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ON RELATIONS AMONG DISPERSIONS OF AN OSCILLATORY DIFFERENTIAL EQUATION $y'' = q(t)y$

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1.1. This paper is a generalization of some results of Laitoch [3] and Barvínek [1].

Consider a differential equation

$$(q) \qquad y'' = q(t) \cdot y, \qquad q \in C^\circ[a, b), b \leqq \infty$$

where $C^n[a, b)$ (n a non-negative integer) is the set of all continuous functions with continuous derivatives up to and including the order n on $[a, b)$. Let (q) be an oscillatory ($t \to b_-$) differential equation on $[a, b)$ (i.e. every non-trivial solution has infinitiely many zeros on every interval of the form $[t_c, b)$, $t_0 \in [a, b)$).

Let y_1, y_2 be non-trivial solutions of (q) and $y_1(t) = 0$, $y_2'(t) = 0$. If $\varphi(t)$ ($\psi(t)$) is the first zero of $y_1(y_2')$ lying on the right of t, then φ (ψ) is called the basic central dispersion of the 1st (2nd) kind (briefly, dispersion of the 1st (2nd) kind). If $\chi(t)$ ($\omega(t)$) is the first zero of $y_1'(y_2)$ lying on the right of t, then $\chi(\omega)$ is called the basic central dispersion of the 3rd (4th) kind (briefly, dispersion of the 3rd (4th) kind).

The properties of dispersions can be found in !2]. If δ is the dispersion of the k-th kind of (q), ($k = 1, 2, 3, 4$), then δ has the following properties:

$$\begin{array}{lll} 1) & \delta \in C^3[a, b) & \text{if } k = 1, \\ & \delta \in C^1[a, b) & \text{if } k \neq 1. \\ 2) & \delta'(t) > 0 & \text{on } [a, b). \\ 3) & \delta(t) > t & \text{on } [a, b). \\ 4) & \lim\limits_{t \to b-} \delta(t) = b. & \end{array} \tag{1}$$

Let y be an arbitrary non-trivial solution of (q) and $q < 0$ on $[a, b)$. Then

$$\begin{aligned} \varphi'(t) &= \frac{y^2(\varphi(t))}{y^2(t)} & \text{for} \quad y(t) \neq 0, \\ &= \frac{y'^2(t)}{y'^2(\varphi(t))} & \text{for} \quad y(t) = 0, \end{aligned} \tag{2}$$

$$\psi'(t) = \frac{q(t)}{q(\psi(t))} \frac{y'^2(\psi(t))}{y'^2(t)} \quad \text{or} \quad y'(t) \neq 0,$$
$$= \frac{q(t)}{q(\psi(t))} \frac{y^2(t)}{y^2(\psi(t))} \quad \text{for} \quad y'(t) = 0, \tag{3}$$

$$\chi'(t) = -\frac{1}{q(\chi(t))} \frac{y'^2(\chi(t))}{y^2(t)} \quad \text{for} \quad y(t) \neq 0,$$
$$= -\frac{1}{q(\chi(t))} \frac{y'^2(t)}{y^2(\chi(t))} \quad \text{for} \quad y(t) = 0, \tag{4}$$

$$\omega'(t) = -q(t) \cdot \frac{y^2(\omega(t))}{y'^2(t)} \quad \text{for} \quad y'(t) \neq 0,$$
$$= -q(t) \cdot \frac{y^2(t)}{y'^2(\omega(t))} \quad \text{for} \quad y'(t) = 0, \tag{5}$$

see [2] § 13.3.

2.1. **Theorem 1.** *Let $\varphi(\psi)$ be the dispersion of the 1st (2nd) kind of an oscillatory ($t \to b_-$) differential equation (q), $q \in C^\circ[a, b)$, $q < 0$.*

Let $t_0 \in [a, b)$.	*Then*
a) $\varphi(t_0) < \psi(t_0)$	*if, and only if $\varphi''(t_0) > 0$*
b) $\varphi(t_0) = \varphi(t_0)$	*if, and only if $\varphi''(t_0) = 0$*
c) $\varphi(t_0) > \psi(t_0)$	*if, and only if $\varphi''(t_0) < 0$.*

Proof. Let y be a non-trivial solution of (q), $y(t) \neq 0$. Then according to (2)

$$\varphi'(t) = \frac{y^2(\varphi(t))}{y^2(t)},$$

in a neighbourhood of the point t. From this

$$\varphi''(t) = 2 \cdot \frac{y^2(\varphi(t))}{y^4(t)} (y'(\varphi(t)) \cdot y(\varphi(t)) - y'(t)\, y(t)). \tag{6}$$

Let us put: $\varphi_0 = \varphi(t_0)$, $\psi_0 = \psi(t_0)$.

a) Let us choose such a solution y of (q) that $y(t_0) > 0$, $y'(t_0) = 0$. Hence $y'(\psi_0) = 0$, $y'(t) < 0$ on (t_0, ψ_0), $y'(t) > 0$ on $(\psi_0, \psi(\psi_0))$, $y(\varphi_0) < 0$ (using separation Theorems, see [2] § 2.3).

Let $\varphi''(t_0) > 0$. Then it follows from (6) that $y'(\varphi_0) \cdot y(\varphi_0) > 0$. As $y(\varphi_0) < 0$, then $y'(\varphi_0) < 0$ and thus $\varphi_0 < \psi_0$.

Let $\varphi_0 < \Psi_0$. Then $y'(\varphi_0) < 0$ and it follows from (6) that

$$\varphi''(t_0) = 2 \cdot \frac{y^2(\varphi_0)}{y^2(t_0)} (y'(\varphi_0) \cdot y(\varphi_0)) > 0.$$

The cases b) c) can be proved in the same way as in a).

Corollary. *Let (q) be an oscillatory* $(t \to b_-)$ *differential equation,* $q \in C^\circ!a, b)$, $q < 0$. *Then its dispersions of the 1st and 2nd kind coincide*

$$\varphi(t) = \psi(t), \qquad t \in [a, b)$$

if, and only if $\varphi''(t) = 0$, $t \in [a, b)$, *i.e.*

$$\varphi = ct + d$$

where c, d are suitable constants.

This is the Theorem of Laitoch, see [2], § 16.1 and [3].

Theorem 2. *Let* $\varphi(\psi)$ *be the dispersion of the 1st (2nd) kind of an oscillatory* $(t \to b)$ *differential equation* (q), $q \in C^\circ[a, b)$, $q < 0$. *Let* $t_0 \in [a, b)$.
Then

a) $\varphi(t_0) \neq \psi(t_0)$ *if, and only if* $\psi'(t_0) \, . \, \varphi'(t_0) < \dfrac{q(t_0)}{q(\psi(t_0))}$,

b) $\varphi(t_0) = \psi(t_0)$ *if, and only if* $\psi'(t_0) \, . \, \varphi'(t_0) = \dfrac{q(t_0)}{q(\psi(t_0))}$.

Proof. Let us choose a solution of (q) such that $y(t_0) = 0$, $y'(t_0) > 0$. Then according to (2) and (3) we have

$$\psi'(t_0) = \frac{q(t_0)}{q(\psi_0)} \cdot \frac{1}{\varphi'(t_0)} \frac{y'^2(\psi_0)}{y'^2(\varphi_0)}, \tag{7}$$

where we put $\varphi_0 = \varphi(t_0)$, $\psi_0 = \psi(t_0)$.

a) Let y be a non-trivial solution of (q), $y(t_0) = 0$, $y'(t_0) > 0$. From this and from Separation Theorems it follows that $t_0 < t_1 < \varphi_0$, $\psi_0 < t_2$, $y(t) > 0$ on $[t_1, \varphi_0)$,

$$y(t) < 0 \text{ on } (\varphi_0, t_2], \; y'(t) < 0 \text{ on } (t_1, t_2),$$

where

$$t_1 = \chi(t_0), \qquad t_2 = \chi(\varphi_0).$$

Thus y is an increasing function on $(\varphi_0, t_2]$ and a decreasing function on $[t_1, \varphi_0)$ (because

$$y''(t) = q(t) \, . \, y(t) > 0 \quad (< 0) \quad \text{on} \quad (\varphi_0, t_2] \; ([t_1, \varphi_0))).$$

Let $\varphi_0 < \psi_0$. Then $\dfrac{y'^2(\psi_0)}{y'^2(\varphi_0)} < 1$ and the statement is valid according to (7).

Let $\varphi_0 > \psi_0$. Then the proof is similar.

Let the inequality

$$\psi'(t_0) \, . \, \varphi'(t_0) < \frac{q(t_0)}{q(\psi_0)},$$

be valid. Then it follows from (7) that $\dfrac{y'^2(\psi_0)}{y'^2(\varphi_0)} < 1$ and thus $\psi_0 \neq \varphi_0$.

b)c) We can prove the statement in the same way as in a).

Remark. 1. If $\varphi = \psi$, $t \in [a, b)$, then $\varphi'(t) = \sqrt{\dfrac{q(t)}{q(\varphi(t))}}$.

As $\varphi' \equiv C = \text{const}$ (Corollary), then

$$\frac{q(t)}{q(ct + d)} = c^2, \tag{8}$$

(see Laitoch [3]).

Remark. 2. If $\varphi \equiv \psi$, then the formula (8) is valid and thus $\varphi \not\equiv \psi$ if

$$\frac{q(t)}{q(ct + d)} \neq c^2; \qquad t \in [a, b),$$

for any constant $c > 0$, d.

This statement can be used for the proof of noncoincidence of dispersions of some special differential equations.

Example. Bessel equation.

$$q(t) = -1 - \frac{c_1}{t^2}, \qquad t \in [a, \infty), a > 0,$$

where $C_1 \neq 0$ is a constant. As the identity

$$c^2 \equiv \frac{q(t)}{g(ct + d)} \equiv 1 + c_1 \frac{(ct + d)^2 - t^2}{t^2(ct + d)^2 + c_1 t^2}, \qquad t \in [a, \infty),$$

is fulfilled only for constants $c = 0$, $d = 0$, we can see that the dispersions φ, ψ of the 1st and 2nd kind of Bessel equation do not coincide on $[a, \infty)$.

2.2. Theorem 3. *Let $\chi(\omega)$ be the dispersion of the 3rd (4th) kind of an oscillatory $(t \to b_-)$ differential equation (q), $q \in C^\circ[a, b)$. Let $q < 0$, $t_0 \in [a, b)$. Then*

a) $\omega(t_0) > \chi(t_0)$ *if, and only if* $\left(\dfrac{q(t_0)}{\omega'(t_0)}\right)' < 0$,

b) $\omega(t_0) = \chi(t_0)$ *if, and only if* $\left(\dfrac{q(t_0)}{\omega'(t_0)}\right)' = 0$,

c) $\omega(t_0) < \chi(t_0)$ *if, and only if* $\left(\dfrac{q(t_0)}{\omega'(t_0)}\right)' > 0$.

Proof. Let y be an arbitrary solution of (q) such that $y'(t) \neq 0$. Then according to (5) we have

$$\left(\frac{q}{\omega'}\right)' = \left(-\frac{y'^2(t)}{y^2(\omega)}\right)' = \frac{2y'^2(t) \,.\, \omega'}{y^4(\omega)} y(\omega)\, y'(\omega) - \frac{2 \,.\, y'(t)\, q(t)\, y(t)}{y^2(\omega)}. \tag{9}$$

Let us put: $\omega_0 = \omega(t_0)$, $\chi_0 = \chi(t_0)$.

Let y be a solution of (q) such that $y(t_0) = 0$, $y'(t_0) > 0$. Hence $y(\omega) > 0$, $y(\chi) = 0$, $y'(t) > 0$ on $[t_0, \chi_0)$ and $y'(t) < 0$ on $(\chi_0, \varphi(t_0))$.

a) Let $\omega_0 > \chi_0$. Then $y'(\omega) < 0$ and according to (9) we have

$$\left(\frac{q}{\omega'}\right)'\Bigg|_{t=t_0} < 0.$$

Let $\left(\frac{q(t_0)}{\omega'(t_0)}\right)' < 0$. Then it follows from (9) that $y'(\omega) < 0$ and thus $\chi_0 < \omega_0$.

b)c) We can prove the statement in the same way as in a).

Remark 3. Proving Theorem 5 of [1] the author has in fact proved a more powerful statement — the case b) from our Theorem 3.

Remark 4. The following statement follows from Theorem 3:

The dispersions of the 3rd and 4th kind of (q) coincide if, and only if

$$\omega'(t) = -\frac{q(t)}{c^2},$$

where C is a suitable constant.

This is the result of Barvínek [1].

Theorem 4. *Let $\chi(\omega)$ be the dispersion of the 3rd (4th) kind of an oscillatory $(t \to b -)$ – differential equation (q), $q \in C^\circ[a, b)$, $q < 0$. Let $t_0 \in [a, b)$.*

Then

$$\text{a) } \omega(t_0) = \chi(t_0) \textit{ if, and only if } \chi'(t_0) \, . \, \omega'(t_0) = \frac{q(t_0)}{q(\omega(t_0))},$$

$$\text{b) } \omega(t_0) \neq \chi(t_0) \textit{ if, and only if } \chi'(t_0) \, . \, \omega'(t_0) < \frac{q(t_0)}{q(\omega(t_0))}.$$

Proof. We can prove the theorem in the same way as Theorem 2.

2.3. Let (q) be a differential equation such that its dispersion of the 3rd and 4th kind coincide, $q \in C^\circ!a, b)$:

$$\omega(t) = \chi(t), \qquad t \in [a, b).$$

Then

$$\varphi(t) = \psi(t), \qquad t \in [a, b)$$

(because $\psi = \omega(\chi) = \chi(\omega) = \varphi$).

It follows from Theorem 4 and Remark 4 that

$$q(t) \, . \, q(\omega) = \frac{1}{c^4},$$

where C is a suitable constant. Thus

$$q(\omega) \cdot q(\omega(\omega)) = \frac{1}{c^4}, \qquad q(t) = q(\varphi).$$

According to Remark 1 we get

$$\varphi = t + d$$

and relation (2) gives us

$$y(t + d) = -y(t).$$

We can see that the following theorem is valid (see [2] § 16.8.).

Theorem 5. *Let (q) be an oscillatory $(t \to b_-)$ differential equation, $q \in C^\circ[a, b)$, such that its dispersions of the 3rd and 4th kind coincide*

$$\chi(t) = \omega(t), \qquad t \in [a, b).$$

Then

$$\varphi(t) = \psi(t) = t + d,$$

$$\omega'(t) = -\frac{q(t)}{c^2},$$

$$y(t + d) = -y(t),$$

$$q(t) = q(t + d), \qquad t \in [a, b)$$

where c, d are convenient constants.

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SHRNUTÍ

O VZTAZÍCH MEZI DISPERSEMI OSCILAČNÍ DIFERENCIÁLNÍ ROVNICE $y = q(t) y$

M. BARTUŠEK

Tato práce se zabývá některými závislostmi mezi základními centrálními dispersemi prvního a druhého, resp. třetího a čtvrtého druhu oscilatorické ($t \to b_-$) diferenciální rovnice

$$(q) \qquad y'' = q(t) y, \qquad q \in C^0[a, b), \qquad b \leqq \infty.$$

Nechť $\varphi(\psi)$ značí základní centrální dispersi prvního (druhého) druhu. Věty 1 a 2 dávají nutnou a postačující podmínku pro to, aby v libovolně zvoleném bodě t definičního intervalu $[a, b)$ platilo $\varphi(t) < \psi(t)$ resp. $\varphi(t) > \psi(t)$ resp. $\varphi(t) = \psi(t)$. Věty 3 a 4 řeší tutéž problematiku, avšak pro základní centrální disperse 3. a 4. druhu. Přímými důsledky uvedených vět jsou některá (již dříve dokázaná jiným způsobem) tvrzení o dispersích diferenciální rovnice (q) se splývajícími dispersemi prvního a druhého, resp. třetího a čtvrtého druhu.

РЕЗЮМЕ

О СООТНОШЕНИЯХ МЕЖДУ ДИСПЕРСИЯМИ ОСЦИЛЛИРУЮЩЕГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ $y = q(t) y$

М. БАРТУШЕК

Эта работа занимается некоторыми отношениями между фундаментальными центральными дисперсиями первого и второго (третьего и;четвертого) рода дифференциального уровнения с колеблющимися ($т$ — $б$) решениями

$$(q) \qquad y'' = q(t) y, \qquad q \in C^0[a, n), \qquad n \leqq \infty.$$

Пусть $\varphi(\psi)$ фундаментальная центральная дисперсия первого (второго) рода. Теоремы 1 и 2 дают нам необходимое и достаточное условие для того, чтобы в произвольно выбранной точке $т$ из интервала определения $[a, б)$ имело силу утверждение $\varphi(T) < \psi(T)$ или же $\varphi(T) > \psi(T)$ или же $\varphi(T) = \psi(T)$. Теоремы 3 и 4 решают эту самую проблематику, но для центральных дисперсий третьего и четвертого рода. Непосредственными соедствиями этих теорем являются некоторые (уже ранше другим образом доказанные) утверждения о дисперсиях дифференциального уровнения (p) со совпадающими дисперсиями первого и второго или же третьего и четвертого рода.