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ON THE THEORY OF PHASES OF ACADEMICAN O. BORŮVKA

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The present paper is concerned with investigating a space R of all solutions for the second-order, linear, ordinary differential equation

$$y'' = q(t)\,y \qquad (q)$$

in a real domain. This equation will be thought of as its carrier $q(t)$ to be a continuous function in an open (either bounded or unbounded) definition interval $j = (a, b)$ with $q(t) < 0$ for each $t \in j$. Throughout, the differential equation (q) will be understood to be oscillatory in j, i.e. any nontrivial solution of (q) vanishes in both directions to the endpoints a, b of j an infitely number of times.

The opening section introduces a scalar product in R. The results obtained here have been used in the second section to establish a 1-1 correspondence between the dispersions of the 1st kind and the special affine transformations of R. Each coset of the 1st kind dispersion factor group $\boldsymbol{D}_1/\boldsymbol{S}_1$ has been proved to be a certain special affine transformation of R, under simultaneous determination of a number of subgroups in it, generated by orthogonal transformations of R, i.e. by rotation and axial symmetry. There is next presented a concrete example of a dispersion group generated by rotation of R through an angle $\frac{\pi}{2}$.

In the final section is shown a group of dispersions generated by rotation of R through an angle $\frac{\pi}{2}$ and by symmetry with respect to the axis enclosing an angle $\frac{\pi}{4}$ with a basis vector of R.

The basic notations and relations of the transformation theory of differential equations have been adopted from [1].

1. Euclidean space R of all solutions of the differential equation (q).

Let $\alpha(t)$ be an arbitrary fixed chosen 1st phase of (q) and $\varphi(t)$ be the 1st kind basic central dispersion of (q). In the space R of all solutions of (q) we shall introduce the following composition

$$(f, g) = \int_t^{\varphi(t)} [\alpha'(\tau)]^2 \, . \, f(\tau) \, . \, g(\tau) \, d\tau, \tag{1}$$

where functions f, g are arbitrary elements of R, t an arbitrary element in $j = (a, b)$.

Theorem 1. The composition (1) *is a scalar product in R.*

Proof. Since $\alpha(t) \in C_3$ (i.e. with the 3rd order continuous derivation) and $\alpha'(t) \neq 0$ for each $t \in j$ (see [1, § 5, 5]), $[\alpha'(t)]^2$ is a continuous real function in $\langle t, \varphi(t) \rangle \subset j$ with values lying between two positive constants. An arbitrary function $f(t) \in R$ has derivatives to the second order and thus it can be integrated in the interval $\langle t, \varphi(t) \rangle$ and so the function $f^2(t)$. The latter, by [3, § 2, 7], implies the proposition.

Remark 1. For a function $\alpha(t)$ there exists a base $u(t)$, $v(t) \in R$ such that

$$u(t) = \frac{\sin \alpha(t)}{\sqrt{|\alpha'(t)|}}, \qquad v(t) = \frac{\cos \alpha(t)}{\sqrt{|\alpha'(t)|}} \tag{2}$$

and $\alpha(t)$ is the 1st phase of this base. (See [1, § 5, 7]).

Theorem 2. The base (2) *is orthogonal with respect to the scalar product* (1).

Proof.

$$(u(t), v(t)) = \int_t^{\varphi(t)} [\alpha'(t)]^2 \, \frac{\sin \alpha(\tau) \cos \alpha(\tau)}{|\alpha'(\tau)|} \, d\tau =$$

$$= \int_t^{\varphi(t)} \frac{1}{4} \operatorname{sgn} \alpha' \, . \, \sin 2\alpha(\tau) \, d\, 2\alpha(\tau) = \left[-\frac{1}{4} \operatorname{sgn} \alpha' \, . \, \cos 2\alpha(\tau) \right]_t^{\varphi(t)} =$$

$$= \frac{1}{4} \operatorname{sgn} \alpha' \, . \, (\cos 2\alpha(t) - \cos 2\alpha(\varphi(t))) = 0 \quad \text{for} \quad \alpha(\varphi(t)) = \alpha(t) + \pi \, . \operatorname{sgn} \alpha'(t).$$

Theorem 3. The base

$$u(t) = \frac{2}{\pi} \frac{\sin \alpha(t)}{\sqrt{|\alpha'(t)|}}, \qquad v(t) = \frac{2}{\pi} \frac{\cos \alpha(t)}{\sqrt{|\alpha'(t)|}} \tag{3}$$

is normed and orthogonal with respect to the scalar product (1).

Proof. An orthogonality of the base (3) follows from Theorem 2. It suffices, therefore, to prove the normalization. It is

$$(u(t), u(t)) = \frac{2}{\pi} \int_t^{\varphi(t)} \frac{\sin^2 \alpha(\tau)}{|\alpha'(\tau)|} [\alpha'(\tau)]^2 \, d\tau = \frac{2}{\pi} \operatorname{sgn} \alpha' \int_t^{\varphi(t)} \frac{1 - \cos 2\alpha(\tau)}{2} \alpha'(\tau) \, d\tau =$$

$$= \frac{\operatorname{sgn} \alpha'}{\pi} . (\alpha(\varphi(t)) - \alpha(t) - \frac{1}{2\pi} \operatorname{sgn} \alpha'(\sin 2\alpha(\varphi(t)) - \sin 2\alpha(t)) = 1$$

for $\alpha(u(t)) = \alpha(t) + \pi \operatorname{sgn} \alpha'(t)$.
Consequently, for the norm $\| u(t) \|$ of a vector $u(t)$

$$\| u(t) \| = \sqrt{(u(t), u(t))} = 1.$$

Likewise $\sqrt{(v(t), v(t))} = \| v(t) \| = 1$.

We can summarize the previous results in the following

Theorem 4. The vector space R of all solutions of (q) with the composition (1) *is an Euclidean space. The base (3) is a orthonormal base of this space.*

2. Orthogonal transformations of the space R.

Let D_1/S_1 be the factor group, where D_1 is the group of the 1st kind dispersions of (q) and S_1 the subgroup of all central dispersions with an even index. Further, let p be a linear mapping of R onto itself. This is a *generating mapping* of exactly one coset $\mathscr{X}_1 \in D_1/S_1$. (See [1, § 20, 4, 1].) Choose a base u, v in R. Then $p(u) = U$, $p(v) = V$, where U, V is also a base in R. Conversely, any dispersion X_1 from a coset generated by the mapping p, transforms the base U, V to a base u, v:

$$\frac{U[X_1]}{\sqrt{|X_1'|}} = u, \qquad \frac{V[X_1]}{\sqrt{|X_1'|}} = v.$$

(See [1, § 21, 2.2]). Taken from the point of view of vector space transformations, the dispersion X_1 is thus a linear transformation of R acting as the inverse mapping of the mapping p.

Linear transformations of a vector space with transformation matrices possessing determinants equal to $+1$ or -1 are termed special affine transformations. Linear transformations of a vector space with transformation matrices possessing determinants $+1$ are termed proper, special, affine transformations.

The following Lemma is obvious:

Lemma 1. All special, affine transformations with a mapping composition constitute a group isomorphic to the group of all unimodular matrices (of the 2nd order). All proper, special, affine transformations constitute a subgroup of this group.

Theorem 5. The factor group $\boldsymbol{D}_1/\boldsymbol{S}_1$ is isomorphic to the group of all special, affine transformations of R. The subgroup of all direct dispersion cosets is isomorphic to the group of all proper special affine transformations of R.

Proof. The group $\boldsymbol{D}_1/\boldsymbol{S}_1$ is isomorphic to the group of all unimodular matrices (see [1, § 21, 6]). From this and from Lemma 1 follows the proposition of our theorem.

It is immediate that each dispersion of the same coset $\mathcal{X}_1 \in \boldsymbol{D}_1/\boldsymbol{S}_1$ is, from the vector space transformation point of view that special affine mapping of R, corresponding to $\mathcal{X}_1$ in the isomorphism of Theorem 5. Consequently and for simplicity, we shall call this coset of dispersion directly the affine transformation of R.

Restricting our considerations to scalar product-preserving transformations only, gives a subgroup of the special affine transformation group formed by *the orthogonal transformations*. The relative matrices are called orthogonal. (They possess the property $\mathbf{A} . \mathbf{A}' = \mathbf{E}$, where $\mathbf{A}'$ stands for a matrix transposed to the matrix $\mathbf{A}$). If the determinant of an orthogonal matrix is equal to -1 and $+1$, the relative transformation is called the *proper orthogonal transformation* or *the rotation* of the space and *the axial symmetry* of the space, respectively.

The following theorem is an immediate consequence of previous considerations.

Theorem 6. In the group $\boldsymbol{D}_1/\boldsymbol{S}_1$ exists a subgroup $\mathfrak{G}_1$ formed by exactly all orthogonal transformations of R.

Note. Throughout this paper $\mathfrak{G}_1$ will denote the group of all cosets of dispersions that are orthogonal transformations of R; rot φ will denote a rotation through an angle φ and similarly sym φ will denote a symmetry with respect to the axis enclosing an angle $\varphi/2$ with the vector x of the base x, y of R.

Theorem 7. In the group $\mathfrak{G}_1$ exists a normal subgroup $\mathfrak{H}_1$ (with index 2) of all dispersion cosets acting as rotations of R. $\mathfrak{H}_1$ consists of exactly all direct dispersion cosets of $\mathfrak{G}_1$.

Proof. Let us consider a mapping rot φ. Then a primary base x, y is mapped onto the following base X, Y by

$$\begin{aligned} X &= x . \cos\varphi - y . \sin\varphi \\ Y &= x . \sin\varphi + y . \cos\varphi \end{aligned} \tag{4}$$

The inverse transformation take the base x, y into a base $\bar{X}, \bar{Y}$ by

$$\begin{aligned} \bar{X} &= x . \cos\varphi + y . \sin\varphi = x . \cos(-\varphi) - y . \sin(-\varphi) \\ \bar{Y} &= -x . \sin\varphi + y . \cos\varphi = x . \sin(-\varphi) + y . \cos(-\varphi) . \end{aligned} \tag{5}$$

The transformation (5) is rot$(-\varphi)$. The matrix of this transformation belongs exactly to one coset $\mathcal{X}_1$ of direct dispersions, because its determinant is equal to $+1$. Conversely, for each coset of direct dispersions of $\mathfrak{G}_1$ there always exists a corresponding matrix of the transformation (5), because only such orthogonal matrices have a determinant equal to $+1$.

Corollary 1. If $p = \text{rot}\,\varphi$, then it represents a generating mapping of exactly one coset $\mathscr{X}_1 \in \mathfrak{H}_1$ ($\mathfrak{H}_1$ is the group of Theorem 7) and $\mathscr{X}_1 = \text{rot}\,(-\varphi)$.

Theorem 8. Let $p = \text{rot}\,\varphi$ *and* $\alpha(t)[A(t)]$ *be a first phase of a primary [new] base of* R. *Then an arbitrary dispersion* $X_1 \in \mathscr{X}_1$ *generated by* p *takes the form* $X_1(t) = A^{-1}(\alpha(t))$, *with*

a) $A(t) = \alpha(t) + \varphi + 2k\pi$, $k = 0, 1, \ldots$ *whereby both* $\alpha(t)$, $A(t)$ *simultaneously are proper or improper, or*

b) $A(t) = \alpha(t) + \varphi + (2k + 1)\pi$, $k = 0, 1, \ldots$ *whereby one of the phases* $\alpha(t)$, $A(t)$ *is proper and the other is improper.*

Proof. Let

$$v(t) = \varepsilon\sqrt{|w|}\,\frac{\cos\alpha(t)}{\sqrt{|\alpha'(t)|}}, \qquad u(t) = \varepsilon\sqrt{|w|}\,\frac{\sin\alpha(t)}{\sqrt{|\alpha'(t)|}} \tag{6}$$

be a primary base and

$$V(t) = E\sqrt{|W|}\,\frac{\cos A(t)}{\sqrt{|A'(t)|}}, \qquad U(t) = E\sqrt{|W|}\,\frac{\sin A(t)}{\sqrt{|A'(t)|}} \tag{7}$$

be a new base of R. (See [1, § 5, 3].)
Substituting (6) and (7) into (4) gives

$$E\sqrt{|W|}\,\frac{\cos A(t)}{\sqrt{|A'(t)|}} = \varepsilon\sqrt{|w|}\,\frac{\cos\alpha(t)}{\sqrt{|\alpha'(t)|}}\cos\varphi - \varepsilon\sqrt{|w|}\,\frac{\sin\alpha(t)}{\sqrt{|\alpha'(t)|}}\sin\varphi$$

$$E\sqrt{|W|}\,\frac{\sin A(t)}{\sqrt{|A'(t)|}} = \varepsilon\sqrt{|w|}\,\frac{\cos\alpha(t)}{\sqrt{|\alpha'(t)|}}\sin\varphi + \varepsilon\sqrt{|w|}\,.\,\frac{\sin\alpha(t)}{\sqrt{|\alpha'(t)|}}\cos\varphi.$$

The base u, v and U, V can be expressed in the form

$$\begin{aligned} u(t) &= \varepsilon\,.\,r(t)\,.\,\sin\alpha(t) \\ v(t) &= \varepsilon\,.\,r(t)\,.\,\cos\alpha(t), \end{aligned} \tag{8}$$

$$\begin{aligned} U(t) &= E\,.\,R(t)\,.\,\sin A(t) \\ V(t) &= E\,.\,R(t)\,.\,\cos A(t). \end{aligned} \tag{9}$$

Since U, V are obtained from u, v by means of rot φ, $r(t) = R(t)$ and following (6) and (7) then

$$\frac{\sqrt{|w|}}{\sqrt{|\alpha'(t)|}} = \frac{\sqrt{|W|}}{\sqrt{|A'(t)|}} \qquad \text{for each } t \in j.$$

Thus

$$\begin{aligned} E\cos A(t) &= \varepsilon\,.\,\cos(\alpha(t) + \varphi) \\ E\sin A(t) &= \varepsilon\,.\,\sin(\alpha(t) + \varphi). \end{aligned}$$

a) $E = \varepsilon$, that is to say both phases are simultaneously proper or improper, then $A(t) = \alpha(t) + \varphi + 2k\pi$, $k = 0, 1, \ldots$;
b) $E = -\varepsilon$, that is to say one of the phases is proper and the other improper, then $A(t) = \alpha(t) + \varphi + (2k + 1)\pi$, $k = 0, 1, \ldots$. Moreover, by [1, § 20, 3], a dispersion $X_1 \in \mathscr{X}_1$ takes the form $X_1(t) = A^{-1}(\alpha(t))$.

Consider now an angle $\frac{2\pi}{n}$, where n is a natural number. rot $\frac{2\pi}{n}$ is a generator of the cyclic group $\mathfrak{D}_1^n$ the elements of which are rot $\frac{2k\pi}{n}$, $k = 0, 1, \ldots, n - 1$. It is valid next that rot $\frac{2\pi}{n}$ is a generating mapping for a coset $\mathscr{X}_1 \in \boldsymbol{D}_1/\boldsymbol{S}_1$, where $\mathscr{X}_1 = \operatorname{rot}\left(-\frac{2\pi}{n}\right)\left(= \operatorname{rot}\frac{2(n-1)\pi}{n}\right)$. The above coset $\mathscr{X}_1$ acts as generator of the same cyclic group $\mathfrak{D}_1^n$. The inverse element to the generator of the cyclic group, namely, is the generator of the same cyclic group.

We recapitulate in the following

Theorem 9. Let $\mathfrak{H}_1$ be the group of Theorem 7. In $\mathfrak{H}_1$, to each natural n exists a cyclic subgroup $\mathfrak{D}_1^n$ of order n with elements rot $\frac{2k\pi}{n}$, $k = 0, 1, \ldots, n - 1$. *The coset $\mathscr{X}_1 \in \mathfrak{H}_1$, where $\mathscr{X}_1 = \operatorname{rot}\frac{2\pi}{n}$, generates the group $\mathfrak{D}_1^n$.*

Let p be a prime number and n a natural one. The group of all p^nth roots of unity for a fixed n is a cyclic group of order p^n with an element $a_n = \cos\frac{2\pi}{p^n} + i \sin\frac{2\pi}{p^n}$ (See [4, III, 3.10].) This group is isomorphic to the group $\mathfrak{P}_1^n$ of all rot $\frac{2k\pi}{p^n}$, $k = 0, 1, \ldots, p^n - 1$ and rot $\frac{2\pi}{p^n}$ generates $\mathfrak{P}_1^n$.

The group of the type p^∞ as a union of an increasing sequence of subgroups $\langle a_n \rangle$, $n = 1, 2, \ldots$ is an Abelian infinite group and the set of elements a_n, $n = 1, 2, \ldots$ is a system of generators of p^∞.

With the properties of isomorphism of groups we can now express the following.

Theorem 10. Let $\mathfrak{H}_1$ be the group of Theorem 7, p an arbitrary but a fixed natural number. Further, let $\mathfrak{P}_1^n \subset \mathfrak{H}_1$ be the group of all rot $\frac{2k\pi}{n}$, $k = 0, 1, \ldots, p^n - 1$. *Then the union of sequence of the groups $\mathfrak{P}_1^n$, $n = 1, 2, \ldots$ is an Abelian infinite group $\mathfrak{P}_1^\infty \subset \mathfrak{H}_1$ and the dispersion cosets rot $\frac{2\pi}{p^n}$, $n = 1, 2, \ldots$ represent a system of generators of $\mathfrak{P}_1^\infty$.*

Now we are going to show a concrete example of a cyclic group with elements being the rotations of R.

Example 1. Let $p = \operatorname{rot}\frac{\pi}{2}$. Then p is a generator of the cyclic group $\mathfrak{D}_1^4$ with elements rot 0, $\operatorname{rot}\frac{\pi}{2}$, rot π, $\operatorname{rot}\frac{3}{2}\pi$. Further p is a generating mapping of the coset $\mathscr{X}_1 \in \boldsymbol{D}_1/\boldsymbol{S}_1$, where $\mathscr{X}_1 = \operatorname{rot}\left(-\frac{\pi}{2}\right)$. The mapping $\operatorname{rot}\left(-\frac{\pi}{2}\right)$ generates the same cyclic group $\mathfrak{D}_1^4$ as also $\operatorname{rot}\frac{\pi}{2}$ does. Let us look now at the elements of $\mathfrak{D}_1^4$ in more detail.

rot 0: For the primary base x, y and the new base X, Y of the space R holds $X = x$, $Y = y$. The transformation matrix

$$\mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

rot 0 generates the coset $\mathscr{X}_1^0$ and $\mathscr{X}_1^0 = \operatorname{rot} 0$. For the first phases $A(t), \alpha(t)$ of the bases X, Y, x, y holds

$$A(t) = \alpha(t) + 2k\pi, \qquad \text{where } \varepsilon = E,$$
$$A(t) = \alpha(t) + (2k+1)\pi, \quad \text{where } \varepsilon = -E.$$

$\operatorname{rot}\frac{\pi}{2}$: x, y and X, Y satisfy $X = y$, $Y = -x$ and the transformation matrix

$$\mathbf{I} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

$\operatorname{rot}\frac{\pi}{2}$ generates the coset $\mathscr{X}_1^{\frac{1}{2}\pi}$ and $\mathscr{X}_1^{\frac{1}{2}\pi} = \operatorname{rot}\left(-\frac{\pi}{2}\right)$. For $A(t)$, $\alpha(t)$ holds

$$A(t) = \alpha(t) + \frac{\pi}{2} + 2k\pi, \qquad \text{where} \quad \varepsilon = E,$$

$$A(t) = \alpha(t) + \frac{\pi}{2} + (2k+1)\pi, \quad \text{where} \quad \varepsilon = -E.$$

rot π: x, y and X, Y satisfy $X = -x$, $Y = -y$, the transformation matrix

$$-\mathbf{E} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

rot π generates the coset $\mathscr{X}_1^\pi$ and $\mathscr{X}_1^\pi = \operatorname{rot}(-\pi)$ (= rot π).
Further

$$A(t) = \alpha(t) + \pi + 2k\pi, \qquad \text{where} \quad \varepsilon = E,$$
$$A(t) = \alpha(t) + \pi + (2k+1)\pi, \quad \text{where} \quad \varepsilon = -E.$$

$\text{rot}\,\frac{3}{2}\pi$: x, y and X, Y satisfy $X = -y$, $Y = -x$ and the transformation matrix

$$-\mathbf{I} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$\text{rot}\,\frac{3}{2}\pi$ generates the coset of dispersions $\mathcal{X}_1^{\frac{3}{2}\pi}$ and $\mathcal{X}_1^{\frac{3}{2}\pi} = \text{rot}\left(-\frac{3}{2}\pi\right)\left(= \text{rot}\,\frac{\pi}{2}\right)$.
For the first phases holds

$$A(t) = \alpha(t) + \frac{3}{2}\pi + 2k\pi, \qquad \text{where} \qquad \varepsilon = E,$$

$$A(t) = \alpha(t) + \frac{3}{2}\pi + (2k + 1)\pi, \qquad \text{where} \qquad \varepsilon = -E.$$

Axial symmetry of R.

We shall now return to the group $\mathfrak{G}_1$ of all dispersion cosets that are orthogonal transformations, i.e. either rotations or axial symmetry. The mapping $\text{sym}\,\frac{\varphi}{2}$ transforms the base x, y onto X, Y in the following way:

$$\begin{aligned} X &= x \,.\, \cos\varphi + y \,.\, \sin\varphi \\ Y &= x \,.\, \sin\varphi - y \,.\, \cos\varphi. \end{aligned} \tag{10}$$

The inverse transformation transfers the base x, y onto a base $\bar{X}, \bar{Y}$ by

$$\begin{aligned} \bar{X} &= x \,.\, \cos\varphi + y \,.\, \sin\varphi \\ \bar{Y} &= x \,.\, \sin\varphi - y \,.\, \cos\varphi. \end{aligned} \tag{11}$$

We now observe that the transformation of axial symmetry is an inverse mapping to itself. The transformation matrix (11) belongs exactly to one coset of indirect dispersions (the determinant of the matrix is equal to -1).

Corollary 2. If $p = \text{sym}\,\frac{\varphi}{2}$, then p is a generating mapping of exactly one coset of the 1st kind indirect dispersions. This coset is also $\text{sym}\,\frac{\varphi}{2}$.

Theorem 11. Let $p = \text{sym}\,\frac{\varphi}{2}$ *and let* $\alpha(t)[A(t)]$ *be a first phase of a primary* [*new*] *base of R. Then an arbitrary dispersion* $X_1 \in \mathcal{X}_1$, *where p is a generating mapping of* $\mathcal{X}_1$, *can be expressed in the form* $X_1(t) = A^{-1}(\alpha(t))$, *with*

a) $A(t) = \varphi - \alpha(t) + 2k\pi$, $k = 0, 1, \ldots$, *whereby both* $\alpha(t)$, $A(t)$ *are simultaneously proper or improper; or*

b) $A(t) = \varphi - \alpha(t) + (2k + 1)\pi$, $k = 0, 1, \dots$, *whereby one of these phases* $\alpha(t)$, $A(t)$ *is proper and the other is improper.*

Proof. Let (6) be the primary base of R and (7) the new one. Put (6) and (7) into (10). We have

$$E \, . \sqrt{|-W|} \, \frac{\cos A(t)}{\sqrt{|A'(t)|}} = \varepsilon \sqrt{|w|} \, \frac{\cos \alpha(t)}{\sqrt{|\alpha'(t)|}} \cos \varphi + \varepsilon \sqrt{|w|} \, \frac{\sin \alpha(t)}{\sqrt{|\alpha'(t)|}} \sin \varphi,$$

$$E \, . \sqrt{|-W|} \, \frac{\sin A(t)}{\sqrt{|A'(t)|}} = \varepsilon \sqrt{|w|} \, \frac{\cos \alpha(t)}{\sqrt{|\alpha'(t)|}} \sin \varphi - \varepsilon \sqrt{|w|} \, \frac{\sin \alpha(t)}{\sqrt{|\alpha'(t)|}} \cos \varphi.$$

Since the above mentioned transformation p is an axial symmetry, it again holds $r(t) = R(t)$ (see the expression of bases of the forms (8) and (9)), therefore

$$\frac{\sqrt{|w|}}{\sqrt{|\alpha'(t)|}} = \frac{\sqrt{|W|}}{\sqrt{|A'(t)|}} \qquad \text{for each } t \in j.$$

Thus

$$E \, . \cos A(t) = \varepsilon \, . \cos (\varphi - \alpha(t)),$$
$$E \, . \sin A(t) = \varepsilon \, . \sin (\varphi - \alpha(t)).$$

a) $E = \varepsilon$, i.e. both phases are simultaneously proper or improper:

$$A(t) = \varphi - \alpha(t) + 2k\pi, \qquad k = 0, 1, \dots;$$

b) $E = -\varepsilon$, i.e. one phase is proper and the other is improper:

$$A(t) = \varphi - \alpha(t) + (2k + 1)\pi, \; k = 0, 1, \dots;$$

moreover, by [1, § 20, 3], a dispersion $X_1 \in \mathscr{X}_1$ satisfies $X_1(t) = A^{-1}(\alpha(t))$.

Theorem 12. Let n be a natural number. Adjoining the transformation $\operatorname{sym} \frac{\pi}{n}$ *to the cyclic group* $\mathfrak{O}_1^n$ *of orthogonal transformations, then the orthogonal transformation group* $\mathfrak{T}_1^n$ *consisting of elements* $\operatorname{rot} \frac{2k\pi}{n}$, $k = 0, 1, \dots, n - 1$, $\operatorname{sym} \frac{k\pi}{n}$, $k = 0, 1, \dots, n - 1$ *is obtained.* $\mathfrak{O}_1^n$ *is a normal subgroup of* $\mathfrak{T}_1^n$,

Proof. a)

$$\begin{pmatrix} \cos \varphi & \sin \varphi \\ \sin \varphi & -\cos \varphi \end{pmatrix} . \begin{pmatrix} \cos \psi & \sin \psi \\ \sin \psi & -\cos \psi \end{pmatrix} =$$

$$= \begin{pmatrix} \cos \varphi \cos \psi + \sin \varphi \sin \psi & \cos \varphi \sin \psi - \sin \varphi \cos \psi \\ \sin \varphi \cos \psi + \cos \varphi \sin \psi & \sin \varphi \sin \psi + \cos \varphi \cos \psi \end{pmatrix} =$$

$$= \begin{pmatrix} \cos (\varphi - \psi) & -\sin (\varphi - \psi) \\ \sin (\varphi - \psi) & \cos (\varphi - \psi) \end{pmatrix};$$

then if we compose sym $\frac{l\pi}{n} \in \mathfrak{T}_1^n$ and sym $\frac{m\pi}{n} \in \mathfrak{T}_1^n$ (m, l integer numbers), we can obtain rot $\frac{2(l-m)}{n}\pi \in \mathfrak{D}_1^n$.

b)

$$\begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix} \cdot \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\varphi\cos\psi + \sin\varphi\sin\psi & -\cos\varphi\sin\psi + \sin\varphi\cos\psi \\ \sin\varphi\cos\psi - \cos\varphi\sin\psi & -\sin\varphi\sin\psi - \cos\varphi\cos\psi \end{pmatrix} =$$

$$= \begin{pmatrix} \cos(\varphi-\psi) & \sin(\varphi-\psi) \\ \sin(\varphi-\psi) & -\cos(\varphi-\psi) \end{pmatrix};$$

composing sym $\frac{l\pi}{n}$ and rot $\frac{2m\pi}{n}$ we have sym $\frac{l-2m}{n}\pi$ (l, m integer numbers).

c)

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \cdot \begin{pmatrix} \cos\psi & \sin\psi \\ \sin\psi & -\cos\psi \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\varphi\cos\psi - \sin\varphi\sin\psi & \cos\varphi\sin\psi + \sin\varphi\cos\psi \\ \sin\varphi\cos\psi + \cos\varphi\sin\psi & \sin\varphi\sin\psi - \cos\varphi\cos\psi \end{pmatrix} =$$

$$= \begin{pmatrix} \cos(\varphi+\psi) & \sin(\varphi+\psi) \\ \sin(\varphi+\psi) & -\cos(\varphi+\psi) \end{pmatrix};$$

composing rot $\frac{2l\pi}{n}$ and sym $\frac{m\pi}{n}$ we obtain sym $\frac{2l-m}{n}\pi$ (l, m integer).

d) An inverse element to sym φ is again sym φ (for arbitrary φ).

e) $\mathfrak{D}_1^n$ is a normal subgroup of $\mathfrak{T}_1^n$ because $\mathfrak{D}_1^n$ has an index 2.

Evidently each axial symmetry together with an identical transformation forms a two-element subgroup with respect to the composition of transformations. The following theorem describes a next type of subgroups of $\mathfrak{G}_1$ containing axial symmetries.

Theorem 13. In the group $\mathfrak{G}_1$ there exists a subgroup $\mathfrak{T}_1^n$ of the 2n-th order for each natural n. The elements of $\mathfrak{T}_1^n$ are rot $\frac{2k\pi}{n}$, $k = 0, 1, \ldots, n-1$ *and* sym $\frac{k\pi}{n}$, $k = 0, 1, \ldots, n-1$. *The cyclic subgroup $\mathfrak{D}_1^n$ is a normal subgroup of $\mathfrak{T}_1^n$. The elements* rot $\frac{2\pi}{n}$ *and* sym $\frac{\pi}{n}$ *generate $\mathfrak{T}_1^n$.*

Let us show finally a concrete example of the groups described above.

Example 2. Let $p =$ sym $\frac{\pi}{4}$. Then the coset in $\boldsymbol{D}_1/\boldsymbol{S}_1$ with the generating mapping p is this mapping. If this transformation is adjoined to the group $\mathfrak{D}_1^4$ we obtain the group $\mathfrak{T}_1^4$ with elements: sym 0, sym $\frac{\pi}{4}$, sym $\frac{\pi}{2}$, sym $\frac{3}{4}\pi$ and rot 0, rot $\frac{\pi}{2}$, rot π, rot $\frac{3}{2}\pi$. The elements of $\mathfrak{D}_1^4$ have been described in the preceding example. Let us now describe the remainder of the group $\mathfrak{T}_1^4$.

sym 0: For the primary base x, y and the new base X, Y of the space R holds: $X = x$, $Y = -y$. The transformation matrix

$$\mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

sym 0 generates the coset $\mathcal{Y}_1^0$ and herein $\mathcal{Y}_1^0$ = sym 0. For the first phases $A(t)$, $\alpha(t)$ of the bases X, Y and x, y holds

$$A(t) = -\alpha(t) + 2k\pi, \qquad \text{where} \quad \varepsilon = E,$$
$$A(t) = -\alpha(t) + (2k + 1)\pi, \qquad \text{where} \quad \varepsilon = -E.$$

sym $\frac{\pi}{4}$: x, y and X, Y satisfy $X = y$, $Y = x$ and the transformation matrix is

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

sym $\frac{\pi}{4}$ generates the coset $\mathcal{Y}_1^{\frac{1}{4}\pi}$ and again $\mathcal{Y}_1^{\frac{1}{4}\pi}$ = sym $\frac{\pi}{4}$. For $A(t)$, $\alpha(t)$ we have

$$A(t) = \frac{\pi}{4} - \alpha(t) + 2k\pi, \qquad \text{where} \quad \varepsilon = E,$$
$$A(t) = \frac{\pi}{4} - \alpha(t) + (2k + 1)\pi, \qquad \text{where} \quad \varepsilon = -E.$$

sym $\frac{\pi}{2}$: x, y and X, Y satisfy $X = -x$, $Y = y$ and the transformation matrix

$$\mathbf{K} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

sym $\frac{\pi}{2}$ generates the coset $\mathcal{Y}_1^{\frac{1}{2}\pi}$ and $\mathcal{Y}_1^{\frac{1}{2}\pi}$ = sym $\frac{\pi}{2}$. For $A(t)$, $\alpha(t)$ holds

$$A(t) = \frac{\pi}{2} - \alpha(t) + 2k\pi, \qquad \text{where} \quad \varepsilon = E,$$
$$A(t) = \frac{\pi}{2} - \alpha(t) + (2k + 1)\pi, \qquad \text{where} \quad \varepsilon = -E.$$

sym $\frac{3}{4}\pi$: x, y and X, Y satisfy $X = -y$, $Y = -x$ and the transformation matrix

$$-\mathbf{J} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

sym $\frac{3}{4}\pi$ generates the coset $\mathcal{Y}_1^{\frac{3}{4}\pi}$ and $\mathcal{Y}_1^{\frac{3}{4}\pi} = \text{sym}\,\frac{3}{4}\pi$. For $A(t)$, $\alpha(t)$ holds

$$A(t) = \frac{3}{4}\pi - \alpha(t) + 2k\pi, \qquad \text{where} \qquad \varepsilon = E,$$

$$A(t) = \frac{3}{4}\pi - \alpha(t) + (2k + 1)\pi, \qquad \text{where} \qquad \varepsilon = -E.$$

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Souhrn

POZNÁMKA K TEORII FÁZÍ O. BORŮVKY

IRENA RACHŮNKOVÁ

V práci je zkoumán prostor R všech řešení obyčejné lineární diferenciální rovnice 2. řádu v reálném oboru $y'' = q(t)\,y$, kde $q(t)$ je spojitá funkce na otevřeném intervalu $j = (a, b)$ (interval může být ohraničený i neohraničený) a $q(t) < 0$ pro každé $t \in j$. O rovnici $y'' = q(t)\,y$ se dále předpokládá, že je oscilující na j.

V úvodní části článku je v prostoru R zaveden skalární součin. Na základě tohoto je pak v další části určena vzájemně jednoznačná korespondence mezi disperzemi 1. druhu a speciálními afinními transformacemi prostoru R. Je dokázáno, že každá třída faktorové grupy $\boldsymbol{D}_1/\boldsymbol{S}_1$ disperzí 1. druhu je jistou speciální afinní transformací prostoru R. Zároveň je v grupě $\boldsymbol{D}_1/\boldsymbol{S}_1$ určena řada podgrup generovaných ortogonálními transformacemi prostoru R (tj. otočením a osovou souměrností). Je předveden konkrétní příklad grupy disperzí generované otočením prostoru R o úhel $\frac{\pi}{2}$ a závěrem je ukázána grupa disperzí generovaná otočením prostoru R o úhel $\frac{\pi}{2}$ a souměrností podle osy svírající s vektorom báze prostoru R úhel $\frac{\pi}{4}$.

Резюме

ЗАМЕТКА К ТЕОРИИ ФАЗ АКАД. О. БОРУВКИ

ИРЕНА РАХУНКОВА

В статье рассматривается простратство R всех решений обыкновенного линейного дифференциального уравнения второго порядка $y'' = q(t)\,y$, где $q(t)$ непрерывная функция на открытом интервале $j = (a, b)$ (ограниченном или неограниченном) и $q(t) < 0$ при всех $t \in j$. Уравнение $y'' = q(t)\,y$ предполагается колебающим на интервале j.

В первой главе введено скаларное произведение в пространстве R.

В дальнейшей главе показано взаимно однозначное соответствие между дисперсиями первого рода и специальными аффинными преобразованиями пространства R. Доказывается, что каждый смежный класс факторгруппы D_1/S_1 дисперсий первого рода является некоторым специальным аффинным преобразованием пространства R. В группе D_1/S_1 тоже определено несколько последовательностей подгрупп порожденных ортогональными преобразованиями простраства R. Кроме того показаны два конкретные примеры таких подгрупп.