Karel Beneš

On modelling dynamic systems excited by the Dirac function

Sborník prací Přírodovědecké fakulty University Palackého v Olomouci. Matematika, Vol. 17 (1978), No. 1, 123--129

Persistent URL: http://dml.cz/dmlcz/120062

1978 — ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM — TOM 57

Katedra algebry a geometrie přírodovědecké fakulty Univerzity Palackého v Olomouci
Vedoucí katedry: prof. RNDr. Lad. Sedláček, CSc.

ON MODELLING DYNAMIC SYSTEMS EXCITED BY THE DIRAC FUNCTION

KAREL BENEŠ

(*Received March 6, 1977*)

In investigating impulse characteristics of dynamic systems we meet with the necessity to generate the Dirac function (the unit impulse), which, however, cannot be generated directly in the form of (2). That is why we seek another equivalent mathematical description of the system investigated, in which either the Dirac function does not occur at all, or—with a smaller demand on the accuracy of the solution—it may be approximated by the rectangular or triangular impulse. This paper presents a way of finding the equivalent mathematical description of the system investigated, and some considerations of linear dynamic systems described by linear differential equations with constant coefficients.

We have the following differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_0 y = \delta(t) \tag{1}$$

with the initial conditions $({}_{(0)}, y'_{(0)}, \dots, y^{(n-1)}_{(0)}$, $a_k =$ constant, where the Dirac function $\delta(t)$ is defined by the relations

$$\delta(t) = \lim_{\varepsilon \to 0} \delta(t, \varepsilon), \tag{2}$$

$$\delta(t, \varepsilon) = 0 \qquad \text{for } t < 0,$$

$$\delta(t, \varepsilon) = \frac{1}{\varepsilon} \qquad \text{for } 0 \leqq t \leqq \varepsilon,$$

$$\delta(t, \varepsilon) = 0 \qquad \text{for } t > \varepsilon,$$

$$\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = \int_0^t \delta(t)\,\mathrm{d}t = \vartheta(t),$$

where

$$\begin{aligned} \vartheta(t) &= 0 \qquad \text{for } t < 0, \\ \vartheta(t) &= 1 \qquad \text{for } t \geqq 0. \end{aligned} \tag{2a}$$

We consider next the differential equation

$$y_1^{(n)} + a_{n-1}y_1^{(n-1)} + \dots + a_0 y = 0 \tag{3}$$

and seek such initial conditions $y_1(0), y_1'(0), \dots, y_{1(0)}^{(n-1)}$ for the solution of (1) to be the same as that of (3), i.e. $y_1 = y$. The Laplace transformation of the equation (1) has the form

$$\begin{aligned} s^n Y(s) - \sum_{i=0}^{n-1} s^{n-1-i} y_{(0)}^{(i} + a_{n-1}(s^{n-1}Y(s) - \\ - \sum_{i=0}^{n-2} s^{n-2-i} y_{(0)}^{(i)}) + \dots + a_1(sY(s) - y_{(0)}) + a_0 Y(s) = 1, \end{aligned} \tag{4}$$

the Laplace transformation of the equation (3) has the form

$$\begin{aligned} s^n Y_1(s) - \sum_{i=0}^{n-1} s^{n-1-i} y(0)^{(i)} + a_{n-1}(s^n Y_1(s) - \\ - \sum_{i=0}^{n-2} s^{n-2-i} y_1(0^{(i)}) + \dots + a_1(sY_1(s) - y_1(0)) + a_0 Y_1(s) = 0. \end{aligned} \tag{5}$$

If it holds

$$y = y_1, \tag{6}$$

then it holds also

$$Y(s) = Y_1(s). \tag{7}$$

To satisfy the conditions of (7) in the equations (4) and (5), it is necessary for the coefficients composed of the initial conditions at the powers s to hold

$$\begin{aligned} s^0: \quad & y_1(0^{(n-1)}) + a_{n-1}y_1(0^{(n-2)}) + \dots + a_2 y_1'(0) + a_1 y_1(0) = \\ & = 1 + y(0^{(n-1)}) + a_{n-1}y(0^{(n-2)}) + \dots + a_2 y'(0) + a_1 y(0), \\ s: \quad & y_1(0^{(n-2)}) + a_{n-1}y_1(0^{(n-3)}) + \dots + a_3 y_1'(0) + a_2 y_1(0) = \\ & = y(0^{(n-2)}) + a_{n-1}y(0^{(n-3)}) + \dots + a_3 y'(0) + a_2 y(0), \\ & \vdots \\ s^{n-2}: \quad & y_1'(0) + a_{n-1}y_1(0) = y'(0) + a_{n-1}y(0), \\ s^{n-1}: \quad & y_1(0) = y(0). \end{aligned}$$

Then we have for the initial conditions of (3)

$$\begin{aligned} y_1(0) &= y(0), \\ y_1'(0) &= y'(0), \\ &\vdots \\ y_1(0^{(n-1)}) &= 1 + y(0^{(n-1)}). \end{aligned} \tag{9}$$

The solution of (1) can be thus carried over to the solution of (3) with the initial conditions of (9). In machine computing there occurs no unrealizable Dirac function –

see fig. 2 — by the above procedure for the solution of $y(t)$ (the response of dynamic system) and its derivatives $y(t^{(j)})$, $j_{\max} = n - 1$. The programme diagram with the possibility of following $y^{(n)}$ is unrealizable, in the relation

$$y^{(n)} = \delta(t) - a_{n-1}y^{(n-1)} - \dots - a_0 y$$

there occurs the unrealizable Dirac function.

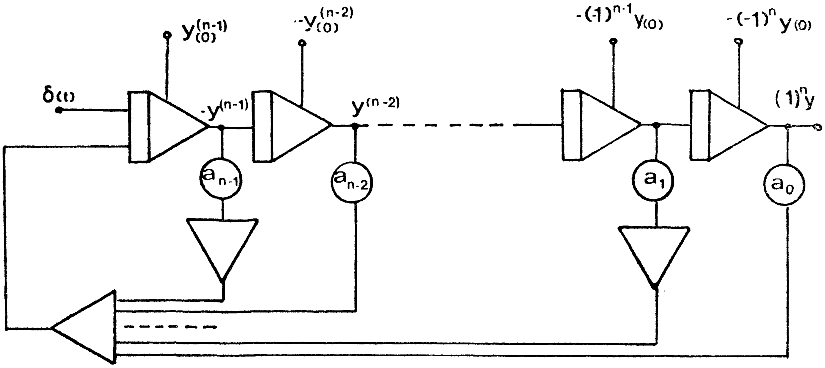

Fig. 1

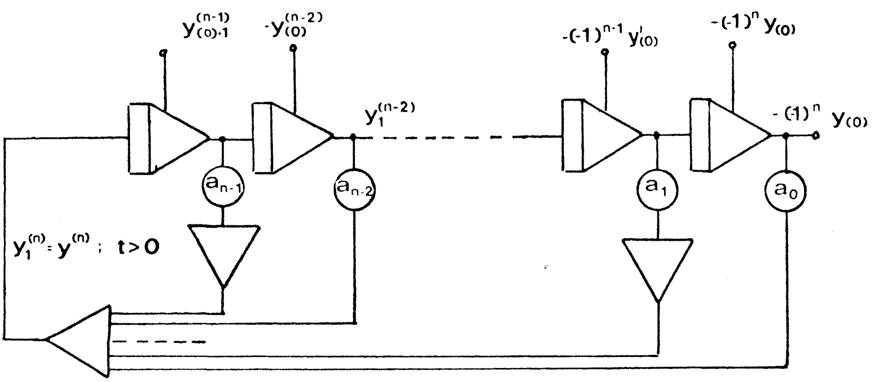

Fig. 2

This conclusion follows even from the properties of the Dirac function and from the programme diagram in fig. 1 and 2 for the solution of (1) and (3). According to (2) we have for the Dirac function

$$\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = \int_{0}^{t} \delta(t)\,\mathrm{d}t = \vartheta(t).$$

If we lead the Dirac function $\delta(t)$ on the input of the integrator, we obtain the jump function $\vartheta(t)$ on its output. Integrating the sum of the functions $u(t)$ and $\delta(t)$ (see fig. 3), we get (10) on the output of the integrator, under the assumption that the integrator changes its sign:

$$v(t) = -\int_0^t (u(t) + \delta(t))\,\mathrm{d}t = -\int_0^t u(t)\,\mathrm{d}t - \vartheta(t), \tag{10}$$

where the jump function $\vartheta(t)$ can be considered to be the initial value of the function $v(t)$ for $t = 0$. ($\vartheta(t) = 1$ for $t \geqq 0$).

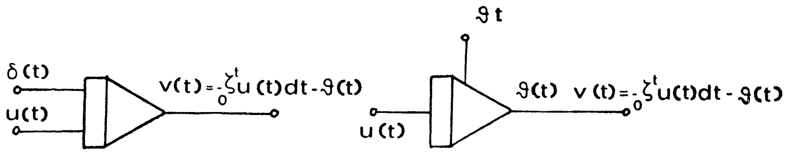

Fig. 3

This procedure is well available in modelling transfer functions of dynamic systems which is equivalent to the solution of the differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_0 y = b_m z^{(m)} + \dots + b_0 z \tag{11}$$

a_k, b_k constant, $n \geqq m$. The equation of (11) is generally written in the form of a system of differential equations

$$\begin{aligned} y_1' &= b_0 z - a_0 y, \\ y_2' &= b_1 z - a_1 y + y_1, \\ y_3' &= b_2 z - a_2 y + y_2, \\ &\vdots \\ y_n' &= b_{n-1} z - a_{n-1} y + y_{n-1}, \\ y &= b_n z + y_n. \end{aligned} \tag{12}$$

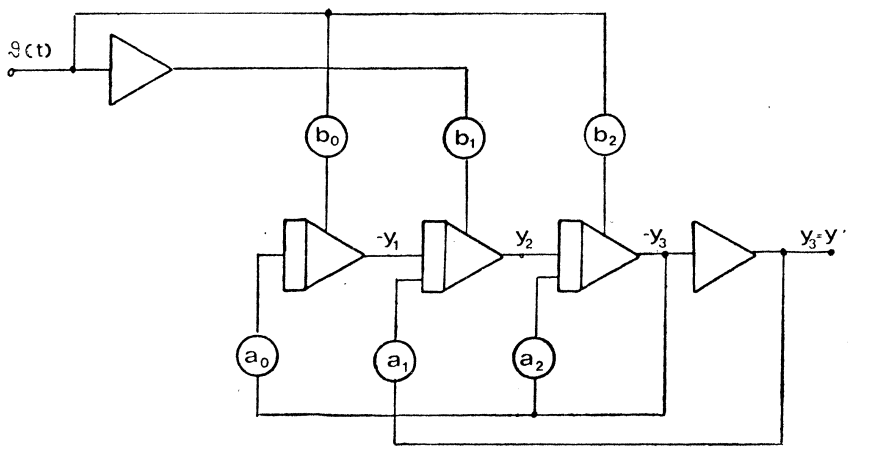

Fig. 4

The programme diagram for the solution of (12) in case of $n = 3$, $m = 2$ the function $\delta(t)$ being realizable is given in fig. 4. The programme diagram for the solution of the same equation by the metod of the equivalent initial conditions on the basis

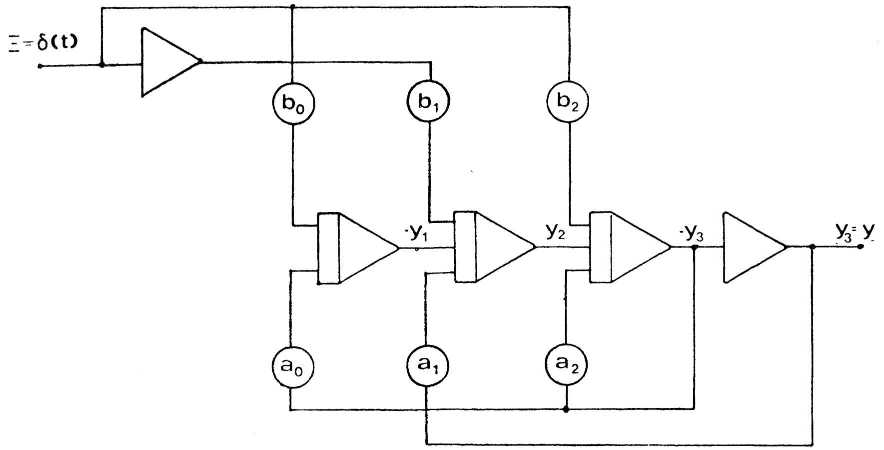

Fig. 4a

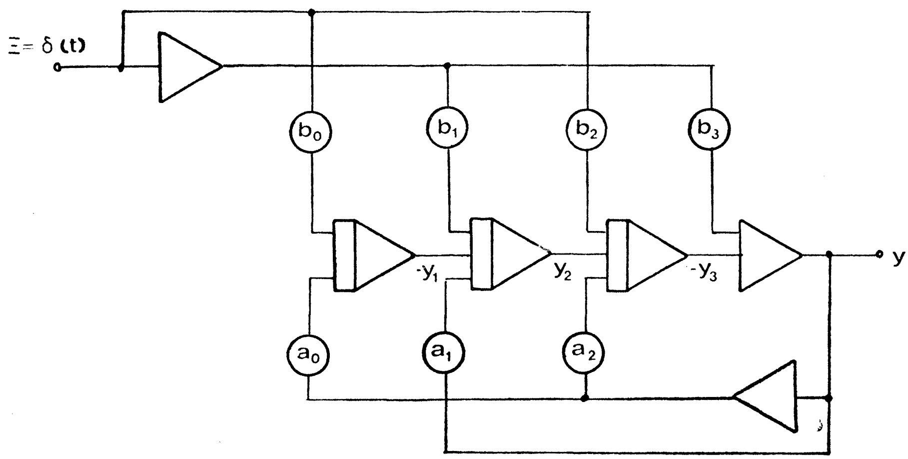

Fig. 5

of (10) is given in figure (4a). For $z(t) = \delta(t)$, there is a solution realizable for $n > m$ on the basis of the last relation of (12), the response y will be obtained on the input of the summary unit. The unrealizable programme diagram for the solution of (12) in case of $n = m = 3$ is given in fig. 5. The programme diagram for the solution of the same equation by the metod on the basis of (10) is given in fig. 6. The response of dynamic system can be followed for $t > 0$ only. Similary $y^{(n)}$ according to fig. 2 can be followed for $t > 0$ only.

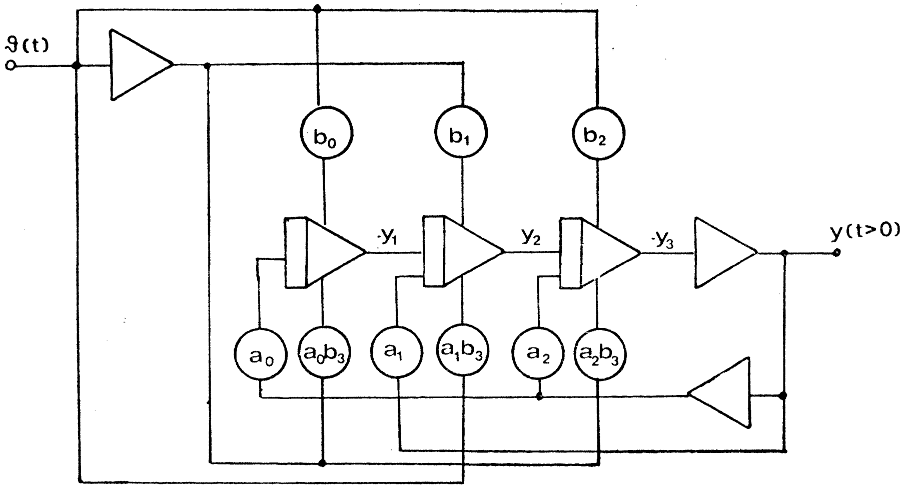

Fig. 6

REFERENCES

[1] Plander I.: *Matematické metódy a programovanie analógových počítačov*. SAV Bratislava, 1969.
[2] Schwarz W.: *Analogprogrammierung*. VEB Fachbuchverlag Leipzig, 1974.

SOUHRN

MODELOVÁNÍ DYNAMICKÝCH SYSTÉMŮ BUZENÝCH DIRACOVOU FUNKCÍ

KAREL BENEŠ

V práci je popsána náhrada matematického popisu dynamického systému buzeného Diracovou funkcí ekvivalentním popisem, ve kterém se Diracova funkce nevyskytuje. Jsou uvedeny úpravy programových schémat s použitím skokové funkce.

РЕЗЮМЕ

МОДЕЛИРОВАНИЕ ДИНАМИЧЕСКИХ СИСТЕМ ВОЗБУЖДАЕМЫХ ФУНКЦИЕЙ ДИРАКА

КАРЕЛ БЕНЕШ

В статье описана замена математического описания динамической системы возбуждаемой функцией Дирака эквивалентным математическим описанием, в котором функция Дирака не находится. В статье осуществлены приспособления програмных схем на программные схемы с применением единичной скачковой функции.