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ON A CERTAIN MODIFICATION OF STURM'S COMPARISON THEOREM

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To professor Miroslav Laitoch on the occasion of his 50[th] birthday

Let's have a differential equation

$$\text{(q)} \qquad y'' = q(t)y\,,$$

whose coefficient $q(t)$ belongs to class C_1 in the interval j, where j is a bounded or unbounded open interval and $q(t) < 0$ for every $t \in j$. Let $y(t)$ be an integral of differential equation (q) defined in the interval j. Let's form a function

$$f(t,y) = \alpha(t)y(t) + \beta(t)y'(t)$$

and call it a linear combination of integral $y(t)$ and its derivative with regard to the weighing function $\alpha(t)$, $\beta(t)$. Let these weighing functions have the following properties:

1° functions $\alpha(t)$, $\beta(t)$ belong to class C_2 in the interval j,

2° functions $\alpha(t)$, $\beta(t)$ don't change their signs in the interval j and at least one of them has no zero point in the interval j,

3° if $\beta(t) \neq 0$ for every $t \in j$, let a function $\frac{\alpha(t)}{\beta(t)}$ be nonincreasing in the interval j. If $\alpha(t) \neq 0$ for every $t \in j$, let a function $\frac{\beta(t)}{\alpha(t)}$ be non-decreasing in the interval j.

Together with diffeiential equation (q) we'll consider a differential equation

$$\text{(Q)} \qquad Y'' = Q(t)Y\,,$$

whose coefficient $Q(t)$ belongs to class C_1 in the interval j and $Q(t) < 0$ for every $t \in j$. Similarly a function

$$F(t,Y) = \mathscr{A}(t)Y(t) + \mathscr{B}(t)Y'(t)$$

is called a linear combination of integral $Y(t)$ and its derivative with regard to the weighing functions $\mathscr{A}(t), \mathscr{B}(t)$. Let these functions have properties 1°, 2°, 3° in the interval j.

Further we shall not take into consideration those integrals of (q), (Q), which are identically equal to zero in the interval j. Instead of „differential equation" we shall say only „equation".

In further consideration functions $\alpha(t)$, $\beta(t)$ resp. $\mathscr{A}(t)$, $\mathscr{B}(t)$ will be arbitrary but firmly chosen weighing functions fulfilling properties 1°, 2°, 3° and in functions $f(t,y)$ resp. $F(t,Y)$ will be y resp. Y mark an arbitrary integral of equation (q) resp. (Q) defined in the interval j.

Lemma: *Let a function $y(t)$ resp. $Y(t)$ be given. Let numbers a, $b \in j$, $a < b$, be neighbouring zero points of $f(t,y)$ and let $F(t,Y) \neq 0$ for every $t \in (a,b)$. Then*

$$\int_a^b \left[f(t,y)\,f''(t,y) - \frac{f^2(t,y)}{F(t,Y)}\,F''(t,Y)\right] dt + \int_a^b \left[f'(t,y) - \frac{f(t,y)}{F(t,Y)}\,F'(t,Y)\right]^2 dt = 0 \ . \tag{1}$$

Equality (1) will be called the arranged Picone's identity (see [1] pg. 186).

Proof: By direct calculation we easy find out that equality

$$\left[\frac{f(t,y)}{F(t,Y)}\big((F(t,y)\,f'(t,y) - (ft,y)\,F'(t,Y)\big)\right]' = f(t,y)\,f''(t,y) - \frac{f^2(t,y)}{F(t,Y)}\,F''(t,Y) + \left[f'(t,y) - \frac{f(t,y)}{F(t,Y)}\,F'(t,Y)\right]^2 \tag{2}$$

always holds where $F(t,Y) \neq 0$. From results of [2] follows that $f'(a,y) \neq 0$ and $f'(b,y) \neq 0$ as well. But $F(t,Y) \neq 0$ in the interval (a,b) by assumption and therefore for x_1, x_2, $a < x_1 < x_2 < b$ from relation (2) after integrating from x_1 to x_2 we obtain

$$\left[\frac{f(t,y)}{F(t,Y)}\big(F(t,Y)\,f'(t,y) - f(t,y)\,F'(t,Y)\big)\right]_{x_1}^{x_2} = \int_{x_1}^{x_2} \left[f(t,y)\,f''(t,y) - \frac{f^2(t,y)}{F(t,Y)}\,F''(t,Y)\right] dt + \int_{x_1}^{x_2} \left[f'(t,y) - \frac{f(t,y)}{F(t,y)}\,F'(t,Y)\right]^2 dt \ . \tag{3}$$

If e. g. $F(b, Y) \neq 0$, so the left-hand side of (3) has its limit for $x_2 \to b_-$ and this limit is equal to zero. If $F(b, Y) = 0$ and then $F'(b, Y) \neq 0$, we have by L'Hôspital's rule

$$\lim \frac{f^2(x_2,y)}{F(x_2,Y)}\,F'(x_2,Y) = 0 \ .$$

Now it is evident that for $x_2 \to b_-$ the left-hand side of (3) has always its limit equal to zero. Similarly we can find out that the same result takes place in the case $x_1 \to a_+$. From the preceding consideration we obtain the validity of (1) and lemma is thus proved.

Lemma 2: *Let functions $f(t, y)$, $F(t, Y)$ be given fulfilling in the interval j the following condition*

$$F(t,Y) = k\,f(t,y), \tag{4}$$

where k is a constant value different from zero. If

$$(5) \qquad \mathscr{A}(t) = p\alpha(t), \qquad \mathscr{B}(t) = p\beta(t)$$

holds in the interval j, so then $Q(t) = q(t)$, $Y = ry$, where p, r are constan. values different from zero and $k = pr$.

Proof of this lemma is evident.

Theorem: *Let functions $f(t, y)$, $F(t, Y)$ be given fulfilling in the interval j the following condition*

$$(6) \qquad \frac{f''(t, y)}{f(t, y)} \geq \frac{F''(t, Y)}{F(t, Y)} .$$

Then either between each two neighbouring zero points a, b, $a < b$, of function $f(t, y)$ there lies at least one zero point of each function $F(t, Y)$ or the functions $f(t, y)$, $F(t, Y)$ differ from each other only by a multiplicative constant value. In this second case equations (q), (Q) under the assumption (5) are identical in the interval j and integrals Y, y differ from each other only by a multiplicative constant value.

Proof: There are two possibilities: either $F(t, Y) = 0$ for certain $\tau \in (a, b)$ and then the first part of the assertion of theorem holds, or $F(t, Y) \neq 0$ for every $t \in (a, b)$ and then by lemma 1 there holds the arranged Picone's identity

$$\int_a^b \left[f(t, y)\, f''(t, y) - \frac{f^2(t, y)}{F(t, Y)}\, F''(t, Y) \right] dt +$$

$$+ \int_a^b \left[f'(t, y) - \frac{f(t, y)}{F(t, Y)}\, F'(t, Y) \right]^2 dt = 0 .$$

As the second term is non-negative and the first one is by (6) non-negative as well, the Picone's identity can hold only under assumption that both integrands are identically equal to zero. Herefrom we get the condition

$$\frac{f''(t, y))}{F''(t, Y)} = \frac{f(t, y)}{F(t, Y)} = \frac{f'(t, Y)}{F'(t, Y)}$$

that holds only if condition (4) holds, i. e.

$$F(t, Y) = k\, f(t, y),$$

where k is a constant value different from zero. Now lemma 2 implies the validity of the rest of the assertion of the theorem.

Note: The preceding theorem is a certain generalization of Sturm's comparison theorem for the equations of Jacobi's type. We obtain it by choosing in relation (6) $\mathscr{A}(t) = \alpha(t) = 1$; $\mathscr{B}(t) = \beta(t) = 0$.

At the end of this paper I should like to express my gratitude to Prof. RNDr. Miroslav Laitoch CSc. for his suggestion to investigate this problem and for his valuable advice.

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[2] *Háčik, M.*: Generalization of Amplitude, Phase and Accompanying Differential Equation. Acta Univ. Palack. Olomucensis (to appear)

SHRNUTÍ

O ISTEJ MODIFIKÁCII STURMOVEJ POROVNÁVACEJ VETY

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V práci sú skúmané namiesto integralov y resp. Y diferenciálnych rovníc (q) resp. (Q) lineárne kombinácie týchto integrálov a ich derivácií v tvare

$$\alpha(t)\,y(t) + \beta(t)\,y'(t) \quad \text{resp.} \quad \mathscr{A}(t)\,Y(t) + \mathscr{B}(t)\,Y'(t)\,,$$

kde funkcie $\alpha, \beta, \mathscr{A}, \mathscr{B}$ splňujú na intervale j vlastnosti 1°, 2°, 3°. V tejto súvislosti sa prichádza k istej modifikácii Sturmovej porovnávacej vety.