha\boldsymbol{S}_1. \tag{7}$$

4. **Lemma 1.** *For each phase $\alpha_i \in Q$ there exists a dispersion $X_1 \in \boldsymbol{D}_1$ such that $\alpha X_1 = \alpha_i$.*

Proof. By [1], $\alpha^{-1}(\alpha_i(t)) \in D_1$. Therefore $X_1 = \alpha^{-1}(\alpha_i(t)) \in \boldsymbol{D}_1$ and $\alpha X_1 = \alpha_i$.

Lemma 2. *Let $\alpha X_1 = \alpha_i$, $\alpha Y_1 = \alpha_j$ with α_i, $\alpha_j \in Q$ and X_1, $Y_1 \in \boldsymbol{D}_1$. Then X_1, Y_1 lie in the same class of the factor group $\boldsymbol{D}_1/\boldsymbol{S}_1$ if and only if $\alpha_i \sim \alpha_j$.*

Proof. a) Let X_1, Y_1 lie in the same class of $\boldsymbol{D}_1/\boldsymbol{S}_1$ which means $X_1\boldsymbol{S}_1 = Y_1\boldsymbol{S}_1$. Then $\alpha_i\boldsymbol{S}_1 = \alpha X_1\boldsymbol{S}_1 = \alpha Y_1\boldsymbol{S}_1 = \alpha_j\boldsymbol{S}_1$ and thus $\alpha_i\varphi_{2m} = \alpha_j\varphi_{2n}$. Hence $\alpha_i = \alpha_j\varphi_{2(n-m)}$, therefore $\alpha_i \sim \alpha_j$

b) Let $\alpha_i \sim \alpha_j$. Then $\alpha_i \boldsymbol{S}_1 = \alpha_j \boldsymbol{S}_1$. This implies $\alpha X_1 \boldsymbol{S}_1 = \alpha_i \boldsymbol{S}_1 = \alpha_j \boldsymbol{S}_1 = \alpha Y_1 \boldsymbol{S}_1$. Since $\alpha' \neq 0$, $X_1 \boldsymbol{S}_1 = Y_1 \boldsymbol{S}_1$. Thus X_1, Y_1 lie in the same class of $\boldsymbol{D}_1/\boldsymbol{S}_1$.

Lemma 3. *For each class $A_i \in Q/\sim$ there exists exactly one class $\mathscr{X}_1 \in \boldsymbol{D}_1/\boldsymbol{S}_1$ such that*

$$\alpha \mathscr{X}_1 = A_i. \tag{8}$$

This lemma is an immediate consequence of those given above. Thus it holds also.

Theorem 1. *There exists a 1 – 1 mapping Φ: $Q/\sim \to \boldsymbol{D}_1/\boldsymbol{S}_1$ given in the following way:*
Let $A_i \in Q/\sim$, then $\Phi A_i = \mathscr{X}_1$ with $\mathscr{X}_1 \in \boldsymbol{D}_1/\boldsymbol{S}_1$ satisfying (8).

Remark. Any class $A_i \in Q/\sim$ can be expressed as follows:

$$A_i = S\alpha_i = \alpha_i S_1 = \alpha \mathscr{X}_1, \tag{9}$$

where $\mathscr{X}_1 \in \boldsymbol{D}_1/\boldsymbol{S}_1$ satisfies (8) and $\alpha_i \in A_i$.

5. Let Ψ: $\boldsymbol{D}_1/\boldsymbol{S}_1 \to \boldsymbol{L}$ be the isomorphism considered in [1, § 21, 6]. In this isomorphism the group $\boldsymbol{S}_1$ corresponds to the unity matrix E and the set $\bar{S}_1$ of central dispersions with an odd index corresponds to the matrix $-E$.

Let us consider a 1 – 1 mapping $\Psi\Phi$: $Q/\sim \to \boldsymbol{L}$. Here the class $A = S\alpha = \alpha \boldsymbol{S}_1$ corresponds to the matrix E and the class $\bar{A} = \{\alpha(t) + (2k + 1)\,\pi, k \in I\}$ corresponds to the matrix $-E$. We can introduce a composition of classes from $Q/\sim$ and $\boldsymbol{D}_1/\boldsymbol{S}_1$. First we have to prove the following.

Lemma. *If we compose any phase $\alpha_i \in A_i$ and any dispersion $X_1 \in \mathscr{X}_1$ we always obtain a phase of the same class $A_j \in Q/\sim$.*

Proof. Let $\alpha_i \in A_i$, $X_1 \in \mathscr{X}_1$. Then $\alpha_i X_1 = \alpha_j \in A_j$. Now let $\alpha_i \sim \bar{\alpha}_i$, $X_1 \sim \bar{X}_1$, that is $\alpha_i \varphi_{2n} = \bar{\alpha}_i$, $X_1 \varphi_{2m} = \bar{X}_1$. Then $\bar{\alpha}_i \bar{X}_1 = \alpha_i \varphi_{2n} X_1 \varphi_{2m} = \alpha_i X_1 \varphi_{2(m+n)}$ and therefore $\bar{\alpha}_i \bar{X}_1 \in A_j$.

Now we introduce the following composition

$$A_i \mathscr{X}_1 = A_j, \tag{10}$$

where A_j is the class from $Q/\sim$ containing the composed function $\alpha_i X_1$, where α_i and X_1 are arbitrary elements of A_i and $\mathscr{X}_1$, respectively.

6. In [2] there is determined a number of subgroups of the group $\boldsymbol{D}_1/\boldsymbol{S}_1$ generated by the orthogonal transformations of R, i.e. by rotation and axial symmetry. We will show a connection between those subgroups and corresponding subsets of $Q/\sim$. It holds (see [2]):

The set $\mathscr{G}_1 \subset \boldsymbol{D}_1/\boldsymbol{S}_1$ of all classes of dispersions which are the orthogonal transformations of R is a group; the set $\mathscr{H}_1 \subset \mathscr{G}_1$ of all classes of dispersions which are the rotations of R is a normal subgroups of $\mathscr{G}_1$. For any natural n the set $\mathscr{O}_1^n$ of all classes of dispersions which are the rotations of R through the angles $2m\pi/n$ (henceforth rot $2m\pi/n$), $m = 0, 1, \ldots, n - 1$ is a cyclic subgroup of $\mathscr{H}_1$ of order n. Adjoinging the axial symmetry with respect to the axis enclosing an angle π/n with the vector u of the base u, v (henceforth sym π/n) to the cyclic group $\mathcal{O}_1^n$, then the orthogonal transformation group $\mathcal{T}_1^n$ consisting of elements rot $2m\pi/n$, sym $m\pi/n$, $m = 0, 1, \ldots, n - 1$, is obtained. $\mathcal{O}_1^n$ is a normal subgroup of $\mathcal{T}_1^n$.

7. We shall need the following

Lemma. *Let* $A_i \in Q/\sim$, $\mathcal{X}_1 \in \boldsymbol{D}_1/\boldsymbol{S}_1$. *If* $\Psi\Phi(A_i) = M$ *and* $\Psi\mathcal{X}_1 = N$, *then* $\Psi\Phi(A_i\mathcal{X}_1) = MN$.

Proof. By p. 4 Lemma 3, there exists $\mathcal{Y}_1 \in \boldsymbol{D}_1/\boldsymbol{S}_1$ such that $A_i = \alpha\mathcal{Y}_1$ and $\Psi\mathcal{Y}_1 = M$. Thus $A_i\mathcal{X}_1 = \alpha\mathcal{Y}_1\mathcal{X}_1$, where $\mathcal{Y}_1\mathcal{X}_1 \in \boldsymbol{D}_1/\boldsymbol{S}_1$. Since Ψ in an isomorphism $\boldsymbol{D}_1/\boldsymbol{S}_1 \to \boldsymbol{L}$, $\Psi\mathcal{Y}_1\mathcal{X}_1 = \Psi\mathcal{Y}_1\Psi\mathcal{X}_1 = MN$. Therefore $\Phi\Psi(A_i\mathcal{X}_1) = MN$.

Theorem 2. *Let us consider all bases of R obtainable from the base u, v by orthogonal transformations. Denote the set of all classes of phases of those bases by G. Then*

$$G = A_i\mathcal{G}_1, \qquad A_i \in G. \tag{11}$$

Proof. G is the set of all classes of phases which the orthogonal matrices (in the mapping $\psi\Phi$) correspond to.

a) Let $A_j \in A_i\mathcal{G}_1$. Then $A_j = A_i\mathcal{Y}_1$, where $\mathcal{Y}_1 \in \mathcal{G}_1$, which means that $\Psi\mathcal{Y}_1$ is an orthogonal matrix. Moreover $A_i \in G$, thus $\Psi\Phi A_i$ is also an orthogonal matrix. Therefore by the foregoing lemma $\Phi\Psi A_j = \Phi\Psi(A_i\mathcal{Y}_1) = \Phi\Psi A_i\Psi\mathcal{Y}_1$. The product of two orthogonal matrices is also an orthogonal matrix and thus $A_j \in G$.

b) Let $A_k \in G$. Then $\Psi\Phi A_k$ is an orthogonal matrix. By p. 4 Lemma 3, $A_k = \alpha\mathcal{Z}_1$, where $\mathcal{Z}_1 \in \boldsymbol{D}_1/\boldsymbol{S}_1$. But $\Phi\Psi A_k = \Psi\mathcal{Z}_1$ and so $\Psi\mathcal{Z}_1$ is an orthogonal matrix. Thus $\mathcal{Z}_1 \in \mathcal{G}_1$ and hence $A_k \in \alpha\mathcal{G}_1$. Let $A_i \in G$. Then $A_i = \alpha\mathcal{X}_1$ with $\mathcal{X}_1 \in \mathcal{G}_1$. Since $\mathcal{G}_1$ is a group, $\mathcal{X}_1\mathcal{G}_1 = \mathcal{G}_1$. Therefore $\alpha G_1 = \alpha\mathcal{X}_1\mathcal{G}_1 = A_i\mathcal{G}_1$. $A_k \in \alpha\mathcal{G}_1$, thus $A_k \in A_i\mathcal{G}_1$.

Theorem 3. *Let us consider all bases of R obtainable from the base u, v by rotations. Denote the set of all classes of phases of those bases by H. Then*

$$H = A_i\mathcal{H}_1, \qquad A_i \in H. \tag{12}$$

Proof. H is the set of all classes of phases which the orthogonal matrices $\begin{pmatrix} \cos p & -\sin p \\ \sin p & \cos p \end{pmatrix}$ correspond to. (p is a real number.)

a) Let $A_j \in A_i\mathcal{H}_1$. Then $A_j = A_i\mathcal{Y}_1$, where $\mathcal{Y}_1 \in \mathcal{H}_1$, which means that $\Psi\mathcal{Y}_1 = \begin{pmatrix} \cos p & -\sin p \\ \sin p & \cos p \end{pmatrix}$. Since $A_i \in H$, $\Phi\Psi A_i = \begin{pmatrix} \cos q & -\sin q \\ \sin q & \cos q \end{pmatrix}$. Therefore $\Phi\Psi A_j = \begin{pmatrix} \cos p & -\sin p \\ \sin p & \cos p \end{pmatrix}\begin{pmatrix} \cos q & -\sin q \\ \sin q & \cos q \end{pmatrix} = \begin{pmatrix} \cos (p+q) & -\sin (p+q) \\ \sin (p+q) & \cos (p+q) \end{pmatrix}$. Thus $A_j \in H$.

b) Let $A_k \in H$, i.e. $\Psi\Phi A_k = \begin{pmatrix} \cos p & -\sin p \\ \sin p & \cos p \end{pmatrix}$.

$A_k = \alpha\mathscr{X}_1$ with $\mathscr{X}_1 \in \mathscr{H}_1$ and hence $A_k \in \alpha\mathscr{H}_1$. $\alpha\mathscr{H}_1 = A_i\mathscr{H}_1$ and thus $A_k \in A_i\mathscr{H}_1$.

Theorem 4. *Let us consider all bases of R obtainable from the base u, v by the transformations* rot $2m\pi/n$, $m = 0, 1, \ldots, n - 1$. *Denote the set of all classes of the proper phases of those bases by* O^n. *Then*

$$O^n = A_i\mathscr{O}_1^n, \qquad A_i \in O^n. \tag{13}$$

Proof. Since O^n is the set of all classes of phases which the orthogonal matrices $\begin{pmatrix}\cos 2m\pi/n & -\sin 2m\pi/n \\ \sin 2m\pi/n & \cos 2m\pi/n\end{pmatrix}$ correspond to ($m = 0, 1, \ldots, n - 1$), the proofs of the inclusions $O^n \subset A_i\mathscr{O}_1^n$ and $A_i\mathscr{O}_1^n \subset O^n$ are analogous to the proofs of Theorem 3.

Theorem 5. *Let us consider all bases of R obtainable from the base u, v by the transformations* rot $2m\pi/n$, sym $m\pi/n$, $m = 0, 1, \ldots, n - 1$. *Denote the set of all classes of proper phases of thoses bases by* T^n. *Then*

$$T^n = A_i\mathscr{T}_1^n, \qquad A_i \in T^n. \tag{14}$$

Proof. Since T^n is the set of all classes of phases which the orthogonal matrices $\begin{pmatrix}\cos 2m\pi/n & -\sin 2m\pi/n \\ \sin 2m\pi/n & \cos 2m\pi/n\end{pmatrix}$ and $\begin{pmatrix}\cos m\pi/n & \sin m\pi/n \\ \sin m\pi/n & -\cos m\pi/n\end{pmatrix}$ correspond to ($m = 0, 1, \ldots, n - 1$), the proof is analogous to that above.

Remark. G consists of the classes $A + p = \{\alpha(t) + p + 2k\pi, k \in I\}$ and $p - A = \{p - \alpha(t) + 2k\pi, k \in I\}$ p is real.

H consists of the classes $A + p$, p real.
O^n consists of the classes $A + 2m\pi/n$, $m = 0, 1, \ldots, n - 1$.
T^n consists of the classes $A + 2m\pi/n$ and $m\pi/n - A$, $m = 0, 1, \ldots, n - 1$ as follows from [2, Theorem 8 and Theorem 11].

8. Let us consider the differential equation

$$y'' = -y. \tag{-1}$$

Denote the group of its increasing dispersions by $\boldsymbol{E}$, the group of its central dispersions by $\boldsymbol{C}$ and the group of its central dispersions with an even index by $\boldsymbol{S}$. It holds

$$\boldsymbol{C} = \{t + k\pi, k \in I\}, \qquad \boldsymbol{S} = \{t + 2k\pi, k \in I\}.$$

We can introduce an equivalence $\sim^*$ in the set Q.
Let $\alpha_i, \alpha_k \in Q$, then $\alpha_i \sim^* \alpha_k$ iff there exists $k \in I$ such that

$$\alpha_i - \alpha_k = k\pi.$$

Moreover, $\alpha_i \sim^* \alpha_k$ iff there exists $\varphi_n \in \boldsymbol{C}_1$ such that $\alpha_i\varphi_n = \varphi_k$.

Theorem 6. *Any class* $\mathscr{A}_i$ of the decomposition $Q/\sim^*$ *can be expressed as follows:*

$$\mathscr{A}_i = C\alpha_i = \alpha_i C_1 = \alpha \mathscr{X}_1, \tag{15}$$

where $\alpha_i \in \mathscr{A}_i$ *and* $\mathscr{X}_1 \in \boldsymbol{D}_1/\boldsymbol{C}_1$.

Proof. (15) can be proved analogous to (9).

In particular $\mathscr{A} = A \cup \bar{A}$ satisfies the equality

$$\mathscr{A} = C\alpha = \alpha C_1. \tag{16}$$

By means of the matrix repıesentation we can prove

Theorem 7. *Let* B *be the set of all classes of increasing phases of* Q. *Then*

$$B = \boldsymbol{E}\alpha = \alpha \boldsymbol{B}_1. \tag{17}$$

Phases of (q) can be understood as functions from $C^3(-\infty, \infty)$ with a derivation different from zero. The set of all phases of the given equation and its significant subsets can be characterized by the foregoing theorems with the aid of the subgroups of the dispersion group of (q) or with the aid of the subgroup of the dispersion group of (-1). This approach to the set of phases can be even used to the convenient classes of the higher order differential equations, since it is not based on the definition of phase (1).

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SOUHRN

ALGEBRAICKÉ VLASTNOSTI FÁZÍ DIFERENCIÁLNÍ ROVNICE $y'' = q(t)\,y$

IRENA RACHŮNKOVÁ

V práci je provedena charakterizace jistých význačných podmnožin množiny fází rovnice (q) pomocí podgrup grupy disperzí rovnice (q), resp. pomocí podgrup grupy disperzí rovnice (-1).

РЕЗЮМЕ

АЛГЕБРАИЧЕСКИЕ СВОЙСТВА ФАЗ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ $y'' = q(t)y$

ИРЕНА РАХУНКОВА

В статье характеризуются некоторые множества фаз уравнения (q) подгруппами группы дисперсий уравнения (q), и тоже подгруппами дисперсий уравнения (-1).