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QUADRATIC FUNCTIONALS WITH EULER'S EQUATION $(py')' + qy = 0$

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1. Introduction.

This paper is concerned with studying a quadratic functional of the form

$$f(u; a, b) = \int_a^b [p(t)u'^2 - q(t)u^2]\,dt, \tag{f}$$

where the integral is defined in the sense of Riemann, making use of central dispersions defined analogous to O. Borůvka. With the results obtained we then further investigate the disconjugacy and the definiteness of selfadjoint linear differential equations of the second order, whose coefficients are the complex functions of a real argument.

By Euler's differential equation of the functional (f) is meant a selfadjoint linear differential equation of the second order:

$$(p(t)\,y')' + q(t)\,y = 0. \tag{pq}$$

The symbol $C_k(M)$ is used to indicate a set of all real functions which are continuous even with the derivatives up to and including the order k on the set M. (k is a non-negative integer).

Throughout this article we assume that

1° the functions $p(t), q(t) \in C_0(j)$,
2° for every solution y of equation (pq) there holds $y \in C_1(i)\ py' \in C_1(j)$,
3° the functions $p(t) \neq 0$ for all $t \in j$, where j is an interval.

Assuming the properties 1°, 2°, and 3° to be satisfied, we can simplify our writing as: $p, q \in v(j)$ or $p, q \in v(j \supset \langle a, b\rangle)$ when the interval j contains a bounded, closed interval $\langle a, b\rangle$.

The nontrivial solution of equation (pq) is called the extremal of (f). The symbol (pq) will be also used to denote the set of all extremals of (f).

The function $u(t)$ possessing the property $u \in C_1(\langle a, b \rangle)$, $u \neq 0$ on the interval $\langle a, b \rangle$ is called the admissible function of (f).

Evidently every $v \in (pq)$ is an admissible function of the functional (f).

The functional (f) is said to be positive definite or negative definite on the set of admissible functions, which have a certain property, according as $f(u; a, b) > 0$ or $f(u; a, b) < 0$ holds for all these admissible functions u. Both the positive definite and negative definite functionals are called definite functionals.

Similarly we can define the non-negativeness and non-positiveness of (f) by the inequalities $f(u; a, b) \geqq 0$ and $f(u; a, b) \leqq 0$, respectively.

2. The interpretation of some values of (f) by means of an extremal.

Let $y \in (pq)$ be arbitrary but fixed chosen extremal of (f). The symbol M_y will stand for a set of all admissible functions of (f) which have on $\langle a, b \rangle$ at least those zeroes as the extremal y has. We introduce now the interpretation of (f) which will be of need in the sequel.

Lemma 1. Let $p, q \in v(\langle a, b \rangle)$, $y \in (pq)$, $u \in M$-; then

$$f(u; a, b) = [pu^2 y'/y]_a^b + \int_a^b p[y(u/y)']^2 \, dt. \tag{1}$$

Proof. Under the assumptions of the theorem, let for $t_0 \in \langle a, b \rangle$ be $y(t_0) \neq 0$. Then for the integrated function of (f) in t_0

$$pu'^2 - qu^2 = [pu^2 y'/y]' + p[y(u/y)']^2. \tag{2}$$

If $y(t_1) = 0$ is for $t_1 \in \langle a, b \rangle$, then also $u(t_1) = 0$ for $u \in M$-. From the property of the solution of (pq) and from the definition of the extremal it follows that $y'(t_1) \neq 0$. It is now easy to see that there exists a finite limit of the function u/y with $t \to t_1$, or a one-sided limit when t_1 represents the end point of $\langle a, b \rangle$. Let us define the value of u/y at the point t_1 by this limit. Then the function $u/y \in C_1(\langle a, b \rangle)$. The relation (2) is thus an identity on $\langle a, b \rangle$ and our interpretation (1) becomes true.

3. The non-negativeness of the functional (f).

Theorem 1. Let $p, q \in v(\langle a, b \rangle)$, $p > 0$. If it holds

1. $y(a) = y(b) = 0$ or 2. $y(a) = y'(b) = 0$ or
3. $y'(a) = y(b) = 0$ or 4. $y'(a) = y'(b) = 0$

for $y \in (pq)$, then the functional (f) on the admissible functions $u \in M_y$ which are satisfying the boundary conditions

$$1.\ u(a) = u(b) = 0, \qquad 2.\ u(a) = u'(b) = 0,$$

$$3.\ u'(a) = u(b) = 0, \qquad 4.\ u'(a) = u'(b) = 0,$$

respectively, is non-negative and the vanishing extreme-minimum is realized just on the admissible functions $u = ky$, where $k \neq 0$ is a constant.

Proof. We express the functional (f) in the form of (1). In all four cases is $[pu^2y'/y]_a^b = 0$; hence it holds

$$f(u; a, b) = \int_a^b p[y(u/y)']^2 \, dt, \tag{3}$$

whence it follows that $f(u; a, b) \geqq 0$ and the extreme $f(u; a, b) = 0$ is realized on $u \in M_y$ if and only if $u = ky$, where $k \neq 0$ is a constant.

Remark. If $p < 0$ were true in the assumption of the theorem, we should obtain the statement on the non-positiveness of the functional (f).

4. Conjugate numbers.

By analogy with [1], I., § 3 we now define the conjugate numbers with respect to (pq).

Let $y_1, y_2 \in (pq)$ and let $y_1(t) = 0$, $y_2'(t) = 0$ for $t \in j$. The number $x \in j$, $x \neq t$ is called conjugate to the number t of the first, second, third, fourth kinds, if $y_1(x) = 0$, $y_2'(x) = 0$, $y_1(x) = 0$, $y_2(x) = 0$ hold, respectively.

If $t < x (t > x)$, we speak of conjugate numbers on the right (on the left). We let $\varphi_n(t)$, $\psi_n(t)$, $\chi_n(t)$, $\omega_n(t)$ denote the n-th ($n \geqq 1$) conjugate number on the right to the number t of the first, second, third, fourth kinds, respectively. We write $\varphi_0(t) = t$, $\psi_0(t) = t$. The first conjugate numbers on the right are customary written without index.

If two solutions of (pq) or their derivatives have a common zero, then they are linearly independent and have therefore all zeroes in common including the zeroes of their derivatives. Consequently the conjugate numbers are independent of the choice of $y_1, y_2 \in (pq)$.

Between two neighbouring zeroes of any solution of (pq) lies at least one zero of the derivative of the respective solution. In such instances when there lies exactly one zero, it is possible to define the functions—the central dispersions of the first, second, third, and fourth kinds with the aid of conjugate numbers, completely analogous to [1], II. § 12. We denote them similarly as the conjugate numbers.

Provided that the central dispersions are defined, the following inequalities hold for any $t \in j$:

(4)
$$t < \chi_2(t) < \varphi_1(t) < \chi_2)t) < \varphi_2(t) < \dots$$
$$t < \omega_1(t) < \psi_1(t) < \omega_2(t) < \psi_2(t) < \dots$$

A sufficient condition that exactly one zero of the derivative y be lying between two neighbouring zeroes of the solution $y(pq)$ is given by

Lemma 2. Let $p, q \in v(j)$ and $q \neq 0$ for all $t \in j$. If for $t_1 \in j$ we have $t_2 = \varphi(t_1) \in j$, then there exists exactly one number $t_3 \in (t_1, t_2)$ for which $y'(t_3) = 0$ holds with $y \in (pq)$ having the property $y(t_1) = 0$.

Proof. For $y \in (pq)$ for which $y(t_1) = 0$ we have $y \neq 0$ for all $t \in (t_1, t_2)$, $y(t_2) = 0$. The existence of the number t_3 is thus obvious and we argue the uniqueness by contradiction. Let $y'(t_3) = y'(t_4) = 0$ hold for $t_3 \neq t_4$, t_3, $t_4 \in (t_1, t_2)$. By integrating (pq) for the y in question, we obtain

$$\int_{t_3}^{t_4} qy \, dt = 0$$

which is a contradiction, since $qy \neq 0$ for all $t \in (t_1 t_2)$.

Remark. It becomes readily apparent from the form of the functional (f) that if $(\operatorname{sgn} p)(\operatorname{sgn} q) = -1$ on the interval $\langle a, b\rangle$, then (f) is positive definite or negative definite on the set of all admissible functions according as $\operatorname{sgn} p = 1$ or $\operatorname{sgn} p = -1$ on the interval $\langle a, b\rangle$.

From this remark and from the statement of Lemma 3 now follows that supposing $\operatorname{sgn} p = \operatorname{sgn} q \neq 0$ on the interval $\langle a, b\rangle$, we can apply conjugate numbers or central dispersions of the second, third, and fourth kinds to the investigation of the definiteness of (f).

5. Sufficient conditions for the positive definiteness of the functional (f).

Theorem 2. Let $p, q \in v(j \supset \langle a, b\rangle)$, $p > 0$ and let $b \in (\varphi_{n-1}(a), \varphi_n(a))$, $n \geqq 1$. Then the functional (f) is positive definite on the admissible functions u that satisfy the conditions u $[\varphi_i(a) = u(b) = 0$, $i = 0, 1, \dots, n-1$.

Proof. Let us take $y \in (pq)$ satisfying the condition $y(a) = 0$. Then $u \in M_y$ holds for arbitrary admissible function u, for which $u[\varphi_i(a) = 0$, $i = 0, 1, \dots, n-1]$. Hence the functional (f) may be expressed by the relation (1), from which we obtain the interpretation (3) with respect to the boundary conditions $u(a) = u(b) = 0$. The possibility $f(u; a, b)$ gives us $u = ky$, where $k \neq 0$ is a constant. Yet the condition $u(b) = 0$ yields $b = \varphi_n(a)$ which is a contradiction and from (3) we arrive to $f(u; a, b)$ 0.

Theorem 3. Let $p, q \in v(j \supset \langle a, b\rangle)$, sgnp $=$ sgn $q = 1$. If $b \in (\varphi_{n-1}(a), \chi_n(a))$, $n \geqq 1$ or $b \in \langle\varphi_{n-1}(a), \chi_n(a))$, $n > 1$, then the functional (f) is positive definite on the admissible functions u that satisfy the conditions $u[\varphi_i(a)] = u'(b) = 0$, $i = 0, 1, \dots, n - 1$.

Proof. Let us take $y \in (pq)$ so that $y(a) = 0$. We examine first the case of $b \in \in (\varphi_{n-1}(a), \chi_n(a))$. This evidently implies that $y \neq 0$ and $\operatorname{sgn} y = \operatorname{sgn} y'$ for all $t \in (\varphi_{n-1}(a), b\rangle$. Lemma 1 with respect to the assumption $\operatorname{sgn} p = 1$ if $u(b) \neq 0$ gives

$$f(u; a, b) \geqq [pu^2 y'/y]_a^b = p(b)\, u^2(b)\, y'(b)/y(b) > 0.$$

If $u(b) = 0$, then the functional (f) takes the form of (3) from which we observe that $f(u; a, b) \geqq 0$. An immediate consequence of the possibility $f(u; a, b) = 0$ is $u = ky$ for all $t \in \langle a, b\rangle$, where $k \neq 0$ is a constant. However, this would imply that $y'(b) = 0$, i.e. $b = \chi_n(a)$ contrary to assumption and therefore $f(u; a, b) > 0$.

Similarly will be shown in case of $n > 1$ when $b\, \varphi_{n-1}(a)$ that $f(u; a, b) = 0$ impossible since it would have to hold $\varphi_{n-1}(a) = \chi_n(a)$ in contradiction to the inequality (4) and the proof is thus complete.

Since the proofs regarding the statements of further theorems would be very much like that of the foregoing, only their main idea will be given here.

Theorem 4. Let $p, q \in v(j \supset \langle a, b\rangle)$, $\operatorname{sgn} p = \operatorname{sgn} q = 1$. If $b \in (\psi_{n-1}(a), \omega_n(a))$ $n \geqq 1$, then the functional (f) is positive definite on the admissible functions u that satisfy the conditions $u'(a) = u[\omega_i(a)] = u(b) = 0$, $i = 1, 2, \dots, n - 1$, for $n > 1$, for $n = 1$ $u'(a) = u(b) = 0$.

Proof. Let us take $y \in (pq)$ for which $y'(a) = 0$. From Lemma 1 we have the interpretation (3) and $f(u; a, b) = 0$ leads to a contradiction. Consequently $f(u; a, b) > > 0$.

Theorem 5. Let $p, q \in v(j \supset \langle a, b\rangle)$, $\operatorname{sgn} p = \operatorname{sgn} q = 1$. If $b \in (\omega_n(a), \psi_n(a)\rangle$, $n \geqq 1$, then the functional (f) is positive definite on the admissible functions u that satisfy the conditions $u'(a) = u[\omega_i(a)] = u(b) = 0$, $i = 1, 2, \dots, n$.

Proof. The proof for $b \in (\omega_n(a), \psi_n(a))$ will be carried out analogous to that of Theorem 4. Since the case $f(u; a, b) = 0$ for $b = \psi_n(a)$ reduces to a contradiction to the inequalities (4), it holds $f(u; a, b) > 0$.

Theorem 6. Let $p, q \in v(i \supset \langle a, b\rangle)$, $\operatorname{sgn} p = \operatorname{sgn} q = 1$. If $b \in (\omega_n(a), \psi_n(a))$, $n \geqq 1$, then the functional (f) is positive definite on the admissible functions u that satisfy the conditions $u'(a) = u[\omega_i(a)] = u'(b) = 0$, $i = 1, 2, \dots, n$.

Proof. Let for $y \in (pq)$ be $y'(a) = 0$. Let us next from arbitrary admissible function u satisfying the conditions $u'(a) = u[\omega_i(a)] = u'(b) = 0$, $i = 1, 2, \dots, n$ form partial functions u_1 and u_2 defined on the interval $\langle a, \omega_n(a)\rangle$ and $\langle \omega_n(a), b\rangle$, respectively. Evidently $f(u; a, b) = f(u_1; a, \omega_n(a)) + f(u_2; \omega_n(a), b)$. By the statement of the third part of Theorem 1 it holds $f(u; a, \omega_n(a)) \geqq 0$. Because of $\psi_n(a) = \chi[\omega_n(a)]$, for $b \in (\omega_n(a), \chi[\omega_n(a)])$, we obtain from Theorem 3 $f(u_2; \omega_n(a), b) > 0$ and thus $f(u; a, b) > 0$ which was to be demonstrated.

6. Necessary and sufficient conditions for the positive definiteness of the functional (f).

Theorem 7. Let $p, q \in v(i \supset \langle a\ b\rangle)$, $p > 0$. The functional f on the admissible functions u that satisfy the conditions $u[\varphi_i(a)] = u(b) = 0$, $i = 0, 1, \dots, n \geqq 1$ is positive definite if and only if the interval $\langle \varphi_{n-1}(a), b\rangle$ does not contain any conjugate number of the first kind to the number $\varphi_{n-1}(a)$.

Proof. 1. If the interval $\langle \varphi_{n-1}(a), b\rangle$, $n \geqq 1$ does not contain any conjugate number to the number $\varphi_{n-1}(a)$ of the first kind, then $b \in (\varphi_{n-1}(a), \varphi_n(a))$ and by the statement of Theorem 2 the functional (f) is positive definite on the admissible functions u that satisfy the conditions $u[\varphi_i(a)] = u(b) = 0$, $i = 0, 1, \dots, n-1$.

2. If the interval $\langle \varphi_{n-1}(a), b\rangle$, $n \geqq 1$ contains a conjugate number of the first kind to the number $\varphi_{n-1}(a)$, e.g. let $b = \varphi_n(a)$, then by the first part of Theorem 1, we have $f(y; a, b) = 0$ for $y \in (pq)$ for which $y(a) = 0$. In other words, the functional (f) is not positive definite on the admissible functions that satisfy the conditions $u[\varphi_i(a)] = u(b) = 0$, $i = 0, 1, \dots, n-1$.

Another way of proving the statement of Theorem 7 for $n = 1$ has been shown in [2], V., § 22.

Theorem 8. Let $p, q \in v(j \supset \langle a, b\rangle)$, $\operatorname{sgn} p = \operatorname{sgn} q = 1$. The functional (f) on the admissible functions u that satisfy the conditions $u'(a) = u[\omega_i(a)] = u(b) = 0$, $i = 1, 2, \dots, n$, $n \geqq 1$, is positive definite if and only if the interval $\langle \omega_n(a), b\rangle$ does not contain any conjugate number of the first kind to the number $\omega_n(a)$.

Proof. If the interval $\langle \omega_n(a), b\rangle$, $n \geqq 1$, does not contain any conjugate number of the first kind to the number $\omega_n(a)$, then $b \in (\omega_n(a), \psi_n(a)\rangle$, or $b \in (\psi_n(a), \omega_{n+1}(a))$ and by the statements of Theorems 5 and 4 the functional (f) is positive definite on the admissible functions u that satisfy the conditions $u'(a) = u[\omega_i(a)] = u(b) = 0$, $i = 1, 2, \dots, n$.

2. If the interval $\langle \omega_n(a), b\rangle$, $n \geqq 1$, contains a conjugate number of the first kind number $\omega_n(a)$, let $b = \varphi[\omega_n(a)]$ i.e. $b = \omega_{n+1}(a)$, then by the third part of Theorem 1 $f(y; a, b) = 0$ holds for $y \in (pq)$ for which $y'(a) = 0$. Hence the functional (f) is not positive definite on the admissible functions that satisfy the conditions $u'(a) = u[\omega_i(a)] = u(b) = 0$, $i = 1, 2, \dots, n$.

7. The definiteness of the differential equation (pq).

The differential equation (pq) is said to be definite on the interval $j_0 \subset j$ if it has no $y \in (pq)$ such that $y(t_1) = y'(t_2) = 0$ for any $t_1 \neq t_2$, $t_1, t_2 \in j_0$.

The definiteness relations of the functional (f) and of the equation (pq) are described by

Theorem 9. Let $p, q \in v(i)$. If the functional (f) is definite on every interval $\langle a, b\rangle \subset j$ on the admissible functions u that satisfy the condition $u(a) = u'(b) = 0$, then the equation (pq) is definite on the interval j.

Proof. Under the assumptions of the theorem, let the equation (pq) be indefinite on j, which implies that there exists a $y \in (pq)$ and $a \neq b$, $a, b \in j$ such that $y(a) = y'(b) = 0$. Substituting this y in (pq) and then multiplying out we obtain a relation which integrated on the interval $\langle a, b \rangle$ gives $f(y; a, b) = 0$ contrary to assumption.

From the statement of the above theorem and from the remark in § 4 we arrive to the following

Corollary. If $p, q \in v(j)$ and $\operatorname{sgn} p = -\operatorname{sgn} q$, then the differential equation (pq) is definite on the interval j

8. The disconjugacy of the differential equation (pq).

The differential equation (pq) is said to be disconjugate on the interval $j_0 \subset j$ if it has no $y \in (pq)$ such that $y(t_1) = y(t_2) = 0$ for any $t_1 \neq t_2$, $t_1, t_2 \in j_0$.

It is evident from the definitions that if the differential equation (pq) is definite on the interval j_0, then it is disconjugate there as well.

Theorem 10. Let $p, q \in v(j \supset j_0)$ and let the central dispersion $\varphi(t)$ be defined on the interval j_0. The differential equation (pq) is disconjugate on the interval j_0 if and only if for every closed bounded interval $\langle a, b \rangle \subset j_0$ the functional (f) on the admissible functions u that satisfy the conditions $u(a) = u(b) = 0$, is definite.

Proof. 1. If the equation (pq) is not disconjugate on the interval j_0, then there exists a, $b = \varphi(a) \in j_0$. For $y \in (pq)$ for which $y(a) = y(b) = 0$ from the interpretation of (f):

$$f(y; a, b) = [pyy']_b^a - \int_a^b [(py')' + qy]\, y \, \mathrm{d}t$$

we obtain $f(y; a, b) = 0$; consequently (f) is indefinite on the admissible functions u that satisfy the conditions $u(a) = u(b) = 0$ whereby the interval $\langle a, b \rangle \subset j_0$.

2. If (pq) is disconjugate on the interval j_0, then for every interval $\langle a, b \rangle \subset j_0$ is $b \in (a, \varphi(a))$ and for $p > 0$, according to theorem 2, the functional (f) on the admissible functions u, for which $u(a) = u(b) = 0$ holds, is positive definite. Similarly to the proof of Theorem 2 the functional (f) on the admissible functions u for which $u(a) = u(b) = 0$ holds, turns out to be negative definite.

The substance of this theorem is frequently called the variational principle, the proof of which has been given in another form in [4], XI., § 6.

9. The disconjugacy of complex differential equations.

We have a differential equation

$$(P(t)\, w')' + Q(t)\, w = 0 \tag{PQ}$$

whose coefficients are complex functions of a real argument:

$$P(t) = P_1(t) + \mathrm{i}\, P_2(t), \qquad Q(t) = Q_1(t) + \mathrm{i}\, Q_2(t).$$

Here the symbols PQ, $V(j)$ will have a similar meaning as (pq), $v(j)$ in the real case. The definition for the disconjugacy of (PQ) is analogous to that for the disconjugacy of (pq) in the real case. (See § 8).

The next step is to prove the sufficient conditions for the disconjugacy of (PQ) by comparing its coefficients with those of the disconjugate equation (pq). Doing this we make use of the properties of (f) and $\varphi(t)$. Here is the growth and continuity involved which will be proved even for the equation (pq) as surely as it has been done for the equation $y'' = q(t)\, y$ in [1], § 13, 1.

Theorem 11. Let $p, q \in v(j \supset \langle a, b \rangle)$, $p > 0$ and let the function $\varphi(t)$ be defined on the interval $\langle a, b)$. Let next $P, Q \in V(j)$. If $b = \varphi(a)$ and a real constant γ exists such that on the interval (a, b)

$$P_1 + \gamma P_2 \geqq p, \qquad Q_1 + \gamma Q_2 \leqq q \tag{5}$$

is true, then the differential equation (PQ) on the intervals $\langle a, b)$, $(a, b\rangle$ is disconjugate.

Proof. Suppose that the equation (PQ) is not disconjugate on the above intervals. Then there exist $w = w_1 + i\, w_2 \in (PQ)$ and the numbers t_1, t_2 in these intervals, $t_1 < t_2$ such that it holds

$$w(t_1) = w(t_2) = 0. \tag{6}$$

After substituting first the w in question into (PQ) and then multiplying it by the function $\bar{w} = w_1 - i\, w_2$ and integrating from t_1 to t_2 we obtain the relation:

$$\int_{t_1}^{t_2} (P \,|\, w' \,|^2 - Q \,|\, w \,|^2)\, \mathrm{d}t = 0 \tag{7}$$

whose real and imaginary part is

$$\int_{t_1}^{t_2} (P_k \,|\, w' \,|^2 - Q_k \,|\, w \,|^2)\, \mathrm{d}t = 0, \qquad k = 1 \text{ and } 2, \text{ respectively}$$

Upon multiplying out the imaginary part by γ and adding the real part to it, we find that

$$\int_{t_1}^{t_2} ((P_1 + \gamma P_2) \,|\, w' |^2 - (Q_1 + \gamma Q_2) \,|\, w \,|^2)\, \mathrm{d}t = 0$$

and with respect to the assumption (5) we arrive at the inequality

$$I = \int_{t_1}^{t_2} (p \mid w' \mid^2 - q \mid w \mid^2) \, dt \leqq 0. \tag{8}$$

The real functions w_1, w_2 are admissible functions of the functional (f) and as can be seen from (6), they satisfy the conditions $w_k(t_1) = w_k(t_2) = 0$, $k = 1, 2$. From the assumption that $t_1, t_2 \in \langle a, b)$ or $(a, b\rangle$, $b = \varphi(a)$ and from the property of the function $\varphi(t)$ we note that $t_2 \in (t_1, \varphi(t_1))$. This implies that by Theorem 2 the following inequalities

$$f(w_k; t_1, t_2) > 0, \qquad k = 1, 2 \tag{9}$$

hold which upon adding give

$$I = \int_{t_1}^{t_2} (p \mid w' \mid^2 - q \mid w \mid^2) \, dt > 0 \tag{10}$$

in contradiction to the relation (8).

Corollary. If at least one of the inequalities (5) sharp for all $t \in (a, b)$ exists in Theorem 11, then the differential equation (PQ) is disconjugate on the interval $\langle a, b\rangle$.

Proof. The procedure of proving this is the same as we have used for Theorem 11, only that we have $I < 0$ instead of the inequality (8), and on the basis of the first part of Theorem 1 when $a \leqq t_1 < t_2 \leqq b$ there hold the inequalities $f(w_k; t_1, t_2) \geqq 0$, $k = 1, 2$, whence instead of (10) we obtain $I \geqq 0$ and thus also the contradiction.

The statements of Theorem 11 and of its Corollary generalize the statements of Theorem 1 and its Corollary from [6] which we obtain when $p \equiv P \equiv 1$ assuming in addition that $y \in (1q)$ and $w \in (1Q)$ belong to the class $C_2(j)$.

10. The definiteness of complex differential equations.

The definition of the definiteness of the differential equation (PQ) is analogous to that of (pq) in § 7.

From here on the writing $p, q \in v_1(j)$ will be used to indicate that for the real functions of the real argument p, q, the following conditions are satisfied:

1° $p \in C_1(j)$, $q \in C_0(j)$,

2° every $y \in (pq)$ belongs to the class $C_2(j)$,

3° $p \neq$ for every $t \in j$.

It is obvious that $p, q \in v(j)$ if $p, q \in v_1(j)$. Now we shall derive the sufficient conditions for the definiteness of the differential equation (PQ) using the property of the functional (f) by means of the central dispersions $\chi(t)$, $\omega(t)$ – of the third and fourth kinds. Their proprerties are discussed in

Lemma 3. Let $p, q \in v_1(j)$, $\operatorname{sgn} p = \operatorname{sgn} q$. Next let X, Ω be arbitrary central dispersion of the third, fourth kinds of the differential equation (pq). The functions X, Ω are in their domains of definition continuous and increasing.

Proof. We shall show the derivatives of X, Ω to be positive. Since the idea of this proof is very much like that given in [1], § 13, 3, we shall only briely demonstrate it for the function Ω.

Let $u, v \in (pq)$ be linearly independent. From the base function of (pq): $F(t, x) = u'(t)\, v(x) - u(x)\, v'(t)$ for which $F(t, \Omega(t)) = 0$ holds if $x = \Omega(t)$ we shall prove the existence of the single function Ω having the derivative:

$$\Omega' = -\frac{u''v(\Omega) - u(\Omega)\, v''}{u'v'(\Omega) - u'(\Omega)\, v'}.$$

From the equation (pq) we have $y'' = -(p'y' + qy)/p$. A similar expression of u'' and v'' produces

$$\Omega' = \frac{q}{p}\,\frac{uv(\Omega) - u(\Omega)\, v}{u'v'(\Omega) - u'(\Omega)\, v'}. \tag{11}$$

Suppose $u' \neq 0$. Then let us multiply numerator and denominatør of the above fraction by it and let us replace $u'v(\Omega)$ by $v'u(\Omega)$ in numerator. The result is

$$\Omega' = \frac{q}{p}\,\frac{u(\Omega)\,(uv' - vu')}{u'^2 v'(\Omega) - v'u'u'(\Omega)}.$$

It follows from $u' \neq 0$ that likewise $u(\Omega) \neq 0$ and after multiplying out numerator and denominator by it and replacing $v'u(\Omega)$ by $u'v(\Omega)$ we obtain

$$\Omega' = \frac{q}{p}\,\frac{u^2(\Omega)\,(uv' - vu')}{u'^2(u(\Omega)\, v'(\Omega) - u'(\Omega)\, v(\Omega))}.$$

The Wronskian of the equation (pq) is $uv' - vu' = c/p$. Thus we obtain

$$\Omega' = \frac{qp(\Omega)}{p^2}\,\frac{u^2(\Omega)}{u'^2} \tag{12}$$

If $u' = 0$, then likewise $u(\Omega) = 0$ and from the relation (11) we arrive at

$$\Omega' = \frac{q}{p}\,\frac{uv(\Omega)}{-u'(\Omega)\, v'}$$

whence on multiplying out numerator and denominator by $uu'(\Omega)$ we have

$$\Omega' = \frac{q}{p(\Omega)}\,\frac{u^2}{u'^2(\Omega)}. \tag{13}$$

In an analogous fashion, we could find for the derivative of X:

$$X' = \begin{cases} \dfrac{p^2(X)}{pq(X)} \dfrac{u^2(X)}{u^2} & \text{for } u \neq 0 \\ \dfrac{p}{q(X)} \dfrac{u^2}{u^2(X)} & \text{for } u = 0 \end{cases} \tag{14}$$

From (12), (13), and (14) we have the statement of our Lemma.

Remark. The above relations equally modified as in [1] will furnish formulas as follow:

$$X' = \frac{p^2(X)}{pq(X)} \frac{s^2(X)}{r^2}, \qquad \Omega' = \frac{p(\Omega)\, q}{p^2} \frac{r^2(\Omega)}{s^2},$$

where $r = \sqrt{u^2 + v^2}$, $s = \sqrt{u'^2 + v'^2}$ and u, v are arbitrary linearly independent solutions of (pq).

For completeness let us note that for arbitrary central dispersion of the first and second orders

$$\Phi' = \frac{p(\Phi)}{p} \frac{r^2(\Phi)}{r^2} \quad \text{and} \quad \Psi' = \frac{p^2(\Psi)}{p^2} \frac{q}{q(\Psi)} \frac{s^2(\Psi)}{s^2}$$

hold, respectively. The latter relation has been proved in [3].

Theorem 12. Let $p, q \in v_1(j \supset \langle a, b \rangle)$, $\operatorname{sgn} p = \operatorname{sgn} q = 1$ and let the functions $\chi(t)$, $\omega(t)$ be defined in the interval $\langle a, b \rangle$. Next let $P, Q \in V(i)$. If $b = \min (\chi(a), \omega(s)$ and there is a real constant γ such that the inequalities (5) are valid in (a, b), then the differential equation (PQ) is definite on the intervals $\langle a, b)$, $(a, b \rangle$.

Proof. Let us consider the case of $b = \chi(a)$, which we are going to argue by contradiction. Suppose now that there exist $w \in (PQ)$ and any t_1, t_2 in each of the intervals stated above, $t_1 < t_2$ such that

$$w(t_1) = w'(t_2) = 0 \tag{6'}$$

is true.

By analogy with the proof of Theorem 11 we now obtain the relations (7) and (8).

From (6') it is clear that $w_k(t_1) = w_k'(t_2) = 0$, $k = 1, 2$. If $a \leqq t_1 < t_2 < b$ $[a < t_1 < t_2 \leqq b]$, then $t_2 \in (t_1 b) \subset (t_1, \chi(t_1))$ $t_2 \in (t_1, b\rangle \subset (t_1 \chi(t_1))$, since from $a < t_1$ bz Lemma 3 there is $\chi\,(a) < \chi(t_1)$, and by Theorem 3 the inequalities (9) are valid, whence we finally get the inequality (10) and thus a contradiction.

Suppose next that there exist $w \in (PQ)$ and any t_1, t_2 in each of the stated intervals, $t_1 < t_2$ such that

$$w'(t_1 = w(t_2) = 0 \tag{6''}$$

holds. Hence it follows $w_k'(t_1) = w_k(t_2) = 0$, $k = 1, 2$. In analogy with the preceding part of the proof we obtain the inequality (8). It can be proved without difficulty that in both cases $a \leqq t_1 < t_2 < b$ and $a < t_1 < t_2 \leqq b$ there is $t_2 \in (t_1, \omega(t_1))$ and by the statement of Theorem 4 we arrive at the inequality (9), whence also at the inequality (10) and thus at the contradiction.

The proof for $b = \omega(a)$ is similar to the foregoing. This completes the proof.

Corollary. If, under the assumptions of Theorem 12, at least one of the inequalities (5) is sharp on the interval (a, b), then the differential equation (PQ) is definite on the interval $\langle a, b \rangle$.

Proof. The idea is exactly the same as in proving Theorem 12 only that we have $I < 0$ instead of the inequality (8) and in case of $a \leqq t_1 < t_2 \leqq b$ we obtain respectively from the second and third part of Theorem 1 the inequality $I \geqq 0$, in analogy with the proof of the Corollary of Theorem 11.

It should be noted here, too, that if $p \equiv P \equiv 1$ on the interval j, then we arrive at the results of [5].

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Resumé

KVADRATICKÉ FUNKCIONÁLY S EULEROVOU ROVNICOU

$$(py')' + qy = 0$$

JAROSLAV KRBIĽA, ŽILINA

V práci sa vyšetrujú kvadratické funkcionály tvaru:

$$f(u; a, b) = \int_a^b (pu'^2 - qu^2)\,\mathrm{d}t. \qquad (f)$$

Dokazujú sa postačujúce podmienky pre nezápornosť a kladnú definitnosť funkcionál (f) na prípustných funkciách u, ktoré splňujú niektorů z podmienok:

$$u[\varphi_i(a)] = u(b) = 0, \quad u[\varphi_i(a)] = u'(b) = 0, \quad i = 0, 1, \ldots, n-1$$
$$u'(a) = u[\omega_i(a)] = u(b) = 0, \quad u'(a) = u[\omega_i(a)] = u'(b) = 0, \quad i = 1, 2, \ldots, n,$$

n prirodzené číslo, pričom sa používajú konjugované čísla, resp. centrálne disperzie $\varphi_i, \psi_i, \chi_i, \omega_i$, zavedené O. Borůvkom.

Odvádzajú sa tiež nutné a postačujúce podmienky pre kladnú definitnosť funkcionály (f).

Ďalej sa vyšetruje vzťah diskonjugovanosti a definitnosti rovnice $(py')' + qy = 0$ s definitnosťou funkcionály (f).

Vlastnosti funkcionál (f) a centrálnych disperzií sa využívajú pri dôkazoch postačujúcich podmienok pre diskonjugovanosť a definitnosť rovnice $(Pw')' + Qw = 0$, ktorej koeficienty sú komplexné funkcie reálneho argumentu.

КВАДРАТИЧЕСКИЕ ФУНКЦИОНАЛЫ С УРАВНЕНИЕМ ЭЙЛЕРА $(py')' + qy = 0$

ЯРОСЛАВ КРБИЛА, ЖИЛИНА

В настоящей работе исследуются квадратические функционалы вида

$$f(u; a, b) = \int_a^b (pu'^2 - qu^2)\,\mathrm{d}t. \qquad (f)$$

Доказываются достаточные условия неотрицательности и положительной определенности функционала (f) на допускаемых функциях u, которые удовлетворяют некоторму из условий:

$$u[\varphi_i(a)] = u(b) = 0, \ u[\varphi_i(a)] = u'(b) = 0, \ i = 0, 1, \ldots, n-1$$
$$u'(a) = u[\omega_i(a)] = u(b) = 0, \ u'(a) = u[\omega_i(a)] = u'(b) = 0,$$

$i = 1, 2, \ldots, n$, u натуральное число, причем используются сопряженные числа или центральные дисперсии $\varphi_i, \psi_i, \chi_i, \omega_i$ введенные О. Борувкой.

Доказываются также необходимые и достаточные условия для положительной определенности функционала (f).

Далее рассматриваются соотнтшение несопряженности и определенности уравнения $(py')' + qy = 0$ с определенностью функицонала (f).

Свойства функционала (f) и центральных дисперсий используются при доказательстве достаточных условий для несопряженности и определенности уравнения $(Pw')' + Qw = 0$, коэффициенты которого являются комплексными функциями действительного аргумента.