Jaroslava Jachanová; Helena Žáková

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ON A DEPENDENCE BETWEEN POLARITIES AND AN ORDER OF A PROJECTIVE PLANE

JAROSLAVA JACHANOVÁ AND HELENA ŽÁKOVÁ

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Introduction. Polarities in incidence projective planes have been studied by many authors. Some of these investigations were directed to finite (infinite) planes and the results cannot be transferred to the infinite (finite) case. Following the results of [1] a natural question arises whether some partial results of [1] may be satisfied also in finite planes.

By a polarity of a projective plane $(\mathscr{P}, \mathscr{L}, \mathscr{I})$ we mean every involutory correlation (a correlation is an isomorphism of the given plane on its dual). Let π be a polarity of a projective plane $(\mathscr{P}, \mathscr{L}, \mathscr{I})$. Then a point $P \in \mathscr{P}$, and a line $l \in \mathscr{L}$, will be termed absolute if $P \mathscr{I} P^\pi$, and $l \mathscr{I} l^\pi$, respectively. Further we term $P \in \mathscr{P}$ interior, and exterior, if no absolute line passes through it, if it is neither absolute nor interior, respectively. Analogously a line $l \in \mathscr{L}$ is termed interior, and exterior, if the point l^π is exterior, and interior, respectively. A line l is called elliptic if it is not absolute but carries absolute points. Lines l, l' are said to be mutually perpendicular if $l^\pi \mathscr{I} l'$ (which also implies $l' \mathscr{I} l$). Denote the set of all interior, exterior, absolute points, lines, respectively, by $\mathscr{P}_{\text{int}}$, $\mathscr{P}_{\text{ext}}$, $\mathscr{P}_{\text{abs}}$, $\mathscr{L}_{\text{int}}$, $\mathscr{L}_{\text{ext}}$, $\mathscr{L}_{\text{abs}}$. It is obvious that the sets $\mathscr{P}_{\text{int}}$, $\mathscr{P}_{\text{ext}}$, $\mathscr{P}_{\text{abs}}$ are mutually disjoint and cover all $\mathscr{P}$. Analogously $\mathscr{L}_{\text{int}}$, $\mathscr{L}_{\text{ext}}$, $\mathscr{L}_{\text{abs}}$ are mutually disjoint and cover all $\mathscr{L}$. A polarity π of a projective plane $(\mathscr{P}, \mathscr{L}, \mathscr{I})$ is said to be regular (see [2], p. 85) if card $(\tilde{l} \cap \mathscr{P}_{\text{abs}})$ is constant for every elliptic line $l \in \mathscr{L}$.*) A polarity is termed admissible whenever any two perpendicular interior as well as any two perpendicular exterior lines meet in an interior point.

(1) Proposition. Let π be a polarity of a projective plane $(\mathscr{P}, \mathscr{L}, \mathscr{I})$. Then

$$\operatorname{card}(\tilde{l} \cap \mathscr{P}_{\text{abs}}) = \operatorname{card}(\tilde{P} \cap \mathscr{L}_{\text{abs}}) = 1$$

for any line $l \in \mathscr{L}_{\text{abs}}$ and for any point $P \in \mathscr{P}_{\text{abs}}$.

Proof. If $l \in \mathscr{L}_{\text{abs}}$, $P \in \mathscr{P}_{\text{abs}}$, $P \neq l^\pi$, then $l^\pi \mathscr{I} P^\pi$ and $P \mathscr{I} P^\pi$. Hence $P^\pi = l$ and $l^\pi = P$, a contradiction. The second part of the proposition follows dually.

*) In the given projective plane $(\mathscr{P}, \mathscr{L}, \mathscr{I})$ we denote: $\tilde{l} := \{P \in \mathscr{P} \mid P \mathscr{I} l\}$ for all $l \in \mathscr{L}$, $\tilde{P} := \{l \in \mathscr{L} \mid P \mathscr{I} l\}$ or all $P \in \mathscr{P}$.

(2) Corollary. For any polarity of a projective plane there are points and lines which are not absolute.

(3) Proposition. For any polarity of a projective plane every interior line is elliptic.

Proof. By definition a point is interior exactly if no absolute line passes through it. Thus if the point P lies on an absolute line l then it is either absolute (so that $l^\pi = P$) or exterior. In the latter case $P^\pi \in \mathscr{L}_{\text{int}}$, and every interior line can be obtained in such a manner.

(4) Proposition. If the order n of the given projective plane is finite, then

$$\operatorname{card} \mathscr{P}_{\text{abs}} \geqq n + 1.$$

(see [2], p. 82).

For an infinite plane it can be valid that $\mathscr{P}_{\text{abs}} = \emptyset$ and it can even happen $\mathscr{P}_{\text{abs}} = 1$ but this case is without interest. Assume in addition that card $\mathscr{P}_{\text{abs}}$ and card $\mathscr{L}_{\text{abs}} \geqq 2$ and that the polarity π is admissible as well as regular.

(5) Proposition. A point $P \in \mathscr{P}$ is exterior if and only if it is a point of intersection of two absolute lines; a line $l \in \mathscr{L}$ is exterior if and only if no absolute point lies on it.

Proof. Let $A \neq B$, $B \in \mathscr{P}_{\text{abs}}$. Thus $A \cup B \notin \mathscr{L}_{\text{abs}}$ and $A \cup B$ is elliptic.*) Since π is a regular polarity we have card $(\mathscr{P}_{\text{abs}} \cap \tilde{l}) \geqq 2$ for every elliptic line and hence for every interior line, too. Let $P \in \mathscr{P}_{\text{ext}}$, then $P^\pi \in \mathscr{L}_{\text{int}}$ and consequently there exist at least two points $A, B \in \mathscr{P}_{\text{abs}}$, $A \neq B$, $A \cup B = P^\pi$. This yields $P \mathscr{I} A^\pi$ and $P \mathscr{I} B^\pi$, where $A \neq B$ and $A^\pi, B^\pi \in \mathscr{L}_{\text{abs}}$. The converse follows from the definitions of absolute and interior points and from (1). Let $l \in \mathscr{L}_{\text{ext}}$. Assuming the existence of a point $P \in \mathscr{P}_{\text{abs}}$, $P \mathscr{I} l$ we have $l^\pi \mathscr{I} P^\pi$ in contradiction to $l^\pi \in \mathscr{P}_{\text{int}}$. The converse follows from the definition of an absolute line and from (3).

(6) Proposition. There is at least one point being not absolute on every line.

Proof. Suppose conversely there exists a line $l \in \mathscr{L}$, $P \in \mathscr{P}_{\text{abs}}$ for any $P \in \tilde{l}$. Then obviously $(\mathscr{P} \setminus \tilde{l}) \cap \mathscr{P}_{\text{abs}} = \emptyset$. ($\tilde{l} \subset \mathscr{P}_{\text{abs}} \Rightarrow (\mathscr{P} \setminus \tilde{l}) \cap \mathscr{P}_{\text{abs}} = \emptyset$ since if $B \in \mathscr{P}_{\text{abs}}$, $B \notin \tilde{l}$, $B^\pi \in \mathscr{L}_{\text{abs}}$, $B^\pi \cap l = B'$, $B' \in \mathscr{P}_{\text{abs}}$ in contradiction to (1).)

By proposition (1), $l \notin \mathscr{L}_{\text{abs}}$ and is an elliptic line. Then there exists a line $k \neq l$ for which $l^\pi \mathscr{I} k$ is not valid, thus $l \cap k = A$, $A \in \mathscr{P}_{\text{abs}}$ and $k^\pi \mathscr{I} k$ is not valid either which means that k is an elliptic line incident with only one absolute point A, a contradiction.

(7) Theorem. $\mathscr{P}_{\text{int}}$, $\mathscr{P}_{\text{ext}}$, $\mathscr{L}_{\text{int}}$, $\mathscr{L}_{\text{ext}}$ are non-empty,

(8) there is at least one interior and at least one exterior point on every interior line.

Proof. By (2) and (3) $\mathscr{P}_{\text{ext}} \neq \emptyset$ and $\mathscr{L}_{\text{int}} \neq \emptyset$. Let $l \in \mathscr{L}_{\text{int}}$, then by (6) there

*) The $h \cap k = P$ ($H \cup K = p$) will mean the point of intersection of lines $h \neq k$ (the line joining points $H \neq K$).

exists a point P on l and $P \notin \mathscr{P}_{\text{abs}}$. (i) Let $P \in \mathscr{P}_{\text{ext}}$. Then $P^\pi \in \mathscr{L}_{\text{int}}$ as well as l. Lines P^π and l are perpendicular, hence $P^\pi \cap l = P'$, $P' \in \mathscr{P}_{\text{int}}$.

(ii) Let $P \in \mathscr{P}_{\text{int}}$. Then $P^\pi \in \mathscr{L}_{\text{ext}}$ and $P' \in \mathscr{P}_{\text{ext}}$, since two perpendicular lines (one of which is exterior and the other is interior) meet in an exterior point as follows from the definition of an admissible polarity. The theorem is proved and at the same time the following corollary is obvious.

(9) Corollary. card $(\tilde{l} \cap \mathscr{P}_{\text{int}})$ = card $(\tilde{l} \cap \mathscr{P}_{\text{ext}})$ for any line l, $l \in \mathscr{L}_{\text{int}}$.

(10) Theorem. If the given projective plane is finite, then card $(\tilde{l} \cap \mathscr{P}_{\text{int}})$ is constant for any line l, $l \in \mathscr{L}_{\text{int}}$.

Proof. Let l, h be different interior lines. It is a consequence of the regularity of the polarity that card $(\tilde{l} \cap \mathscr{P}_{\text{abs}})$ = card $(\tilde{h} \cap \mathscr{P}_{\text{abs}})$. Hence card $(\tilde{l} \cap (\mathscr{P} \setminus \mathscr{P}_{\text{abs}}))$ = = card $(\tilde{h} \cap (\mathscr{P} \setminus \mathscr{P}_{\text{abs}}))$. Now the result follows immediately from (9).

(11) Theorem. If the given projective plane is finite, then card $(\tilde{l} \cap \mathscr{P}_{\text{abs}}) \leqq$ $\leqq$ card $(\tilde{l} \cap \mathscr{P}_{\text{int}}) + 1$ for any line l, $l \in \mathscr{L}_{\text{int}}$.

Proof. Let $l \in \mathscr{L}_{\text{int}}$, $A \mathscr{I} l$, $A \in \mathscr{P}_{\text{abs}}$; then there exists a line k, $k \in \mathscr{L}_{\text{int}}$, $k \neq l$, $A \mathscr{I} k$ and obviously $l^\pi \mathscr{I} k$ is not valid. Since for any point B, $B \in (\tilde{l} \cap \mathscr{P}_{\text{abs}})$, $B \neq A$, there exists exactly one point B', $B' = k \cap (B \cup l)^\pi$, $B' \neq A$, $B \cup l^\pi \in \mathscr{L}_{\text{abs}} \Rightarrow B' \in \mathscr{P}_{\text{ext}}$.

Remark: If there exists k, $k \in \mathscr{L}_{\text{int}}$, $k \neq l$, $l \cap k \notin \mathscr{P}_{\text{abs}}$, and $l^\pi \mathscr{I} k$ is not valid then in the contention of theorem (11) even the sharp inequality is valid.

Now we will consider the points which may be incident with an exterior line. We know from (5) that $\tilde{l} \cap \mathscr{P}_{\text{abs}} = \emptyset$ for every exterior line l.

(12) Theorem. If the given projective plane is finite, then card $(\tilde{l} \cap \mathscr{P}_{\text{int}})$ and card $(\tilde{l} \cap \mathscr{P}_{\text{ext}})$ are even for any line l, $l \in \mathscr{L}_{\text{ext}}$.

Proof. Suppose that $l \in \mathscr{L}_{\text{ext}}$. If there exists $P \in \mathscr{P}_{\text{int}}$, $P \mathscr{I} l$, then $P' := l \cap P^\pi$, with $P' \in \mathscr{P}_{\text{int}}$. If there exists $R \in \mathscr{P}_{\text{ext}}$, $R \mathscr{I} l$, then also $R' := l \cap R^\pi$, with $R' \in \mathscr{P}_{\text{ext}}$. This means that both interior and exterior points are always two by two which was to be proved.

For the following theorem leave out the previous presumption concerning the cardinality of $\mathscr{P}_{\text{abs}}$ and $\mathscr{L}_{\text{abs}}$.

(13) Theorem. If there exists an exterior line with interior points only then the sets $\mathscr{P}_{\text{abs}} = \mathscr{L}_{\text{abs}} = \emptyset$ and the given projective plane is infinite.

Proof. Let l is an exterior line with interior points only. Then $\mathscr{L}_{\text{abs}} = \emptyset$ because of $l \cap h \in \mathscr{P}_{\text{int}}$ for any $h \in \mathscr{L}$, $h \neq l$. Hence also $\mathscr{P}_{\text{abs}} = \emptyset$. The plane is obviously infinite (see (4)).

(14) Theorem. If there lie only exterior points on every exterior line then the given projective plane is infinite.

Proof. (cf. [1]) Suppose the assertion of the theorem is false. As every line has exactly $n + 1$ points, thus by (10) the card $(\tilde{l} \cap \mathscr{P}_{\text{int}}) = m$, $(l \in \mathscr{L}_{\text{int}})$ is also finite and

$2m \leqq n + 1$. The cardinality of $\mathscr{P}_{\mathrm{int}}$ can be observed in the following two ways:

(i) If $P \in \mathscr{P}_{\mathrm{abs}}$, then $(\breve{P} \setminus P^{\pi}) \subseteq \mathscr{L}_{\mathrm{int}}$, card $(\widetilde{P} \setminus P^{\pi}) = n$ and card $(\widetilde{l} \cap \mathscr{P}_{\mathrm{int}}) = m$ for every $l \in \mathscr{L}_{\mathrm{int}}$ and hence card $\mathscr{P}_{\mathrm{int}} = nm$.

(ii) If $R \in \mathscr{P}_{\mathrm{int}}$, then $\widetilde{R} \subset \mathscr{L}_{\mathrm{int}}$, card $\widetilde{R} = n + 1$ and thus card $\mathscr{P}_{\mathrm{int}} = 1 + (n + 1)(m - 1)$.

Thus we have found that

$$n \cdot m = 1 + (n + 1)(m - 1),$$

which yields

$$n = m,$$

but

$$2m \leqq n + 1,$$

thus

$$n \leqq 1,$$

a contradiction.

(15) Corollary. If the order of the given projective plane is finite, then it is necessarily odd.

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Shrnutí

VZTAH MEZI POLARITAMI A ŘÁDEM PROJEKTIVNÍ ROVINY

JAROSLAVA JACHANOVÁ A HELENA ŽAKOVÁ

V práci se studují vlastnosti polarit v incidenčních projektivních rovinách konečného řádu. Ukazuje se, že některé výsledky práce [1], týkající se nekonečných rovin, platí také v rovinách konečných, jestliže na rozdíl od R. Baera nevylučuje se esistence vnitřních bodů na vnějších přímkách.

Je-li daná projektivní rovina konečná, potom na každé vnější přímce je sudý počet vnitřních i vnějších bodů. Existuje-li vnější přímka obsahující pouze vnitřní body, potom neexistují absolutní body a přímky, tudíž daná projektivní rovina je nekonečná. Leží-li na každé vnější přímce pouze vnější body, pak daná projektivní rovina je rovněž nekonečná.

Резюме

ВЗАИМООТНОШЕНИЕ МЕЖДУ ПОЛЯРНОСТЯМИ И ПОРЯДКОМ ПРОЕКТИВНОЙ ПЛОСКОСТИ

ЯРОСЛАВА ЯХАНОВА И ЕЛЕНА ЖАКОВА

В этой работе исследуются качества полярностей в инцидентных проективных плоскостях конечного и тоже бесконечного порядка. Показывается, что некоторые результаты работы [1], касающиеся бесконечных плоскостей, имеют место тоже в конечных плоскостях, если в отличие от Р. Бэра допускается наличие внутренних точек находящихся на внешних прямых.

Если проективная плоскость конечная, то каждая внешняя прямая инцидентна чётному числу внешних и тоже чётному числу внутренних точек. Если существует внешняя прямая, все точки которой внутренние, тогда абсолютные точки и прямые не существуют и поэтому проективная плоскость бесконечна. Если каждая внещняя прямая содержит только внешние точки, тогда проективная плоскость бесконечна.