Karel Beneš

The distribution of the break points by the approximation of a square root by means of a broken line

Sborník prací Přírodovědecké fakulty University Palackého v Olomouci. Matematika, Vol. 16 (1977), No. 1, 161--165

1977 — ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM — TOM 53

Katedra algebry a geometrie přírodovědecké fakulty
Vedoucí katedry: prof. RNDr. Ladislav Sedláček, CSc.

THE DISTRIBUTION OF THE BREAK POINTS BY THE APPROXIMATION OF A SQUARE ROOT BY MEANS OF A BROKEN LINE

KAREL BENEŠ

(*Received on January 21st, 1976*)

The functional dependences occuring in electronic analog computers are usually created by the so-called function transformers. The diode functional transformers approximate the given functional dependence by a broken line. In this article we are going to introduce the calculation of the distribution of the break points by means of the approximation of a square root by a broken line in accordance with the requirement of the best uniform approximation when the maximal absolute errors are in all sections the same. It is investigated the case when the break points are lying on the greph of the function $z = \sqrt{x}$, $x \in \langle 0; 1 \rangle$, the number of line segments being $k = 5$.

By a polynomical approximation of the function $f(x)$ by a function $g(x)$ in the interval of approximation $\langle a; b \rangle$ the error of the approximation $\varepsilon(x) = f(x) - g(x)$ is given by

$$\varepsilon(x) = \frac{f(\zeta)^{(n+1)}}{(n+1)!} (x - x_0)(x - x_1) \ldots (x - x_n), \tag{1}$$

where $x_0, x_1, \ldots, x_n$ are the knots of the approximation and ζ is a certain point of the interval $\langle a; b \rangle$, n being the degree of the approximating polynomial. Then the maximat absolute error of the approximation satisfies the inequality

$$| \varepsilon(x) | \max \leqq \frac{M_{n+1}}{(n+1)!} \varphi(x, x_0, x_1, \ldots, x_n), \tag{2}$$

where $M_{n+1} = \max\limits_{x \in \langle a; b \rangle} | f(x)^{(n+1)} |$, $\varphi(x, x_0, x_1, \ldots, x_n) = \max\limits_{x \in \langle a; b \rangle} | (x - x_0)(x_1 - x_1) \ldots$
$\ldots (x - x_n) |$.

By the approximation of the function $f(x)$ by a broken line the error is in accordance with (1) given by the expression

$$\varepsilon(x) = \frac{f''(\zeta)}{2!}(x - x_{j-1})(x - x_j), \tag{3}$$

$$\zeta \in \langle x_{j-1}; x_j \rangle, \qquad j = 1, 2, \ldots, k.$$

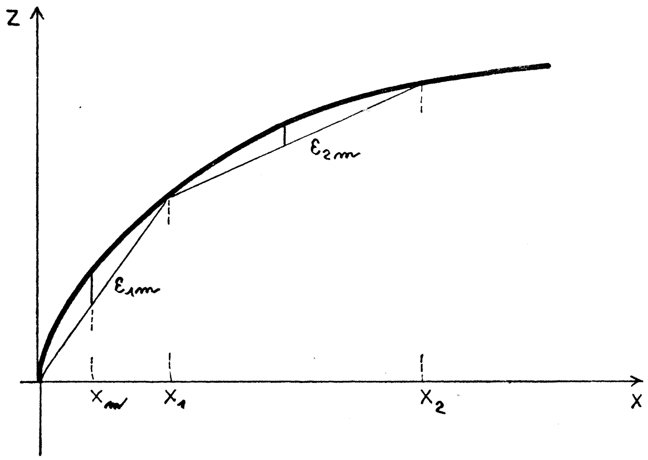
The expression $(x - x_{j-1})(x - x_j)$ attains its maximum at the point $x = \frac{x_{j-1} + x_j}{2}$, according to (2) so that we get

$$|\varepsilon(x)| \max \leqq \frac{M_2}{2!}\left(\frac{x_j - x_{j-1}}{2}\right)^2 \tag{4}$$

for the maximal absolute value of the error, where $M_2 = \max |z''(x)|$, $j = 1, 2, \ldots, 5$, $x \in \langle x_{j-1}; x_j \rangle$. Because of $z'' = -\frac{1}{4\sqrt{x^3}}$, the maximal absolute value of the second derivative $|z''| \max$ for $x \in \langle x_{j-1}; x_j \rangle$ will always be at the starting point of the corresponding segment. For the maximal absolute error in the j-th segment

$$|\varepsilon_j| \max \leqq \frac{1}{32\sqrt{x_{j-1}^3}}(x_j - x_{j-1})^2 \tag{5}$$

will take place. We cannot use this relation for the evaluation of the error in the first segment as $x_0 = 0$ and the function $z = \sqrt{x}$ has at this point the improper derivative.



Pict. 1.

According to Pict. 1, we express the absolute error $|\varepsilon_1|$ in the first segment as a function of the coordinate x_1 of the first break point. The absolute error $|\varepsilon_1|$ in the first segment is given by

$$|\varepsilon_1| = \sqrt{x} - \frac{\sqrt{x_1}}{x_1}x. \tag{6}$$

The point x. at which the absolute error attains its maximum in the first segment is to be determined from condition

$$\frac{\mathrm{d}\,|\varepsilon_1|}{\mathrm{d}x} = 0,$$

i.e., from the condition

$$\frac{1}{2}\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x_1}} = 0.$$

From this we get $x = x_m = \frac{1}{4}x_1$. Substituting into (6) we have

$$|\varepsilon_1|\max = \frac{1}{4}x_1. \tag{7}$$

For the best uniform approximation we get

$$|\varepsilon_1|\max = |\varepsilon_2|\max = \dots = |\varepsilon_5|\max. \tag{8}$$

The relation (4) does not exhibit the value of the error in the respective segments but just the upper bound of this error. Let us suppose that the step $h = x_j - x_{j-1}$ is small enough to ensure

$$f''(\zeta) \doteq \text{const.} = M_2, \tag{9}$$

$$\zeta \in \langle x_{j-1}; x_j \rangle, \qquad x \in \langle x_{j-1}; x_j \rangle.$$

Then according to (4) and (5), the maximal absolute error in the interval $\langle x_{j-1}; x_j \rangle$ will satisfy

$$|\varepsilon(x)|\max = \frac{1}{32}\frac{1}{\sqrt{x_{j-1}^3}}(x_j - x_{j-1}). \tag{10}$$

Applying the relations (7) and (10) the equation (8) takes the form

$$\frac{1}{4}\sqrt{x_1} = \frac{x_1^{-3/2}}{32}(x_2 - x_1) = \frac{x_2^{-3/2}}{32}(x_3 - x_2) =$$
$$= \frac{x_3^{-3/2}}{32}(x_4 - x_3) = \frac{x_4^{-3/2}}{32}(1 - x_4). \tag{11}$$

Transforming this into a system of four equations for the unknown x_1, x_2, x_3 and x_4 res., we get

$$\begin{aligned}
\frac{1}{4}\sqrt{x_1} &= \frac{x_1^{-3/2}}{32}(x_2 - x_1)^2,\\
\frac{x_1^{-3/2}}{32}(x_2 - x_1) &= \frac{x_2^{-3/2}}{32}(x_3 - x_2)^2,\\
\frac{x_2^{-3/2}}{32}(x_3 - x_2)^2 &= \frac{x_3^{-3/2}}{32}(x_4 - x_3)^2,\\
\frac{x_3^{-3/2}}{32}(x_4 - x_3)^2 &= \frac{x_4^{-3/2}}{32}(1 - x_4)^2.
\end{aligned} \tag{12}$$

From the first equation (12), let us express x_2 by means of x_1:

$$x_2 = x_1(1 + \sqrt{8}) = x_1 C_1. \tag{13}$$

Substituting (13) into (12_2), we obtain x_3 in a similar manner:

$$\frac{(x_1 C_1 - x_1)^2}{\sqrt{x_1^3}} = \frac{(x_3 - x_1 C_1)^2}{\sqrt{x_1^3 C_1^3}},$$

after a modification

$$x_1(C_1 - 1) = (x_3 - x_1 C_1) \frac{1}{\sqrt[4]{C_1^3}}.$$

From this we come to

$$x_3 = x_1[\sqrt[4]{C_1^3}(C_1 - 1) + C_1] = x_1 C_2. \tag{14}$$

Analogously we obtain by means of x_1, x_4 from (12_3):

$$x_4 = x_1 \left[\sqrt[4]{\frac{C_2^3}{C_1^3}} (C_2 - C_1) + C_2 \right] = x_1 C_3, \tag{15}$$

finally, substituting x_3 and x_4 from (14) and (15) resp., the fourth equation (12) yields

$$x_1 = \frac{1}{\sqrt[4]{\frac{C_3^3}{C_2^3}} (C_3 - C_2) + C_3}. \tag{16}$$

After calculating the coefficients C_1, C_2, C_3 and substituting them into equations (16), (13), (14) and (15) resp., we get for the x-coordinates of the break points the values

$$\begin{aligned} x_1 &= 0{,}0154, \\ x_2 &= 0{,}0592, \\ x_3 &= 0{,}178, \\ x_4 &= 0{,}452, \\ x_5 &= 1. \end{aligned} \tag{17}$$

The maximal absolute errors in the respective segments are, according to (8) and (7),

$$| \varepsilon | \max = 0{,}031. \tag{18}$$

This way of calculation is not, of course, absolutely exact because the relation (4) does not give the absolute value of the error in respective segments but the upper bound of this error, only. The exactness of this method is increasing when diminishing the step $h = x_j - x_{j-1}$; in this case, the value of the second derivative in the step h doesn't change too much.

Shrnutí

ROZLOŽENÍ BODŮ ZLOMŮ PŘI APROXIMACI ODMOCNINY LOMENOU ČAROU

Karel Beneš

V práci je popsán způsob výpočtu rozložení bodů zlomů při aproximaci odmocniny lomenou čarou při splnění požadavku nejlepší stejnoměrné aproximace.

Резюме

РАСПРЕДЕЛЕНИЕ ТОЧЕК ИЗЛОМА ПРИ АППРОКСИМАЦИИ ИЗВЛЕЧЕНИЯ КОРНЯ ЗЛОМНОЙ КРИВОЙ

Карэл Бэнэш

В статье описывается способ вычисления распределения точек излома при аппроксимации извлечения корня зломной кривой по требованию найлучшей аппроксимации.