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Vladimír Vlček

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ON A DISTRIBUTION OF ZEROS OF SOLUTIONS OF A FOURTH-ORDER ITERATED LINEAR DIFFERENTIAL EQUATION

VLADIMÍR VLČEK

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INTRODUCTION

We consider a fourth order linear homogeneous differential equation of the form

$$y^{(IV)}(t) + 10[q(t)\,y'(t)]' + 3[3q^2(t) + q''(t)]\,y(t) = 0, \tag{1}$$

where the function $q(t)$ defined on the interval $\boldsymbol{I} = (-\infty, +\infty)$ is understood to be continuous together with its derivatives up to and including the *2nd* order, i.e. $q(t) \in \boldsymbol{C}_{\boldsymbol{I}}^{(2)}$ and $q(t) > 0$ for all $t \in (-\infty, +\infty)$ such that the *2nd* order homogeneous linear differential equation of the form

$$y''(t) + q(t)\,y(t) = 0 \tag{2}$$

is oscillatory in terms of [2], which means that to each $t \in (-\infty, +\infty)$ there exist infinitely many zeros of an arbitrary nontrivial solution of differential equation (2) lying to the right and to the left of the point t.

The differential equation (1) is called iterated with respect to the differential equation (2). As is known, if $[u(t), v(t)]$ is the basis of the differential equation (2), then $[u^3(t), u^2(t)\,v(t), u(t)\,v^2(t), v^3(t)]$ is the basis of the equation (1). Hence, every nontrivial solution of the equation (1) is of the form

$$y(t) = \sum_{i=1}^{4} C_i u^{4-i}(t)\,v^{i-1}(t), \tag{3}$$

with $C_i \in \boldsymbol{R}$, $i = 1, \ldots, 4$, $\sum_{i=1}^{4} C_i^2 > 0$, whereby under the assumption of oscillation of differential equation (2) it proves to be oscillatory as well (for brevity, the equation (1) will be called oscillatory too).

Since the differential equation (1) is of the fourth order, one notes that every zero of its nontrivial solutions can be of multiplicity 3 at the highest. Throughout this article every solution both of (2) and of (1) is understood to be nontrivial only.

Lemma 1: Let $t_0 \in (-\infty, +\infty)$ be an arbitrary firmly chosen point. Then every oscillatory solution $y(t)$ of the differential equation (1) which vanishes at the point t_0 is of the form

1. $y(t) = C_1 u^3(t) + C_2 u^2(t)\, v(t) + C_3 u(t)\, v^2(t)$, $C_3 \neq 0$, iff t_0 is a simple zero of the solution $y(t)$,
2. $y(t) = C_1 u^3(t) + C_2 u^2(t)\, v(t)$, $C_2 \neq 0$, iff t_0 is a double zero of the solution $y(t)$,
3. $y(t) = C_1 u^3(t)$, $C_1 \neq 0$, iff t_0 is a triple zero of the solution $y(t)$,

with $[u(t), v(t)]$ being such a basis of the differential equation (2) that $u(t_0) = 0$.

Proof: As said before, every solution of the differential equation (1) is of the form (3) with $C_i \in \boldsymbol{R}$, $i = 1, \ldots, 4$, $\sum_{i=1}^{4} C_i^2 > 0$, where $[u(t), v(t)]$ is the basis of the oscillatory differential equation (2).

Let $t_0 \in (-\infty, +\infty)$ be an arbitrary zero of the solution $y(t)$ of the differential equation (1) and let $[u(t), v(t)]$ be the basis of the oscillatory differential equation (2) such that

$$u(t_0) = v'(t_0) = 0, \tag{P}$$

[so that $u'(t_0) \neq 0$, $v(t_0) \neq 0$ and thus the point t_0 is a simple zero of the function $u(t)$]. Then the system of all solutions $y(t)$ of the dif. equation (1) which vanish at the point t_0 is exactly of the form

$$y(t) = \sum_{i=1}^{3} C_i u^{4-i}(t)\, v^{i-1}(t), \tag{4}$$

with $C_i \in \boldsymbol{R}$, $i = 1, 2, 3$, $\sum_{i=1}^{3} C_i^2 > 0$ being arbitrary constants.

We distinguish first all the alternatives for the values of constants $C_i \in \boldsymbol{R}$, $i = 1, 2, 3$, in view of the condition $\sum_{i=1}^{3} C_i^2 > 0$ and in agreement with the multiplicities $\nu = 1, 2, 3$ of the point t_0.

1.1. Let $C_3 \neq 0$, i.e. $y(t) = u(t) \sum_{i=1}^{3} C_i u^{3-i}(t)\, v^{i-1}(t)$. Since $y'(t) = u(t)\,\{3C_1 u(t)\, u'(t) + C_2[2u'(t)\, v(t) + u(t)\, v'(t)] + 2C_3 v(t)\, v'(t)\} + C_3 u'(t)\, v^2(t)$ and it holds $y(t_0) = 0$ with respect to the assumption (P), but $y'(t_0) = C_3 u'(t_0)\, v^2(t_0) \neq 0$, the point t_0 is a simple zero of the solution $y(t)$.

1.2. Let $C_3 = 0$, $C_2 \neq 0$, i.e. $y(t) = u^2(t)\,[C_1 u(t) + C_2 v(t)]$. Since $y'(t) = u(t)\{3C_1 u(t)\, u'(t) + C_2[u'(t)\, v(t) + u(t)\, v'(t)]\}$, $y''(t) = u(t)\,\{3C_1[2u'^2(t) + u(t)\, u''(t)] + C_2[2u''(t)\, v(t) + 4u'(t)\, v'(t) + u(t)\, v''(t)]\} + 2C_2 u'^2(t)\, v(t)$, so that with respect to the assumption (P) we have $y(t_0) = y'(t_0) = 0$ while $y''(t_0) = 2C_2 u'^2(t_0)\, v(t_0) \neq 0$, the point t_0 is a double zero of the solution $y(t)$.

1.3. Let $C_3 = C_2 = 0$, $C_1 \neq 0$, i.e. $y(t) = C_1 u^3(t)$. Since

$$\begin{aligned} y'(t) &= 3C_1 u^2(t)\, u'(t), \\ y''(t) &= 3C_1 u(t)\,[2u'^2(t) + u(t)\, u''(t)], \\ y'''(t) &= 3C_1\{2u'^3(t) + u(t)\,[6u'(t)\, u''(t) + u(t)\, u'''(t)]\}, \end{aligned}$$

so that with respect to the assumption (P) we get $y(t_0) = y'(t_0) = y''(t_0) = 0$, but $y'''(t_0) = 6C_1 u'^3(t_0) \neq 0$, the point t_0 is a triple zero of the solution $y(t)$.

Conversely:

2.1. if $\nu = 1$, then $C_3 \neq 0$ (because in case of $C_3 = 0$ it follows by the condition $\sum_{i=1}^{3} C_i^2 > 0$ and by 1.2. or 1.3. that the point t_0 is of multiplicity 2 or 3).

2.2. if $\nu = 2$, then $C_2 \neq 0$ (because in case of $C_2 = 0$ it follows by the condition $\sum_{i=1}^{3} C_i^2 > 0$ and by 1.1. or 1.3. that the point t_0 is of multiplicity 1 or 3).

2.3. if $\nu = 3$, then $C_1 \neq 0$ (because in case of $C_1 = 0$ it follows by the condition $\sum_{i=1}^{3} C_i^2 > 0$ and by 1.1. or 1.2. that the point t_0 is of multiplicity 1 or 2).

§ 1. CONJUGATE POINTS

Definition 1.1. Let $t_0 \in (-\infty, +\infty)$ be an arbitrary point and let $y(t)$ be an arbitrary solution of the differential equation (1) which vanishes in itself (we indicate this by writing ${}^{\nu}t_0$, where $\nu = 1, 2, 3$ denotes the multiplicity of the point t_0).

Then the first conjugate point from the right to the point ${}^{\nu}t_0$ will be called the first zero of the solution $y(t)$ lying to the right of the point ${}^{\nu}t_0$ (we indicate this by writing ${}^{\mu}t_1$, where $\mu = 1, 2, 3$ denotes its multiplicity).

Since every solution of the differential equation (1) is oscillatory (see the remark in the introduction), it is obvious that there always exists to an arbitrary point ${}^{\nu}t_0 \in (-\infty, +\infty)$, $\nu = 1, 2, 3$, the first conjugate point ${}^{\mu}t_1$ from the right to the point ${}^{\nu}t_0$ of a suitable multiplicity $\mu = 1, 2, 3$.

Theorem 1.1: Let ${}^{\mu}t_1 \in (-\infty, +\infty)$ be the first conjugate point from the right to the point ${}^{\nu}t_0 \in (-\infty, +\infty)$, $\mu, \nu \in \{1, 2, 3\}$.

Then it holds:

I. if $\nu = 1$, then either $\mu = 1$ or $\mu = 2$,
II. if $\nu = 2$, then $\mu = 1$,
III. if $\nu = 3$, then $\mu = 3$.

Proof: Let $t_0 \in (-\infty, +\infty)$ be an arbitrary firmly chosen point; we choose the basis $[u(t), v(t)]$ of the oscillatory differential equation (2) so that both functions $u(t)$,

$v(t)$ and their first derivatives $u'(t)$, $v'(t)$ satisfy the conditions (P) at the point t_0. Let $y(t)$ be such a solution of the differential equation (1) that the point t_0 is its ν-multiple zero (enabling us to write ${}^{\nu}t_0$, where $\nu = 1, 2, 3$).

I. Let $\nu = 1$; then by **Lemma 1,** every oscillatory solution of (1) vanishing together with the function $u(t)$ at the simple point ${}^1t_0$, is exactly of the form

$$y(t) = u(t)\left[C_1u^2(t) + C_2u(t)\,v(t) + C_3v^2(t)\right],$$

with $C_i \in \boldsymbol{R}$, $i = 1, 2, 3$, $C_3 \neq 0$, being arbitrary constants. Let T_1 stand for the neighbouring (simple) zero of the function $u(t)$ lying to the right after the point t_0, i.e. $T_1 > t_0$, so that

$$y(t_0) = u(t_0) = 0, \qquad y(T_1) = u(T_1) = 0,$$

which gives $u(t) \neq 0$ for all $t \in (t_0, T_1)$ [due to the continuity of the function $u(t)$, there is either $u(t) > 0$ or $u(t) < 0$ on the interval (t_0, T_1)]. The existence of zeros relating to the solution $y(t)$ of the dif. equation (1) on (t_0, T_1) will be decided by investigating the zeros of a three-parametric system of all functions having the form

$$y^*(t) = C_1u^2(t) + C_2u(t)\,v(t) + C_3v^2(t),$$

being always uniquely determined by the choice of the constants $C_i \in \boldsymbol{R}$, $i = 1, 2, 3$, $C_3 \neq 0$.

It is obvious above all that no zero—in so far as such exist—of an arbitrary function from the system of functions $y^*(t)$ cannot be at the same time the zero of the function $u(t)$ [as follows from the assumption saying that both functions $u(t)$, $v(t)$ form the basis of the dif. equation (2) and from the Sturm-theorem on the mutual separation of all zeros of two oscillatory linear independent solutions of the dif. equation (2)].

Restricting the values of the argument t to the open interval (t_0, T_1) enables us to write in place of the equation of zeros of $y^*(t)$

$$C_1u^2(t) + C_2u(t)\,v(t) + C_3v^2(t) = 0$$

an equivalent equation

$$C_3\left[\frac{v(t)}{u(t)}\right]^2 + C_2\frac{v(t)}{u(t)} + C_1 = 0.$$

Putting

$$\frac{v(t)}{u(t)} = \operatorname{cotg}\alpha(t),$$

where the function $\alpha(t)$ denotes the first phase of the basis $[u(t), v(t)]$ of the oscillatory dif. equation (2) [2], enables us to write the above equation in the form

$$C_3 \operatorname{cotg}^2\alpha(t) + C_2 \operatorname{cotg}\alpha(t) + C_1 = 0. \tag{*}$$

There need not be done any distinction among the following three possible cases:

1. If $C_2^2 - 4C_1C_3 > 0$, then

$$\operatorname{cotg}\alpha(t) = \frac{-C_2 - \sqrt{C_2^2 - 4C_1C_3}}{2C_3}$$

or

$$\operatorname{cotg}\alpha(t) = \frac{-C_2 + \sqrt{C_2^2 - 4C_1C_3}}{2C_3},$$

wherefrom we get the following expression for the values of the 1*st* phase $\alpha(t)$ of the basis $[u(t), v(t)]$ of the differential equation (2) at the points t' or t'', with $t', t'' \in (t_0, T_1)$ and $t' \neq t''$:

$$\alpha_n(t) = \operatorname{arccotg}\left(\frac{-C_2 - \sqrt{C_2^2 - 4C_1C_3}}{2C_3}\right) + n\pi$$

at a suitable $n = 0, \pm 1, \pm 2, \dots$ or

$$\alpha_k(t) = \operatorname{arccotg}\left(\frac{-C_2 + \sqrt{C_2^2 - 4C_1C_3}}{2C_3}\right) + k\pi$$

at a suitable $k = 0, \pm 1, \pm 2, \dots$ If we assume $C_3 > 0$ in the equation (*) and denote

$$\tilde{C}_1 = \frac{-C_2 - \sqrt{C_2^2 - 4C_1C_3}}{2C_3}, \qquad \tilde{C}_2 = \frac{-C_2 + \sqrt{C_2^2 - 4C_1C_3}}{2C_3},$$

so that $\tilde{C}_1 < \tilde{C}_2$, then there may occur the following possibilities:

a) $\tilde{C}_1 < \tilde{C}_2 < 0$; then we take both $\operatorname{arccotg} \tilde{C}_1 \in \left(\frac{\pi}{2}, \pi\right)$ and $\operatorname{arccotg} \tilde{C}_2 \in \left(\frac{\pi}{2}, \pi\right)$ with the inequalities

$$\frac{\pi}{2} < \operatorname{arccotg} \tilde{C}_2 < \operatorname{arccotg} \tilde{C}_1 < \pi$$

and thus for all (suitable) $k = n = 0, \pm 1, \pm 2, \dots$:

$$\left(n + \frac{1}{2}\right)\pi < \alpha_n(t'') < \alpha_n(t') < (n + 1)\pi.$$

For both simple zeros $t', t'' \in (t_0, T_1)$ of the functions relative to the system $y^*(t)$ we get

$$t_0 < t' < t'' < t^* < T_1,$$

[where $t^* \in (t_0, T_1)$ is denoted a zero of the function $v(t)$], so that ${}^1t_1 = t'$ holds for the first conjugate point from the right to the point ${}^1t_0$, in which the solution $y(t)$ of the dif. equation (1) vanishes.

b) $\tilde{C}_1 < \tilde{C}_2 = 0$ so that $C_1 = 0$, $C_2 > 0$ and the system of functions $y^*(t)$ may be written in the form

$$y^*(t) = v(t)\,[C_2u(t) + C_3v(t)].$$

Then we take arccotg $\tilde{C}_1 \in \left(\frac{\pi}{2}, \pi\right)$, arccotg $\tilde{C}_2 = \frac{\pi}{2}$, with

$$\frac{\pi}{2} = \text{arccotg}\, \tilde{C}_2 < \text{arccotg}\, \tilde{C}_1 < \pi$$

and thus for all (suitable) $k = n = 0, \pm 1, \pm 2, \ldots$:

$$\left(n + \frac{1}{2}\right)\pi = \alpha_n(t'') < \alpha_n(t') < (n + 1)\,\pi.$$

For both simple zeros $t', t'' \in (t_0, T_1)$, $t' \neq t''$, of the functions from the system $y^*(t)$ we become

$$t_0 < t' < t'' < T_1,$$

where t'' or t' is the zero of the function $v(t)$ or $C_2 u(t) + C_3 v(t)$. Thus it holds ${}^1t_1 = t'$ for the first conjugate point from the right to the point ${}^1t_0$ at which the solution $y(t)$ of the dif. equation (1) vanishes.

c) $\tilde{C}_1 < 0 < \tilde{C}_2$; then we take arccotg $\tilde{C}_2 \in \left(0, \frac{\pi}{2}\right)$ while arccotg $\tilde{C}_1 \in \left(\frac{\pi}{2}, \pi\right)$ and there holds the inequalities

$$0 < \text{arccotg}\, \tilde{C}_2 < \frac{\pi}{2} < \text{arccotg}\, \tilde{C}_1 < \pi$$

and consequently for all (suitable) $k = n = 0, \pm 1, \pm 2, \ldots$;

$$n\pi < \alpha_n(t'') < \left(n + \frac{1}{2}\right)\pi < \alpha_n(t') < (n + 1)\,\pi.$$

We become

$$t_0 < t' < t^* < t'' < T_1$$

for both simple zeros $t', t'' \in (t_0, T_1)$, $t' \neq t''$, of the functions from the system $y^*(t)$ [where $t^* \in (t_0, T_1)$ is denoted a zero of the function $v(t)$]. It holds ${}^1t_1 = t'$ for the first conjugate point from the right to the point ${}^1t_0$, at which the solution $y(t)$ of the dif. equation (1) vanishes.
In particular, if $\tilde{C}_1 < 0 < \tilde{C}_2$ and $\tilde{C}_1 + \tilde{C}_2 = 0$ (so that $C_2 = 0$ and thus due to the assumption $C_3 > 0$ there must be $C_1 < 0$), the system of functions $y^*(t)$ written in the form

$$y^*(t) = C_1 u^2(t) + C_3 v^2(t) =$$
$$= [\sqrt{-C_1}\,u(t) - \sqrt{C_3}\,v(t)]\,[\sqrt{-C_1}\,u(t) + \sqrt{C_3}\,v(t)]$$

has in the interval (t_0, T_1) both simple zeros t', t'' being different to each other and symmetric by the point $t^* \in (t_3, T_1)$ wherein $v(t^*) = 0$, i.e. between their numerical values $t', t'', t^* \in \boldsymbol{R}$ holds the relation $t^* - t' = t'' - t^*$ or $t^* = \frac{1}{2}(t' + t'')$.

d) $0 = \tilde{C}_1 < \tilde{C}_2$, so that $C_1 = 0$, $C_2 < 0$ and the system of functions $y^*(t)$ may be written in the form

$$y^*(t) = v(t)[C_2u(t) + C_3v(t)].$$

We take then $\operatorname{arccotg} \tilde{C}_1 = \frac{\pi}{2}$, $\operatorname{arccotg} \tilde{C}_2 \in \left(0, \frac{\pi}{2}\right)$ with

$$0 < \operatorname{arccotg} \tilde{C}_2 < \operatorname{arccotg} \tilde{C}_1 = \frac{\pi}{2}$$

and thus for all (suitable) $k = n = 0, \pm 1, \pm 2, \dots$:

$$n\pi < \alpha_n(t'') < \alpha_n(t') = \left(n + \frac{1}{2}\right)\pi.$$

We become for both simple zeros $t', t'' \in (t_0, T_1)$, $t' \neq t''$, of the functions from the system $y^*(t)$

$$t_0 < t' < t'' < T_1,$$

where t' or t'' is the zero of the function $v(t)$ or $C_2u(t) + C_3v(t)$. Hence it holds ${}^1t_1 = t'$ for the first conjugate point from the right to the point ${}^1t_0$ at which the solution $y(t)$ of the dif. equation (1) vanishes.

e) $0 < \tilde{C}_1 < \tilde{C}_2$; then we take both $\operatorname{arccotg} \tilde{C}_1 \in \left(0, \frac{\pi}{2}\right)$ and $\operatorname{arccotg} \tilde{C}_2 \in \left(0, \frac{\pi}{2}\right)$, hereby hold the inequalities

$$0 < \operatorname{arccotg} \tilde{C}_2 < \operatorname{arccotg} \tilde{C}_1 < \frac{\pi}{2}$$

and thus for all (suitable) $k = n = 0, \pm 1, \pm 2, \dots$:

$$n\pi < \alpha_n(t'') < \alpha_n(t') < \left(n + \frac{1}{2}\right)\pi.$$

We become

$$t_0 < t^* < t' < t'' < T_1$$

for both simple zeros $t', t'' \in (t_0, T_1)$, $t' \neq t''$, of the functions from the system $y^*(t)$ [where $t^* \in (t_0, T_1)$ denotes a zero of the function $v(t)$], so that it holds ${}^1t_1 = t'$ for the first conjugate point from the right to the point ${}^1t_0$ at which the solution $y(t)$ of the dif. equation (1) vanishes.

Remark. In case of $C_3 < 0$ in the equation (*), the order of the values of the functions $\alpha_n(t')$, $\alpha_n(t'')$ [for suitable $k = n = 0, \pm 1, \pm 2, \dots$] will be reversed in the interval $\left(\left(n + \frac{1}{2}\right)\pi, (n + 1)\pi\right)$, $\left\langle\left(n + \frac{1}{2}\right)\pi, (n + 1)\pi\right)$, $(n\pi, (n + 1)\pi)$, $\left(n\pi, \left(n + \frac{1}{2}\right)\pi\right\rangle$, $\left(n\pi, \left(n + \frac{1}{2}\right)\pi\right)$ respectively. Accordingly (with a suitable value of number n) the order of the corresponding simple and from each other different zeros t', t'' of the functions from the system $y^*(t)$ on the open interval (t_0, T_1), will be reversed as well.

2. If $C_2^2 - 4C_1C_3 = 0$, then due to the assumption $C_3 \neq 0$, the equation (*) may be written as

$$\operatorname{cotg} \alpha(t) = -\frac{C_2}{2C_3}.$$

Herefrom we get the same expression for the common values of two coincident phases $\alpha(t)$ in the form

$$\alpha_n(t') = \operatorname{arccotg}\left(-\frac{C_2}{2C_3}\right) + n\pi$$

with suitable $n = 0, \pm 1, \pm 2, \ldots$, where we take

$$\operatorname{arccotg}\left(-\frac{C_2}{2C_3}\right) \in \left(0, \frac{\pi}{2}\right) \quad \text{if} \quad \operatorname{sgn} C_3 \neq \operatorname{sgn} C_2,$$

$$\operatorname{arccotg}\left(-\frac{C_2}{2C_3}\right) \in \left(\frac{\pi}{2}, \pi\right) \quad \text{if} \quad \operatorname{sgn} C_3 = \operatorname{sgn} C_2,$$

$$\operatorname{arccotg}\left(-\frac{C_2}{2C_3}\right) = \frac{\pi}{2} \quad \text{if} \quad C_2 = 0.$$

Thereby it holds for all values t' of the function invers to the function $\alpha_n(t')$ that (with a suitable value of the number n) $t' \in (t_0, T_1)$ and represent the double zeros of the function system $y^*(t)$.

In particular, in case of $C_2 = 0$ (consequently also $C_1 = 0$), the function system $y^*(t)$ is of the form $y^*(t) = C_3 v^2(t)$ and therefore the double zero t' of the system $y^*(t)$ – and in this way simultaneously even the solution $y(t)$ of the dif. equation (1) – on the interval (t_0, T_1) coincides with the single zero of the function $v(t)$ lying on this open interval.

Hence it holds ${}^2t_1 = t'$ for the first conjugate point from the right to the point ${}^1t_0$ wherein the solution $y(t)$ of the dif. equation (1) vanishes.

3. In case of $C_2^2 - 4C_1C_3 < 0$ the equation (*) has no solution, which means that the system of functions $y^*(t)$ has no zeros. The only zeros of the solution $y(t)$ of the dif. equation (1) in this case are exactly all (simple) zeros of the function $u(t)$.

It holds therefore ${}^1t_1 = T_1$ for the first conjugate point from the right to the point ${}^1t_0$ wherein the solution $y(t)$ of the dif. equation (1) vanishes.

II. Let $\nu = 2$; then by **Lemma 1,** every oscillatory solution of the dif. equation (1) vanishing together with the function $u^2(t)$ at the double point ${}^2t_0$, is exactly of the form

$$y(t) = u^2(t)\left[C_1 u(t) + C_2 v(t)\right]$$

with $C_i \in \boldsymbol{R}$, $i = 1, 2$, $C_2 \neq 0$, being arbitrary constants. If we denote by T_1 the neighbouring zero of the function $u(t)$ (evidently, the zeros of the functions $u(t)$

and $u^2(t)$ coincide; they differ from each other in multiplicity only — while all zeros of the function $u(t)$ are simple, those of the function $u^2(t)$ are double), lying to the right after the point t_0, i.e. $T_1 > t_0$, then repeatedly

$$y(t_0) = u(t_0) = 0, \qquad y(T_1) = u(T_1) = 0$$

with $u(t) \neq 0$ [either $u(t) > 0$ or $u(t) < 0$] holding for all $t \in (t_0, T_1)$, while on (t_0, T_1) there always lies exactly one (simple) zero of every function from the two-parametric system

$$y^*(t) = C_1 u(t) + C_2 v(t),$$

which is always uniquely determined by the choice of the constants $C_i \in \boldsymbol{R}$, $i = 1, 2$, hereby $C_2 \neq 0$, as every such function together with the function $u(t)$ form a pair of a linear independent solutions (i.e. a basis) of the dif. equation (2), all zeros of which — according to the Sturm theorem — separate each other. In particular, if $C_1 = 0$, the solution $y(t) = C_2 u^2(t)\, v(t)$, $C_2 \neq 0$, has on the open interval (t_0, T_1) exactly one (simple) zero t' relating to the function $v(t)$, i.e. holds $v(t') = 0$.

Therefore it holds ${}^1t_1 = t'$ for the first conjugate point ${}^2t_0$ wherein the solution $y(t)$ of the dif. equation (1) vanishes together with the function $u^2(t)$.

III. Let $v = 3$; then by **Lemma 1** every oscillatory solution of the dif. equation (1) vanishing together with the function $u^3(t)$ at the triple point ${}^3t_0$ is exactly of the form

$$y(t) = C_1 u^3(t)$$

with $C_1 \in \boldsymbol{R} - \{0\}$ being an arbitrary constant. If we denote by T_1 the neighbouring zero of the function $u(t)$ lying to the right after the point t_0, i.e. $T_1 > t_0$, then repeatedly

$$y(t_0) = u(t_0) = 0, \qquad y(T_1) = u(T_1) = 0.$$

Then it holds both $u(t) \neq 0$ and $y(t) \neq 0$ for all $t \in (t_0, T_1)$ so that there lies no zero of the solution $y(t)$ on the interval (t_0, T_1) because all zeros of the solution $y(t)$ coincide with the zeros of the function $u(t)$ and with respect to the form of the solution $y(t)$ all the zeros are triple.

Thus it holds ${}^3t_1 = T_1$ for the first conjugate point from the right to the point ${}^3t_0$ wherein the solution $y(t)$ of the dif. equation (1) vanishes together with the function $u^3(t)$.

In this way the **Theorem 1.1** is completely proved.

Definition 1.1. can be extended in a natural manner to

Definition 1.2: Let $t_0 \in (-\infty, +\infty)$ be an arbitrary firmly chosen point and let $y(t)$ be an arbitrary solution of the differential equation (1) vanishing at the point t_0 (we write ${}^vt_0$ whereby the point t_0 is of multiplicity $v = 1, 2, 3$).

Then under the *n-th* ($n = 1, 2, \ldots$) conjugate point from the right [from the left] to the point ${}^vt_0$ ($v = 1, 2, 3$) it will be understood the *n-th* zero of the solution $y(t)$

lying to the right [to the left] of the point ${}^{\nu}t_0$ written as ${}^{\mu}t_n[{}^{\mu}t_{-n}]$, where $\mu = 1, 2, 3$ denotes the multiplicity of this point, respectively.

In analogy with the remark to **definition 1.1** it becomes evident with respect to the oscillation of every solution of the dif. equation (1) that, to an arbitrary point ${}^{\nu}t_0 \in (-\infty, +\infty)$, $\nu = 1, 2, 3$, there always exists a $|k|$-*th* conjugate point ${}^{\nu}t_k \in (-\infty, +\infty)$, where $k = \pm 1, \pm 2, \dots$, from the right (for $k > 0$) or from the left (for $k < 0$).

Remark on the conjunction of points:

Let $t \in (-\infty, +\infty)$ be an arbitrary firmly chosen point; then

1. any point $t^* \in (-\infty, +\infty)$ conjugate to the point t is conjugate to itself (i.e. the property of the conjunction is reflexive),

2. if the point $t^* \in (-\infty, +\infty)$ is a conjugate point to the point t, then the point t is also a conjugate point to the point t^* (i.e. the property of the conjunction is symmetric); one speaks therefore of mutually conjugate points,

3. if the point $t^* \in (-\infty, +\infty)$ is a conjugate point to the point t, and $t^{**} \in (-\infty, +\infty)$ is a conjugate point to the point t^*, then the point t^{**} is a conjugate point to the point t (i.e. the property of conjunction is transitive).

Using the just in **definition 1.2** installed notion of the *n-th* conjugate point, we may express especially by the help of the *2-nd* conjugate point some evident statements (as a consequence of the preceding **Theorem 1.1**) in

Theorem 1.2: Let $t_0 \in (-\infty, +\infty)$ be a zero of an arbitrary solution $y(t)$ of the differential equation (1) of multiplicity $\nu = 1, 2, 3$, i.e. $y({}^{\nu}t_0) = 0$. Then the set of all first conjugate points ${}^{\mu}t_1$ ($\mu = 1, 2, 3$) from the right to the point ${}^{\nu}t_0$ in case of

1. $\nu = 3$ is one-point and exactly ${}^{3}t_1$,
2. $\nu = 2$ is an open interval $({}^{2}t_0, {}^{2}t_2)$, i.e. ${}^{1}t_1 \in ({}^{2}t_0, {}^{2}t_2)$,
3. $\nu = 1$ and $\mu = 2$ is an open interval $({}^{1}t_0, {}^{1}t_2)$, i.e. ${}^{2}t_1 \in ({}^{1}t_0, {}^{1}t_2)$,
4. $\nu = 1$ and $\mu = 1$ is either one-point and exactly ${}^{1}t_1$ or an open interval $({}^{1}t_0, {}^{1}t_2)$, i.e. ${}^{1}t_1 \in ({}^{1}t_0, {}^{1}t_2)$.

The results of **Theorems 1.1** and **1.2** are summarized and with respect to **definition 1.2** generalized in the following

Theorem 1.3: Let ${}^{\nu}t_0 \in (-\infty, +\infty)$, $\nu = 1, 2, 3$, be an arbitrary firmly chosen point and let $y(t)$ be such a solution of the differential equation (1) that the point ${}^{\nu}t_0$ is its ν-multiple zero.

Then it holds:

1. Every $|k|^{th}$ conjugate point ${}^{\mu}t_k$ to the point ${}^{3}t_0$, where $k = \pm 1, \pm 2, \dots$, is uniquely given with $\mu = 3$; it holds the inequality

$$ {}^{3}t_k < {}^{3}t_{k+1} $$

at the same time.

2. Every $2\,|\,k\,|^{th}$ conjugate point ${}^{\mu}t_{2k}$ to the point ${}^{2}t_0$, where $k = \pm 1, \pm 2, \ldots$, is uniquely given with $\mu = 2$ and the set of all $|\,2k + 1\,|^{st}$ conjugate points ${}^{\mu}t_{2k+1}$ to the point ${}^{2}t_0$, where $k = \pm 1, \pm 2, \ldots$, forms an open interval $({}^{2}t_{2k}, {}^{2}t_{2k+2})$ with $\mu = 1$. There hold at the same time the inequalities

$${}^{2}t_{2k} < {}^{1}t_{2k+1} < {}^{2}t_{2k+2}.$$

3. a) If the first conjugate point ${}^{\mu}t_1$ to the point ${}^{1}t_0$ is given uniquely, then an arbitrary $|\,k\,|^{th}$ conjugate point ${}^{\mu}t_k$ with $k = \pm 1, \pm 2, \ldots$ is also given uniquely whereby $\mu = 1$. There holds at the same time the inequality

$${}^{1}t_k < {}^{1}t_{k+1}.$$

3. b) If the set of all first conjugate points ${}^{\mu}t_1$ to the point ${}^{1}t_0$ forms an open interval whereby $\mu = 2$, then any $2\,|\,k\,|^{th}$ conjugate point ${}^{\varepsilon}t_{2k}$ to the point ${}^{1}t_0$, where $k = \pm 1, \pm 2, \ldots$, is given uniquely, whereby $\varepsilon = 1$, and the set of all $|\,2k + 1\,|^{st}$ conjugate points ${}^{\varepsilon}t_{2k+1}$, where $k = \pm 1, \pm 2, \ldots$, to the point ${}^{1}t_0$ forms an open interval $({}^{1}t_{2k}, {}^{1}t_{2k+2})$, whereby $\varepsilon = 2$. There hold at the same time the inequalities

$${}^{1}t_{2k} < {}^{2}t_{2k+1} < {}^{1}t_{2k+2}.$$

3. c) If the set of all first conjugate points ${}^{\mu}t_1$ to the point ${}^{1}t_0$ forms an open interval whereby $\mu = 1$, then any $3\,|\,k\,|^{th}$ conjugate point ${}^{\varepsilon}t_{3k}$, where $k = \pm 1, \pm 2, \ldots$, to the point ${}^{1}t_0$ is given uniquely, whereby $\varepsilon = 1$, the set of all $|\,3k + 1\,|^{st}$ conjugate points ${}^{\varepsilon}t_{3k+1}$, where $k = \pm 1, \pm 2, \ldots$, to the point ${}^{1}t_0$ forms an open interval $({}^{1}t_{3k}, {}^{1}t_{3k+2})$, whereby $\varepsilon = 1$, the set of all $|\,3k + 2\,|^{st}$ conjugate points ${}^{\varepsilon}t_{3k+2}$, where $k = \pm 1, \pm 2, \ldots$, to the point ${}^{1}t_0$ forms an open interval $({}^{1}t_{3k+1}, {}^{1}t_{3k+3})$, whereby $\varepsilon = 1$ and there hold the inequalities

$${}^{1}t_{3k} < {}^{1}t_{3k+1} < {}^{1}t_{3k+2} < {}^{1}t_{3k+3}$$

at the same time.

§ 2. STRONGLY AND WEAKLY CONJUGATE POINTS

Definition 2.1. Let the points ${}^{\nu}t_0$, ${}^{\mu}t_k \in (-\infty, +\infty)$, where $\nu, \mu \in \{1, 2, 3\}$, $k = \pm 1, \pm 2, \ldots$, be conjugate points of the solution $y_0(t)$ of the differential equation (1).

We say, that the point ${}^{\mu}t_k$ is a strongly conjugate point to the point ${}^{\nu}t_0$ exactly if all solutions $y(t)$ of the dif. equation (1) vanishing ν-times at the point ${}^{\nu}t_0$, vanish at the point ${}^{\mu}t_k$.

Any conjugate point to the point ${}^{\nu}t_0$, which is not strongly conjugate to the point ${}^{\nu}t_0$, will be called a weakly conjugate point to the point ${}^{\nu}t_0$.

With the above definition we see that it holds:

the point $t_k^* \in (-\infty, +\infty)$, where $k = \pm 1, \pm 2, \ldots$, is the weakly conjugate point to the point ${}^{\nu}t_0 \in (-\infty, +\infty)$, where $\nu \in \{1, 2, 3\}$, exactly if there exist at least two solutions – among all solutions $y(t)$ of the dif. equation (1) vanishing ν-times at the point ${}^{\nu}t_0$ – such that one of them vanishes at the point t_k^* while the other does not.

The following theorem proves that the multiplicities ν, μ of the points ${}^{\nu}t_0 \in \in (-\infty, +\infty)$ and ${}^{\mu}t_k \in (-\infty, +\infty)$ – the latter is strongly conjugate to the former – always coincide for any of the values ν, $\mu = 1, 2, 3$ and for all $k = \pm 1, \pm 2, \ldots$

Theorem 2.1: Let ${}^{\nu}t_0, {}^{\mu}t_k \in (-\infty, +\infty)$, where ν, $\mu \in \{1, 2, 3\}$, $k = \pm 1, \pm 2, \ldots$, be two conjugate points of the solution $y_0(t)$ of the differential equation (1). Then the point ${}^{\mu}t_k$ is a strongly conjugate point to the point ${}^{\nu}t_0$ exactly if

1. either $\mu = \nu = 3$, $k = \pm 1, \pm 2, \ldots$
2. or $\mu = \nu = 2$ and $k = 2m$, $m = \pm 1, \pm 2, \ldots$
3. or $\mu = \nu = 1$ and

a) $k = 3m$, $m = \pm 1, \pm 2, \ldots$, if there exist simple weakly conjugate points to the points ${}^{\nu}t_0$, ${}^{\mu}t_k$,

b) $k = 2m$, $m = \pm 1, \pm 2, \ldots$, if there exist double weakly conjugate points to the points ${}^{\nu}t_0$, ${}^{\mu}t_k$,

c) $k = m$, $m = \pm 1, \pm 2, \ldots$, if there exist no weakly conjugate points to the points ${}^{\nu}t_0$, ${}^{\mu}t_k$.

Proof: Let $t_0 \in (-\infty, +\infty)$ be an arbitrary firmly chosen point and let $[u(t), v(t)]$ be a basis of the oscillatory dif. equation (2) such that both functions $u(t)$, $v(t)$ together with their first derivatives $u'(t)$, $v'(t)$ satisfy the conditions (P) at the point t_0. Let $y(t)$ be such a solution of the dif. equation (1) which vanishes at the point t_0 ν-times, $\nu = 1, 2, 3$, enabling us to write ${}^{\nu}t_0$.

Then it immediately follows from the statement of **Theorem 1.3**, by using **Lemma 1**, and with respect to the definition on strongly conjugate points that

1. if $\nu = 3$, then any zero of the solution $y(t)$ of the dif. equation (1) from the system

$$y(t) = C_1 u^3(t), \qquad C_1 \neq 0,$$

is at the same time a triple zero of the function $u^3(t)$, so that every $|k|^{th}$, $k = \pm 1, \pm 2, \ldots$, conjugate point ${}^{\mu}t_k$ to the point ${}^{3}t_0$ is simultaneously the $|k|^{th}$ strongly conjugate point to ${}^{3}t_0$, where $\mu = 3$,

2. if $\nu = 2$, then all double zeros of the solution $y(t)$ of the dif. equation (1) from the system

$$y(t) = u^2(t)\,[C_1 u(t) + C_2 v(t)],$$

$C_i \in \boldsymbol{R}$, $i = 1, 2$, $C_2 \neq 0$, are at the same time the double zeros of the function $u^2(t)$. Here between any two neighbouring double zeros of the solution $y(t)$ there lies always exactly one simple zero of the two-parametric function subsystem

$$y^*(t) = C_1 u(t) + C_2 v(t),$$

so that every $|k|^{th}$ conjugate point ${}^{\mu}t_k$ (where $k = 2m$, $m = \pm 1, \pm 2, \ldots$) to the point ${}^{2}t_0$ is simultaneously a strongly conjugate point to it, whereby $\mu = 2$,

3. if $v = 1$, then for the point ${}^1t_0$, at which every solution $y(t)$ of the dif. equation (1) from the system

$$y(t) = u(t)\left[C_1u^2(t) + C_2u(t)\,v(t) + C_3v^2(t)\right],$$

$C_i \in \boldsymbol{R}$, $i = 1, 2, 3$, $C_3 \neq 0$, together with the function $u(t)$ vanishes, it holds: every $|k|^{th}$ conjugate point ${}^{\mu}t_k$ – where $k = 3m$ or $k = 2m$ or $k = m$, $m = \pm 1, \pm 2, \ldots$ – to the point ${}^1t_0$ is simultaneously a strongly conjugate point to it, whereby $\mu = 1$, exactly if between any two neighbouring (simple) zeros of the function $u(t)$ there exist two simple zeros different from each other [or one double zero, or there exists no zero] of the three-parametric function subsystem

$$y^{**}(t) = C_1u^2(t) + C_2u(t)\,v(t) + C_3v^2(t).$$

Remark 2.1 – on the strong conjunction of points:

Let $t_0 \in (-\infty, +\infty)$ be an arbitrary firmly chosen point. Then

1. every point $t^* \in (-\infty, +\infty)$ strongly conjugate to the point t_0 is strongly conjugate to itself (i.e. the property of the strong conjunction is reflexive),

2. if the point $t^* \in (-\infty, +\infty)$ is a strongly conjugate point to the point t_0, then the point t_0 is a strongly conjugate point to the point t^* (i.e. the property of the strong conjunction is symmetric); we speak therefore about mutually strongly conjugate points,

3. if the point $t^* \in (-\infty, +\infty)$ is a strongly conjugate point to the point t_0 and the point $t^{**} \in (-\infty, +\infty)$ is a strongly conjugate point to the point t^* (with respect to retaining the same multiplicity of the point t^* as in case of the pair t_0 and t^*), then the point t^{**} is a strongly conjugate point to the point t_0 (i.e. the property of the strong conjunction is transitive).

Remark 2.2 – on the weak conjunction of points:

Let $t_0 \in (-\infty, +\infty)$ be an arbitrary firmly chosen point and let $t^* \in (-\infty, +\infty)$ be a weakly conjugate point to the point t_0. Then

1. the point t_0 is a weakly conjugate point to the point t^* (i.e. the property of the weak conjunction of the two conjugate points is symmetric),

2. if the point $t^{**} \in (-\infty, +\infty)$ is a strongly conjugate point to the point t^*, then the point t^* is a weakly conjugate point to the point t_0.,

3. the property of transitivity of the weak conjunction for the two pairs of points t_0, t^* and t^*, t^{**} (mutually weakly conjugate) does not hold in general (i.e. from the weak conjunction of the points t_0, t^* and t^*, t^{**} does not follow in general that the points t_0 and t^{**} are weakly conjugate).

From the above Theorem, the following statements are immediate:

1. There exist no weakly conjugate points to the point ${}^3t_0 \in (-\infty, +\infty)$, i.e. every conjugate point to its is strongly conjugate.

2. There always exist weakly conjugate points to the point ${}^2t_0 \in (-\infty, +\infty)$, each of which is exactly of multiplicity $\mu = 1$.

3. The weakly conjugate points to the point ${}^1t_0 \in (-\infty, +\infty)$ either do not exist or they exist and this either all with multiplicity exactly $\mu = 1$ or all with multiplicity exactly $\mu = 2$.

From **Theorem 1.3** it follows simultaneously that none of the weakly conjugate points to the point ${}^\nu t_0 \in (-\infty, +\infty)$, $\nu = 1, 2$, is given uniquely and even: all weakly conjugate points to the point ${}^\nu t_0 \in (-\infty, +\infty)$, $\nu = 1, 2$, constitute throughout open intervals with end points, which are always represented by two mutually neighbouring strongly conjugate points (or by their subintervals wherein one of their end points is always that one of the given pair of the mutually neighbouring strongly conjugate points).

The basic information on the coexistence of the strongly and weakly conjugate points of an arbitrary solution $y(t)$ of the differential equation (1) states the following

Theorem 2.2: Let ${}^\nu t^*, {}^\nu t^{**} \in (-\infty, +\infty)$, where $\nu = 1, 2, 3$, be any two neighbouring strongly conjugate points of the solution $y(t)$ of the differential equation (1). Then, there may lie at most two weakly conjugate points of the solution $y(t)$ between them, i.e. either none or exactly one, or exactly two; and that:

1. if $\nu = 3$, then there lies no weakly conjugate point of the solution $y(t)$, between ${}^3t^*, {}^3t^{**}$;

2. if $\nu = 2$, then there lies one point between ${}^2t^*, {}^2t^{**}$ being simple, weakly conjugate of the solution $y(t)$,

3. if $\nu = 1$, then there lies no weakly conjugate point between ${}^1t^*, {}^1t^{**}$, or there lies exactly one point being double, weakly conjugate, or there lie exactly two points different from each other being simple, weakly conjugate of the solution $y(t)$.

Proof: For the proof of statement 1. or 2. or 3. see the part III. or II. or I. the proof of the fundamental **Theorem 1.1** with respect to the **definition 2.1** of the strongly (weakly) conjugate points respectively.

A more detailed account on the distribution of the weakly conjugate points of any solution $y(t)$ of the differential equation (1) vanishing either at the simple or double points ${}^\nu t \in (-\infty, +\infty)$, $\nu = 1, 2$, is given by the theorem below, which immediately follows from **Theorem 1.3.**

Theorem 2.3: Let $k = 0, \pm 1, \pm 2, \ldots$

1. Let $y(t)$ be a solution of the dif. equation (1) vanishing at the double strongly conjugate points. Then, for any simple, weakly conjugate point at which this solution vanishes, we have

$$ {}^1t_{2k+1} \in ({}^2t_{2k}, {}^2t_{2k+2}), $$

where ${}^2t_{2k}, {}^2t_{2k+2}$ are two mutually neighbouring strongly conjugate points of this solution.

2. Let $y(t)$ be a solution of the dif. equation (1) vanishing at one of the simple strongly conjugate points. Then

a) for any double weakly conjugate point, at which this solution vanishes, we have

$$^{2}t_{2k+1} \in (^{1}t_{2k}, {}^{1}t_{2k+2}),$$

where $^{1}t_{2k}$, $^{1}t_{2k+2}$ are two mutually neighbouring strongly conjugate points of this solution,

b) for any simple weakly conjugate point, at which this solution vanishes, we have either

$$^{1}t_{3k+1} \in (^{1}t_{3k}, {}^{1}t_{3k+2}) \subset (^{1}t_{3k}, {}^{1}t_{3k+3})$$

or

$$^{1}t_{3k+2} \in (^{1}t_{3k+1}, {}^{1}t_{3k+3}) \subset (^{1}t_{3k}, {}^{1}t_{3k+3}),$$

where $^{1}t_{3k}$, $^{1}t_{3k+3}$ are two mutually neighbouring strongly conjugat points of this solution.

§ 3. CONJUGATE POINTS WITH INDEX

In the definition below we give a certain generalization of the concept of the first conjugate point introduced by T. L. Sherman in [1].

Definition 3.1. Let $t_0 \in (-\infty, +\infty)$ be an arbitrary firmly chosen point and let $n = 1, 2, \ldots, \mu = 1, 2, 3.$

Then the n^{th} conjugate point with index μ from the right [or from the left] to the point t_0 will be called and denoted by $^{(\mu)}t_n$ [or $^{(\mu)}t_{-n}$] the smallest [or the greatest] of all numbers $t > t_0$ [or $t < t_0$], such that there exists a (nontrivial) solution of the differential equation (1) with

$$y(t_0) = y(^{(\mu)}t_n) = 0 \; [\text{or } y(t_0) = y(^{(\mu)}t_{-n}) = 0],$$

whereby on the interval $\langle t_0, {}^{(\mu)}t_n \rangle$ [or $\langle {}^{(\mu)}t_{-n}, t_0 \rangle$] there lie exactly $3n + \mu$ zeros of the solution $y(t)$ including their multiplicities.

Remark. For $n = 1$ and $\mu = 1$ we have the definition of the first conjugate point in [1].

The existence of the $|k|^{th}$, $k = \pm 1, \pm 2, \ldots,$ conjugate point $^{(\mu)}t_k \in (-\infty, +\infty)$ with index μ, $\mu = 1, 2, 3$, to the point $t_0 \in (-\infty, +\infty)$ from the right (for $k > 0$) or from the left (for $k < 0$) follows from the statement of **Theorem 3.2** (see later).

Remark 3.1 – on the conjunction of points with index $\mu \in \{1, 2, 3\}$:

Let $t_0 \in (-\infty, +\infty)$ be an arbitrary firmly chosen point. Then

1. every point $t^* \in (-\infty, +\infty)$ conjugate with index μ to the point t_0 is conjugate to itself with the same index μ (i.e. the property of the conjunction with index μ is reflexive)

2. if the point t^* is a $|k|^{th}$ conjugate point, $k = \pm 1, \pm 2, \ldots,$ with index μ to the point t_0, then the point t_0 is a $|-k|^{th}$ conjugate point with the same index μ from

the left (for $k > 0$) or from the right (for $k < 0$) to the point t^* (i.e. the property of the conjunction with index μ is symmetric); one speaks therefore of mutually conjugate points with index

3. if the point t^* is a $|k|^{th}$, $k = \pm 1, \pm 2, \ldots$, conjugate point with index μ to the point t_0 and the point $t^{**} \in (-\infty, +\infty)$ an $|m|^{th}$, $m = \pm 1, \pm 2, \ldots$, $m \neq -k$, conjugate point with the same index μ to the point t^*, then the point t^{**} is a $|k+m|^{th}$ conjugate point with the same index μ to the point t_0 (i.e. the property of the conjunction with index μ is transitive).

Theorem 3.1: Let ${}^{\nu}t_0 \in (-\infty, +\infty)$, $\nu = 1, 2, 3$, be an arbitrary firmly chosen point being a zero of multiplicity ν of the solution $y(t)$ of the differential equation (1) and let the point ${}^{(\mu)}t_k \in (-\infty, +\infty)$, $\mu = 1, 2, 3$, $k = \pm 1, \pm 2, \ldots$, be a $|k|^{th}$ conjugate point with index μ from the right (for $k > 0$) from the left (for $k < 0$) to the point ${}^{\nu}t_0$.

Then, for all $k = \pm 1, \pm 2, \ldots$ we have $\mu = \nu$.

Proof: Let ${}^{\nu}t_0 \in (-\infty, +\infty)$ be an arbitrary firmly chosen point at which the solution $y(t)$ of the differential equation (1) together with the function $u(t)$ are vanishing with the multiplicity $\nu \in \{1, 2, 3\}$; for the multiplicity μ of the $|k|^{th}$ conjugate point ${}^{(\mu)}t_k \in (-\infty, +\infty)$, $k = \pm 1, \pm 2, \ldots$, to the point ${}^{\nu}t_0$ let us consider the value $\mu = 3$, $\mu = 2$ and $\mu = 1$ respectively.

1. Let $\mu = 3$; it can be seen from *Lemma 1*, from the IIIrd part of the proof to *Theorem 1.1* and further [in view of the discrimination between the strongly and weakly conjugate points relative to an arbitrary bundle of the solutions $y(t)$ of (1)] from the 1*st* part of the proof to *Theorem 2.1* that there exists one and only one bundle of $y(t)$ of the differential equation (1), whose (even all) zeros are triple and namely the oneparametric bundle exactly of the form

$$y(t) = C_1 u^3(t), \tag{S$_1$}$$

where $C_1 \in \mathbf{R} - \{0\}$ is an arbitrary parameter.

Thus, it is necessary for $\mu = 3$ to search for the $|n|^{th}$, $n = \pm 1, \pm 2, \ldots$, conjugate point ${}^{(3)}t_n \in (-\infty, +\infty)$ to the point ${}^{3}t_0$ just among the mutually strongly conjugate zeros ${}^{3}t_k$, $k = \pm 1, \pm 2, \ldots$, of such a bundle (S$_1$). On the purpose that for $n = 1, 2, 3, \ldots$ there consecutively exists (by *Definition 3.1*) the smallest of intervals $\langle {}^{3}t_0, {}^{(3)}t_n \rangle$ or $\langle {}^{(3)}t_{-n}, {}^{3}t_0 \rangle$, on which (consecutively) lie exactly 6, 9, 12, ..., i.e. generally 3(n + 1) zeros of $y(t)$ of the differential equation (1) from (S$_1$), such a point must be exactly the k^{th}, $k = \pm 1, \pm 2, \ldots$, conjugate point to the point ${}^{3}t_0$. Hence

$${}^{(3)}t_n = {}^{3}t_k,$$

where $n = k = \pm 1, \pm 2, \ldots$

2. Let $\mu = 2$; it can be seen from *Lemma 1.* from the IInd part of the proof to *Theorem 1.1* and further from the 2nd part of the proof to *Theorem 2.1* that the bundle of all solutions $y(t)$ of the differential equation (1), vanishing under the given condi-

tions at the point ${}^2t_0$, is (up to an arbitrary nonzero multiplicative constant $C \in \mathbf{R}$) a twoparametric bundle having exactly the form

$$y(t) = u^2(t)\left[C_1u(t) + C_2v(t)\right], \tag{S$_2$}$$

where $C_i \in \boldsymbol{R}$, $i = 1, 2$, $C_2 \neq 0$, are arbitrary parameters.

All its zeros are on one hand double (mutually strongly conjugate) at which it is vanishing together with the function $u^2(t)$ and on the other hand simple, at which it is vanishing together with the system of functions $C_1u(t) + C_2v(t)$, and which are to these double zeros altogether weakly conjugate. Between arbitrary two neighbouring double zeros ${}^2t_{2k}$, ${}^2t_{2k+2}$, $k = 0, \pm 1, \pm 2, \ldots$, from ($S_2$) there always lies exactly one simple zero ${}^1t_{2k+1}$, $k = 0, \pm 1, \pm 2, \ldots$, of an arbitrary solution $y_0(t)$ relative to (1), which we get from such a bundle (S_2) in a firm choice of both constants $C_i \in \mathbf{R}$, $i = 1, 2$, $C_2 \neq 0$.

Thus, it is necessary for $\mu = 2$ to search for the $|\,n\,|^{th}$, $n = \pm 1, \pm 2, \ldots$, conjugate point ${}^{(2)}t_n \in (-\infty, +\infty)$ to the point ${}^2t_0$ just among its double (strongly conjugate) zeros ${}^2t_{2k}$, $k = \pm 1, \pm 2, \ldots$

On the purpose that for $n = 1, 2, 3, \ldots$ there consecutively exists (by *Definition 3.1*) the smallest of intervals $\langle {}^2t_0, {}^{(2)}t_n \rangle$ or $\langle {}^{(2)}t_{-n}, {}^2t_0 \rangle$ on which (consecutively) lie exactly 5, 8, 11, …, i.e. generally $3n + 2$ zeros of $y(t)$ relative to (1) from the bundle (S_2), such a point must be exactly $k = 2n^{th}$, $n = \pm 1, \pm 2, \ldots$, conjugate point to ${}^2t_0$; hence

$${}^{(2)}t_n = {}^2t_k,$$

where $k = 2n = \pm 2, \pm 4, \ldots$

Remark. The double zeros are contained in the twoparametrical bundle (S_2) of solutions $y(t)$ relative to (1) and besides also in the threeparametrical bundle (S_3) written onwards as 3b), which is simply vanishing at all zeros of the function $u(t)$ [thus also at ${}^1t_0$]; between arbitrary two neighbouring simple zeros ${}^1t_{2k}$, ${}^1t_{2k+2}$ ($k = 0, \pm 1, \pm 2, \ldots$), being the strongly conjugate points of this bundle, there always lies exactly one double zero ${}^2t_{2k+1}$ ($k = 0, \pm 1, \pm 2, \ldots$) of an arbitrary solution $y_0(t)$ relative to (1) which we get from such a bundle (S_3) in a firm choice of constants $C_i \in \mathbf{R}$, $i = 1, 2, 3$, (naturally satisfying the given conditions $C_3 \neq 0$ and $C_2^2 - 4C_1C_3 = 0$).

However, in this case there consecutively lie on every interval $\langle {}^1t_0, {}^2t_{2n-1} \rangle$ or $\langle {}^2t_{-2n+1}, {}^1t_0 \rangle$, where $n = 1, 2, \ldots$, for $n = 1, 2, 3, \ldots$ exactly 3, 6, 9, …, i.e. generally $3n$ zeros of an arbitrary solution $y(t)$ relative to (1) from the bundle (S_3), which is the number of points (in comparison with the corresponding number of zeros given on the previously found intervals of the minimal length) always by 2 smaller than requested in *Definition 3.1* for the $|\,k\,|^{th}$ conjugate point ${}^{(2)}t_k$, $k = \pm 1, \pm 2, \ldots$, to the point ${}^1t_0$.

If the intervals $\langle {}^1t_0, {}^2t_{2n+1} \rangle$ or $\langle {}^2t_{-2n-1}, {}^1t_0 \rangle$ were considered, where $n = 1, 2, \ldots$, then there would consecutively lie for $n = 1, 2, 3, \ldots$ on them 6, 9, 12, …, i.e. generally $3(n + 1)$ zeros of an arbitrary solution $y(t)$ relative to (1) from the bundle (S_3), which is the number of points always by 1 greater than is required in *Definition 3.1.*

3. Let $\mu = 1$; it can be seen from *Lemma 1,* from the I^{st} part of the proof to *Theorem 1.1* and further from the 3^{rd} part of the proof to *Theorem 2.1* that the bundle of all solutions $y(t)$ relative to (1) vanishing under the given conditions at the point ${}^1t_0$ is (up to an arbitrary nonzero multiplicative constant $C \in \mathbf{R}$) a threeparametrical bundle having exactly the form

$$y(t) = u(t)\,[C_1u^2(t) + C_2u(t)\,v(t) + C_3v^2(t)],$$

where $C_i \in \mathbf{R}$, $i = 1, 2, 3$, $C_3 \neq 0$, are arbitrary parameters.

a) If $C_2^2 - 4C_1C_3 > 0$, then all zeros of every solution $y(t)$ from the bundle (S_3) are simple. Hereby it holds for $k = 0, \pm 1, \pm 2, \ldots$: all simple zeros ${}^1t_{3k}$ from (S_3) coinciding with all zeros of the function $u(t)$ are mutually strongly conjugate, while all the others, i.e. the points ${}^1t_{3k+1}$, ${}^1t_{3k+2}$, are simple zeros relative to the system of functions $C_1u^2(t) + C_2u(t)\,v(t) + C_3v^2(t)$ and they are to the foregoing zeros altogether weakly conjugate. Among the arbitrary two neighbouring strongly conjugate points ${}^1t_{3k}$, ${}^1t_{3k+3}$, $k = 0, \pm 1, \pm 2, \ldots$, from the bundle (S_3) there always lies exactly one zero, both ${}^1t_{3k+1}$ and ${}^1t_{3k+2}$ (for every $k = 0, \pm 1, \pm 2, \ldots$) belonging to an arbitrary solution $y_0(t)$ relative to (1) which we get from the bundle (S_3) in a firm choice of constants $C_i \in \mathbf{R}$, $i = 1, 2, 3$, $C_3 \neq 0$.

Let us remark that allowing for the multiplicities, then precisely this case simultaneously represents all solutions $y(t)$ relative to (1) satisfying the introductory conditions, possessing on every interval $\langle T_s, T_{s+1}\rangle$ [where $T_s, T_{s+1} \in (-\infty, +\infty)$, $T_s < T_{s+1}$, $s = 0, \pm 1, \pm 2, \ldots$, are two arbitrary neighbouring zeros of the function $u(t)$] the greatest possible number of zeros – exactly 4 – different from each other, i.e. it represents the bundle of solutions $y(t)$ relative to (1) with the densest decomposition of zeros.

On the purpose that for $n = 1, 2, 3, \ldots$ there consecutively exists (by *Definition 3.1*) the smallest of intervals $\langle {}^1t_0, {}^{(1)}t_n\rangle$ or $\langle {}^{(1)}t_{-n}, {}^1t_0\rangle$ on which (consecutively) lie exactly 4, 7, 10, ..., i.e. generally $3n + 1$ zeros of the solution $y(t)$ relative to (1) from the bundle (S_3), the point ${}^{(1)}t_n$, $n = \pm 1, \pm 2, \ldots$, must be in this case exactly $k = 3n^{th}$, $n = \pm 1, \pm 2, \ldots$, conjugate point to the point ${}^1t_0$, i.e. it holds

$${}^{(1)}t_n = {}^1t_k,$$

where $k = 3n = \pm 3, \pm 6, \ldots$

b) If $C_2^2 - 4C_1C_3 = 0$, then all zeros from (S_3) are on the one hand simple (mutually strongly conjugate) at which the bundle (S_3) vanishes together with the function $u(t)$ and on the other hand double, at which the bundle (S_3) vanishes together with the system of functions $C_1u^2(t) + C_2u(t)\,v(t) + C_3v^2(t)$, which are to these simple zeros altogether weakly conjugate. Between the arbitrary two neighbouring simple zeros ${}^1t_{2k}$, ${}^1t_{2k+2}$, $k = 0, \pm 1, \pm 2, \ldots$, from the bundle (S_3)

there always lies exactly one double zero ${}^{2}t_{2k+1}$ ($k = 0, \pm 1, \pm 2, \ldots$) of an arbitrary solution $y_0(t)$ relative to (1), which we get from the bundle (S_3) in a firm choice of constants $C_i \in \mathbf{R}$, $i = 1, 2, 3$, $C_3 \neq 0$.

Thus, at this bundle for $\mu = 1$ it is necessary to look for the $| n |^{th}$, $n = \pm 1, \pm 2, \ldots$, conjugate point ${}^{(1)}t_n \in (-\infty, +\infty)$ to the point ${}^{1}t_0$ among its simple (strongly conjugate) zeros ${}^{1}t_{2k}$, $k = \pm 1, \pm 2, \ldots$, only.

On the purpose that for $n = 1, 2, 3, \ldots$ there consecutively exists (by *Definition 3.1*) the smallest interval $\langle {}^{1}t_0, {}^{(1)}t_n \rangle$ or $\langle {}^{(1)}t_{-n}, {}^{1}t_0 \rangle$ on which (consecutively) lie exactly 4, 7, 10, ..., i.e. generally $3n + 1$ zeros of the solution $y(t)$ relative to (1) from the bundle (S_3), the point ${}^{(1)}t_n$, $n = \pm 1, \pm 2, \ldots$, must be in this case precisely the $k = 2n^{th}$, $n = \pm 1, \pm 2, \ldots$, conjugate point to the point ${}^{1}t_0$, i.e. it holds

$$ {}^{(1)}t_n = {}^{1}t_k, $$

where $k = 2n = \pm 2, \pm 4, \ldots$

c) If $C_2^2 - 4C_1C_3 < 0$, then all zeros from (S_3) are but simple (mutually strongly conjugate) at which the bundle (S_3) vanishes at the same time together with the function $u(t)$; in this case there doesnot exist any weakly conjugate points from (S_3) to them since the system of functions $C_1u^2(t) + C_2u(t)\,v(t) + C_3v^2(t)$ for every (admissible) choice of constants $C_i \in K$, $i = 1, 2, 3$, $C_3 \neq 0$, doesnot possess any zeros on the interval $(-\infty, +\infty)$.

However, in such a case, the numbers $3n + 1$, $n = 1, 2, \ldots$, of zeros of an arbitrary solution $y(t)$ relative to (1) from the bundle (S_3) as required by *Definition 3.1*, are lying only on the intervals $\langle {}^{1}t_0, {}^{1}t_n \rangle$ or $\langle {}^{1}t_{-n}, {}^{1}t_0 \rangle$, where $n = 3k$, $k = 1, 2, \ldots$, which in comparison with the corresponding intervals introduced in 3a) and 3b) are not minimal.

Remark. It should be mentioned here that besides the three possible types of (S_3) introduced in 3a), b) and c), also the twoparametrical bundle (S_2) of solutions $y(t)$ relative to (1) may possess the simple zeros with properties described in more detail in (2).

But on every interval of the form $\langle {}^{2}t_0, {}^{1}t_{2n-1} \rangle$ or $\langle {}^{1}t_{-2n+1}, {}^{2}t_0 \rangle$ there always lie (consecutively) for $n = 1, 2, 3, \ldots$ exactly 3, 6, 9, ..., i.e. generally 3n of zeros of an arbitrary solution $y(t)$ relative to (1) from the bundle (S_2) which represent the number of points – in comparison with the corresponding number of zeros stated on the intervals of a minimal length as shown in 3a) and 3b) – always by 1 less then required in *Definition 3.1* for the $| k |^{th}$ conjugate point ${}^{(1)}t_k$, $k = 2n - 1$, $n = 0, \pm 1, \pm 2, \ldots$, to the point ${}^{2}t_0$.

If we considered the intervals of the form $\langle {}^{2}t_0, {}^{1}t_{2n+1} \rangle$ or $\langle {}^{1}t_{-2n-1}, {}^{2}t_0 \rangle$, where $n = 1, 2, \ldots$, then there would (consecutively) lie for $n = 1, 2, 3, \ldots$ on them 6, 9, 12, ..., i.e. generally $3(n + 1)$ zeros of an arbitrary solution $y(t)$ relative to (1) from the bundle (S_2), which is the number of points always by 2 greater than required by *Definition 3.1*.

The theorem is completely proved.

Corollary:

It is immediate from the foregoing **Theorem 3.1** that if the point ${}^{(\mu)}t_k \in (-\infty, +\infty)$, $\mu = 1, 2, 3$, $k = \pm 1, \pm 2, \ldots$, is a $|k|^{th}$ conjugate point with index μ from the right (for $k > 0$) or from the left (for $k < 0$) to the point ${}^{\nu}t_0 \in (-\infty, +\infty)$, $\nu = 1, 2, 3$, then this point ${}^{(\mu)}t_k$ is a strongly conjugate point to the point ${}^{\nu}t_0$.

The situation will be made precise by the following

Theorem 3.2: Let ${}^{\nu}t_0 \in (-\infty, +\infty)$ be an arbitrary firmly chosen point which is a zero of the solution $y(t)$ of the differential equation (1) with multiplicity ν, $\nu = 1, 2, 3$.

Then:

1. the $|k|^{th}$ conjugate point ${}^{(\nu)}t_k \in (-\infty, +\infty)$, $k = \pm 1, \pm 2, \ldots$, with index $\nu = 2, 3$ from the right (for $k > 0$) or from the left (for $k < 0$) is a $|k|^{th}$ strongly conjugate point ${}^{\nu}t_k$ from the right (for $k > 0$) or from the left (for $k < 0$) to the point ${}^{\nu}t_0$, $\nu = 2, 3$, i.e. it holds

$$ {}^{\nu}t_k = {}^{(\nu)}t_k $$

2) the $|k|^{th}$ conjugate point ${}^{(1)}t_k$, $k = \pm 1, \pm 2, \ldots$, with index $\nu = 1$ from the right (for $k > 0$) or from the left (for $k < 0$) is a $|k|^{th}$ strongly conjugate point ${}^{1}t_k$ from the right (for $k > 0$) or from the left (for $k < 0$) to the point ${}^{1}t_0$, i.e. it holds

$$ {}^{1}t_k = {}^{(1)}t_k $$

exactly if there exist (besides the simple strongly conjugate points) also weakly conjugate points (either simple or double) to the point ${}^{1}t_0$.

With respect to **definition 3.1** the foregoing **Theorem 3.2** is equivalent to

Theorem 3.3: Let ${}^{\nu}t_0 \in (-\infty, +\infty)$, $\nu = 1, 2, 3$, be an arbitrary firmly chosen point and let $y(t)$ be a solution of the differential equation (1) such that the point ${}^{\nu}t_0$ is its zero of multiplicity ν. Then the point ${}^{(\nu)}t_n$ [or ${}^{(\nu)}t_{-n}$], where $n = 1, 2, \ldots$, is the n^{th} conjugate point with index ν from the right [or from the left] to the point ${}^{\nu}t_0$ exactly if on the interval $\langle {}^{\nu}t_0, {}^{\nu}t_n)$ [or $({}^{\nu}t_{-n}, {}^{\nu}t_0\rangle$] – where ${}^{\nu}t_n$ [or ${}^{\nu}t_{-n}$] is the n^{th} strongly conjugate point from the right [or from the left] to the point ${}^{\nu}t_0$ – there lie exactly $3n$ zeros of the solution $y(t)$ including multiplicities, whereby

$$ {}^{\nu}t_n = {}^{(\nu)}t_n \quad [\text{or } {}^{\nu}t_{-n} = {}^{(\nu)}t_{-n}]. $$

§ 4. THE FUNCTIONS OF THE DISTRIBUTION OF ZEROS

In investigating the distribution of zeros of solutions of the oscillatory differential equation (2) we used the function $\varphi(t)$ called the basic central dispersion of the first kind, introduced in [2]. This function assigns a first conjugate point lying to the right of the point t to every $t \in (-\infty, +\infty)$. In [2] there is likewise defined the $|k|^{th}$ basic central dispersion of the 1*st* kind as a function $\varphi_k(t)$, $k = 0, \pm 1, \pm 2, \ldots$, assigning the $|k|^{th}$ conjugate point to any point $t \in (-\infty, +\infty)$. This function has the following properties:

1. the interval $\boldsymbol{I} = (-\infty, +\infty)$ is the range of definition and the range of function values of $\varphi_k(t)$ for $k = 0, \pm 1, \pm 2, \ldots$; $\varphi_0(t) = t$

2. $\lim_{t \to -\infty} \varphi_k(t) = -\infty$, $\lim_{t \to +\infty} \varphi_k(t) = +\infty$ for $k = 0, \pm 1, \pm 2, \ldots$

3. $[\varphi_k(t) - t] \operatorname{sgn} k > 0$ for $k = \pm 1, \pm 2, \ldots$

4. $\varphi_k(t) \in \boldsymbol{C}_{\boldsymbol{I}}^3$ for $k = 0, \pm 1, \pm 2, \ldots$

5. $\varphi'(t) > 0$ for all $t \in (-\infty, +\infty)$ and for $k = 0, \pm 1, \pm 2, \ldots$

6. all functions $\varphi_k(t)$, $k = 0, \pm 1, \pm 2, \ldots$, constitute on the interval $\boldsymbol{I}$ a group with respect to the rule of composition and it holds

$$\varphi_k(t) = \varphi^k(t),$$

whereby repeatedly $\varphi_0(t) = \varphi^0(t) = t$ for $k = 0$. If $k > 0$, then $\varphi^k(t)$ stands for the k^{th} iteration of the function $\varphi(t)$;

if $k = -1$, then $\varphi^{-1}(t)$ stands for the function inverse to $\varphi(t)$ which exists due to property 5. and likewise maps the interval $\boldsymbol{I} = (-\infty, +\infty)$ onto itself.

In keeping with [2] let us introduce the function describing the distribution of the strongly conjugate points by the following

Definition 4.1. Let $k = 0, \pm 1, \pm 2, \ldots$ Denote by $\eta_k(t)$ the function assigning the $|k|^{th}$ strongly conjugate point to the point $t \in (-\infty, +\infty)$, lying to the right (when $k > 0$) or to the left (when $k < 0$) of the point t. We put hereby $\eta_0(t) = t$.

With respect to the function introduced in this way, **definition 2.1** may be set up as follows: the points $t, t^* \in (-\infty, +\infty)$, $t \neq t^*$, are strongly conjugate numbers of the solution $y(t)$ of the differential equation (1) exactly if there exists a number $k = \pm 1, \pm 2, \ldots$, such that

$$t^* = \eta_k(t).$$

Theorem 4.1: It holds $\eta_k(t) = \varphi_k(t)$ for all $t \in (-\infty, +\infty)$ and for $k = 0, \pm 1, \pm 2, \ldots$

Proof: Let ${}^{\nu}t_0 \in (-\infty, +\infty)$ be an arbitrary firmly chosen point ($\nu = 1, 2, 3$) and let $y(t)$ be such a solution of the differential equation (1) vanishing ν-times at it; then there exists such a basis $[u(t), v(t)]$ of the differential equation (2) that ${}^{\nu}t_0$ is a zero of the solution $u(t)$ of the dif. equation (2) cf. the condition (P) in the proof of **Lemma 1.**

Reversely:

If the point $t_0 \in (-\infty, +\infty)$ is a zero of the solution $u(t)$ of the differential equation (2) with the basis $[u(t), v(t)]$, then there exists a solution $y(t)$ of the differential equation (1) such that the point t_0 is its zero of multiplicity $\nu = 1, 2, 3$ [enabling us to write ${}^{\nu}t_0$].

Let ${}^{\nu}t_k \in (-\infty, +\infty)$, $\nu = 1, 2, 3$, $k = 0, \pm 1, \pm 2, \ldots$, be the $|k|^{th}$ strongly conjugate point to the point ${}^{\nu}t_0 \in (-\infty, +\infty)$ at which the solution $y(t)$ of the dif.

equation (1) vanishes ν-times. Then it follows from the proof of **Theorem 1.1** that the points t_0, t_k are at the same time simple zeros of the solution $u(t)$ of the dif. equation (2). Here the point t_k is the $|k|^{th}$ zero of the solution $u(t)$ lying to the right (if $k > 0$) or to the left (if $k < 0$) of the point t_0, which proves the statement of the theorem in keeping with the introduction to § 4.

It follows from the just proved coincidence of both functions η and φ that the function η introduced in the above **definition 4.1** has the same properties as has the function φ, stated for all $k = 0, \pm 1, \pm 2, \dots$ at the beginning of § 4.

BIBLIOGRAPHY

[1] Sherman T. L.: *Properties of solutions of N^{th}order linear differential equations*, Pacific J. Math., Vol. 15, No. 3, 1965.

[2] Borůvka O.: *Lineare Differentialtransformationen 2. Ordnung*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1967.

SOUHRN

O ROZLOŽENÍ NULOVÝCH BODŮ ŘEŠENÍ ITEROVANÉ LINEÁRNÍ DIFERENCIÁLNÍ ROVNICE 4. ŘÁDU

VLADIMÍR VLČEK

V práci je studováno rozložení nulových bodů obecného řešení obyčejné homogenní iterované lineární diferenciální rovnice čtvrtého řádu (1) vybraného tak, aby se anulovalo v libovolném bodě $t_0 \in (-\infty, +\infty)$ spolu s funkcí $u(t)$ [dvojice lineárně nezávislých funkcí $u(t)$, $v(t)$ tvoří bázi všech řešení obyčejné homogenní lineární diferenciální rovnice 2. řádu v Jacobiho tvaru (2), splňující v bodě t_0 podmínku $u(t_0) = v'(t_0) = 0$]. Vzhledem k násobnostem $\nu = 1, 2, 3$ nulového bodu t_0 netriviálního řešení $y(t)$ jsou vyšetřovány nulové body tří- resp. dvou- resp. jednoparametrického systému řešení diferenciální rovnice (1), jehož tvar je uveden (postupně) v tvrzení 1 resp. 2 resp. 3 **Lemmy 1.** V základní **Větě 1.1** je dokázána násobnost μ tzv. 1. konjugovaného bodu (zprava resp. zleva) k bodu t_0 s ohledem na jeho násobnost ν (pořadě $\nu = 1, 2, 3$). **Věta 1.3** shrnuje a zobecňuje výsledky dosažené v základní větě na libovolný $|k|$-tý, $k = \pm 1, \pm 2, \dots$, konjugovaný bod zprava (při $k > 0$) resp. zleva (při $k < 0$) k bodu t_0.

Dále jsou **Definicí 2.1** rozlišeny tzv. silně a slabě konjugované body a ukázáno, že násobnosti $\nu, \mu \in \{1, 2, 3\}$ navzájem silně konjugovaných bodů vždy splývají. Základní informaci o koexistenci (a násobnostech) silně a slabě konjugovaných bodů libovolného řešení $y(t)$ dif. rovnice (1) podává **Věta 2.2**; podrobný přehled o rozložení

slabě konjugovaných bodů libovolného řešení $y(t)$ dif. rovnice (1), jež se anuluje buď v jednoduchých nebo dvojnásobných bodech ${}^{\nu}t_0 \in (-\infty, +\infty)$, $\nu = 1, 2$, poskytuje pak **Věta 2.3.**

V § 3. o tzv. konjugovaných bodech indexu μ $(= 1, 2, 3)$ je podáno zobecnění definice prvního konjugovaného bodu zavedeného T. L. Shermanem v [1] a studována existence a rozložení konjugovaných bodů s indexem μ jakožto speciálního typu silně konjugovaných bodů.

V závěrečném § 4. této práce je definována funkce η popisující rozložení silně konjugovaných bodů libovolného řešení $y(t)$ dif. rovnice (1) a dokázána koincidence této funkce η s funkcí φ (tzv. základní centrální disperzí) definovanou akad. Borůvkou v [2].

РЕЗЮМЕ

О РАЗЛОЖЕНИИ НУЛЕВЫХ ТОЧЕК РЕШЕНИЙ ИТЕРИРОВАННОГО ЛИНЕЙНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ 4-ГО ПОРЯДКА

ВЛАДИМИР ВЛЧЕК

В работе изучается разложение нулевых точек общего решения обыкновенного однородного итерированного линейного дифференциального уравнения четвертого порядка (1) выбранного так, чтобы оно аннулировалось в любой точке $t_0 \in (-\infty, +\infty)$ вместе с функцией $u(t)$ [пара линейно независимых функций $u(t)$, $v(t)$ составляет базис всех решений обыкновенного однородного линейного дифференциального уравнения второго порядка в форме ЯКОБИ (2) и удовлетворяет в точке t_0 условию $u(t_0) = v'(t_0) = 0$].

Относительно кратностей $\nu = 1, 2, 3$ нулевой точки t_0 нетривиального решения $y(t)$ разысканы нулевые точки трех- или двух- или одно-параметрической системы решений диф. уравнения (1), наличие которого назначено (по очереди) в утверждении 1 или 2 или 3 леммы 1. В главной теореме 1.1 доказывается кратность μ так называемой первой сопряженной точки (справа или слева) к точке t_0 в согласию с его кратностью ν [по очереди $\nu = 1, 2, 3$]. **Теорема** 1.3 группирует результаты достигнутые в главной теореме и вообщает их на любую $|k|$-ую, $k = \pm 1, \pm 2, \ldots$, сопряженную точку справа (при $k > 0$) или слева (при $k < 0$) к точке t_0.

Далее в **определении** 2.1 различаются так называемые сильно сопряженные от слабо сопряженных точек и показано, что кратности $\nu, \mu \in \{1, 2, 3\}$ взаимно сильно сопряженных точек совсем совпадают. Основную информацию о совместном существовании (и кратностьях) сильно и слабо сопряженных точек

любого решения $y(t)$ диф. уравнения (1) дает **теорема** 2.2; подробное обозрение об разложении слабо сопряженных точек любого решения $y(t)$ диф. уравнения (1), которое исчезает или в простых или в двухкратных точках ${}^{\nu}t_0 \in (-\infty, +\infty)$, $\nu = 1, 2$, предлагается в **теореме** 2.3.

В § 3. о так называемых сопряженных точках индекса μ (= 1, 2, 3) дается обобщение определения первой сопряженной точки введенной Т. Л. ШЕРМЕНОМ (T. L. SHERMAN) в [1] и изучается существование и разложение сопряженных точек с индексом μ как специального типа сильно сопряженных точек.

В заключительном § 4. этой работы определена функция обозначенная η и описывающая разложение сильно сопряженных точек любого решения $y(t)$ диф. уравнения (1) и доказано совпадение этой функции η с функцией φ [так называемой центральной дисперсией] введенной акад. БОРУВКОМ (BORŮVKA) в [2].