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On ω-resolvable and almost-ω-resolvable spaces

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Abstract. We continue the study of almost-ω-resolvable spaces beginning in A. Tamariz-Mascarúa, H. Villegas-Rodríguez, *Spaces of continuous functions, box products and almost-ω-resoluble spaces*, Comment. Math. Univ. Carolin. **43** (2002), no. 4, 687–705. We prove in ZFC: (1) every crowded T_0 space with countable tightness and every T_1 space with π-weight $\leq \aleph_1$ is hereditarily almost-ω-resolvable, (2) every crowded paracompact T_2 space which is the closed preimage of a crowded Fréchet T_2 space in such a way that the crowded part of each fiber is ω-resolvable, has this property too, and (3) every Baire dense-hereditarily almost-ω-resolvable space is ω-resolvable. Moreover, by using the concept of almost-ω-resolvability, we obtain two results due the first one to O. Pavlov and the other to V.I. Malykhin: (1) $V = L$ implies that every crowded Baire space is ω-resolvable, and (2) $V = L$ implies that the product of two crowded spaces is resolvable. Finally, we prove that the product of two almost resolvable spaces is resolvable.

Keywords: Baire spaces, resolvable spaces, almost resolvable spaces, almost-ω-resolvable spaces, tightness, π-weight

Classification: Primary 54D10, 54E52, 54A35; Secondary 54C05, 54A10

1. Introduction

The relations between the topological properties of a crowded space X and the space $C_\square(X)$ of real-valued continuous functions defined on X with its box topology were analyzed in [TV]. In particular, the authors determined for which crowded spaces X, $C_\square(X)$ is a discrete space. This analysis led the authors of [TV] to consider the resolvable and irresolvable spaces and the measurable cardinals. Resolvable and irresolvable spaces were studied extensively first by Hewitt [H]. Later, El'kin and Malykhin published a number of papers on these subjects and their connections with various topological problems. One of the problems considered by Malykhin [M1] refers to the existence of irresolvable spaces satisfying the Baire Category Theorem. He proved that there is such a space if and only if there is a space X on which every real-valued function is continuous at some point. The question about the existence of a Hausdorff space on which every real-valued function is continuous at some point was posed by M. Katětov in [K]. Bolstein introduced in [B] the spaces X on which it is possible to define a real-valued function f with countable range and such that f is discontinuous at every point of X (he called these spaces *almost resolvable*), and proved that every resolvable space satisfies this condition (in [FL] it is proved that X is almost resolvable iff there is a function $f : X \to \mathbb{R}$ such that f is discontinuous at every point of X). *Almost-ω-resolvable* spaces were introduced in [TV]; these are spaces on which it is possible to define a real-valued function f with countable range, and such that $r \circ f$ is discontinuous in every point of X, for every real-valued finite-to-one function r. It was proved in that article that for a Tychonoff space X, $C_{\square}(X)$ is discrete if and only if X is almost-ω-resolvable, some relations were analysed between almost-ω-resolvable spaces and spaces having resolvable-like properties, and it was also proved that the existence of a measurable cardinal is equiconsistent with the existence of a Tychonoff space without isolated points which is not almost-ω-resolvable, and that, on the contrary, if $V = L$ then every crowded space is almost-ω-resolvable.

In this article we are going to continue the study of almost-ω-resolvable, almost resolvable and resolvable spaces, and we will solve some problems related to these posed in [TV]. *In this work, each space will be considered a T_0 topological space.*

Section 2 is devoted to establishing basic definitions, examples and results. In Section 3 we prove that every space with countable tightness and every T_1 space with π-weight $\leq \aleph_1$ is hereditarily almost-ω-resolvable. Furthermore, we are going to see in Section 4 that every crowded paracompact T_2 space which is the closed preimage of a crowded Fréchet T_2 space in such a way that the crowded part of each fiber is ω-resolvable, has this property too. Next, in Section 5, we prove that every Baire dense-hereditarily almost-ω-resolvable space is ω-resolvable, and that every crowded space which does not contain a Baire open subspace is almost-ω-resolvable. In the last section we prove, using the concept of almost-ω-resolvability, that the product of two almost resolvable spaces is resolvable.

The basic terms not defined and considered in this paper are presented as in [W] with the difference that we are adding to all spaces the T_0 separation axiom. And all the basic definitions about the topological cardinal functions can be found in [Ho].

2. Basic definitions and preliminaries

A point x in a space X is an *isolated point* of X if $\{x\}$ is open in X, and a space X is *crowded* if it does not contain isolated points. Observe that every crowded space is infinite and every open subset of a crowded space is still crowded.

A space X is *resolvable* if it is the union of two disjoint dense subsets. We say that X is *irresolvable* if it is crowded and it is not resolvable. For a cardinal number $\kappa > 1$, we say that X is *κ-resolvable* if X is the union of κ pairwise disjoint dense subsets.

The *dispersion character* $\Delta(X)$ of a space X is the minimum of the cardinalities of nonempty open subsets of X. If X is $\Delta(X)$-resolvable, then we say that X is *maximally resolvable*. A space X is *hereditarily irresolvable* if it is crowded and every subspace of X is irresolvable. And X is *open-hereditarily irresolvable* if it is crowded and every open subspace of X is irresolvable.

We call a space (X,t) *maximal* if (X,t) is crowded and (X,t') contains at least one isolated point when t' strictly contains the topology t. And a space X is *submaximal* if it is crowded and every dense subset of X is open. Of course, every κ-resolvable space is τ-resolvable if $\tau \leq \kappa$. Moreover, every maximal space is submaximal, and these are hereditarily irresolvable spaces, which in turn are open-hereditarily irresolvable.

In [B] it was proved that the following formulation can be given as a definition of almost resolvable space: A space X is called *almost resolvable* if X is the union of a countable collection of subsets each of them with an empty interior. And in [TV] it was proved that the following formulation can be given as a definition of almost-ω-resolvable space: A space X is called *almost-ω-resolvable* if X is the union of a countable collection $\{X_n : n < \omega\}$ of subsets in such a way that for each $m < \omega$, $\text{int}(\bigcup_{i\leq m} X_i) = \emptyset$. In particular, every almost-ω-resolvable space is almost resolvable, every ω-resolvable space is almost-ω-resolvable, every (T_0) almost resolvable space is infinite and crowded, and every T_1 separable crowded space is almost-ω-resolvable.

We shall say that a space X is *hereditarily almost-ω-resolvable* if each crowded subspace of X is almost-ω-resolvable, and X is *dense-hereditarily almost-ω-resolvable* if each crowded dense subspace of X is almost-ω-resolvable.

Bolstein [B] proved that a space X is resolvable if it is the union of a finite collection formed by subsets with an empty interior in each. That is, every resolvable space is almost resolvable.

2.1 Examples. It was proved in Theorem 4.4 of [KST] that the existence of an ω_1-complete ideal $\mathcal{I}$ over ω_1 which has a dense set of size ω_1 implies the existence of a T_2 Baire open-hereditarily irresolvable topology $\mathcal{T}$ on ω_1. On the other hand, it was proved in [TV, Corollary 4.9] that every Baire irresolvable space is not almost resolvable (see Proposition 5.10 below). Therefore, $(\omega_1, \mathcal{T})$ is not almost resolvable; and, of course, it is not almost-ω-resolvable. (The existence of a Baire irresolvable space is equiconsistent with the existence of a measurable cardinal, see Corollary 3.6 in [KST] and Theorem 28 in [H]. Besides, each one of these conditions is equiconsistent with the existence of a non-almost-ω-resolvable Tychonoff space; see [TV, Theorem 4.16].)

The classes of almost resolvable, almost-ω-resolvable and resolvable spaces do not coincide in ZFC (but in ZFC + V = L, almost resolvability and almost-ω-resolvability are the same concept; see Theorem 5.11 above). First of all, if there is a measurable cardinal α, then there is a resolvable space X which is not almost-ω-resolvable. Indeed, let α be a non countable Ulam-measurable cardinal, and let p be a free ultrafilter on α w^+-complete. Let $X = \alpha \cup \{p\}$. We define a topology t for X as follows: $A \in t \setminus \{\emptyset\}$ if and only if $p \in A$ and $A \cap \alpha \in p$. This space is resolvable and non-almost-ω-resolvable. (Moreover, this space X is Baire and

$\Delta(X) = \alpha$. See Example 4.2 in [TV].)

Furthermore, in ZFC, there are almost-ω-resolvable spaces which are not resolvable. To see this, observe that the union of Tychonoff (resp., regular) crowded topologies in $\mathbb{Q}$ generates a Tychonoff (resp., regular) crowded topology. By Zorn's Lemma, we can consider a maximal Tychonoff topology $\mathcal{T}_0$ (resp., a maximal regular topology $\mathcal{T}_1$) in $\mathbb{Q}$. It happens that $(\mathbb{Q}, \mathcal{T}_0)$ is hereditarily irresolvable ([H, Theorems 15 and 26]), and it is almost-ω-resolvable because $\mathbb{Q}$ is countable. Moreover, in Example 3.3 in [vD], van Douwen proved that the subspace $\theta = \{x \in \mathbb{Q} :$ there is no nowhere dense subset A of $\mathbb{Q}$ such that $x \in \operatorname{cl} A \setminus A\}$ of $(\mathbb{Q}, \mathcal{T}_1)$ is maximal (and, of course, it is regular and almost-ω-resolvable).

Finally, there is a "concrete" (in the sense that we can say how its open sets look like) countable irresolvable space. Indeed, in [A], the authors construct by transfinite recursion a countable dense subset X of the space $2^{\mathfrak{c}}$ which is irresolvable. (Even more, for every cardinal number κ, it was constructed in Example 4.1 of [TV] a Tychonoff space X which is almost-ω-resolvable, hereditarily irresolvable and $\Delta(X) \geq \kappa$.)

On the other hand, the class of resolvable spaces includes spaces with well known properties:

2.2 Theorem. *All the spaces considered in this theorem are crowded.*

(1) *If X has a π-network $\mathcal{N}$ such that $|\mathcal{N}| \leq \Delta(X)$ and each $N \in \mathcal{N}$ satisfies $|N| \geq \Delta(X)$, then X is maximally resolvable* [E2].
(2) *The locally compact Hausdorff spaces are maximally resolvable* [H].
(3) *First countable T_2 spaces are maximally resolvable* [E1].
(4) *Hausdorff k-spaces are maximally resolvable* [P] (*in particular, metrizable spaces are maximally resolvable* [H]).
(5) *Countably compact regular T_1 spaces are ω-resolvable* [CGF].
(6) *Arc connected spaces are ω-resolvable* (*in particular, every topological vector space over $\mathbb{R}$ is ω-resolvable*).
(7) *Every biradial space* (*in particular, every linearly orderable topological space*) *is maximally resolvable* [Vi2].
(8) *Every homogeneous space containing a non trivial convergent sequence is ω-resolvable* [Vi1].
(9) *If G is an uncountable $\aleph_0$-bounded topological group, then G is $\aleph_1$-resolvable* [Vi2].

The following basic result will be very helpful (see, for example, [CF]).

2.3 Proposition. *If X is the union of κ-resolvable* (*resp., almost-resolvable, almost-ω-resolvable*) *subspaces, then X has the same property.*

2.4 Remarks. (1) *Every open and every regular closed subset of a κ-resolvable* (*resp., almost resolvable, almost-ω-resolvable*) *space shares this property.*

(2) *The free topological sum* $\bigoplus_{j\in J} X_j$ *of crowded spaces is* κ*-resolvable* (*resp., almost resolvable, almost-*ω*-resolvable*) *if and only if each* X_j *satisfies the same property.*
(3) *If* X *is crowded and* Y *is an arbitrary topological space, then* $X \times Y$ *is crowded.*
(4) *Let* X *be a space which contains a dense subset which is* κ*-resoluble* (*resp., almost resolvable, almost-*ω*-resolvable*). *Then,* X *satisfies this property too.*

We cannot ameliorate Proposition 1 in the previous remarks by writing "a G_δ-set" or even "a zero set" instead of "an open set". In fact, if X is a crowded space and does not satisfy $\mathcal{P} \in \{\kappa$-resolvability, almost resolvability, almost-ω-resolvability$\}$, then $X \times \mathbb{R}$ is 2^ω-resolvable (see, for example, [TV]) and $X \times \{0\}$ is a zero-set (then, it is a closed G_δ-set).

2.5 Corollary. *Let* $\mathcal{P}$ *be a topological property such that if* X *satisfies* $\mathcal{P}$*, then any open subset does. Then, the expression "Every crowded topological space with* $\mathcal{P}$ *is* κ*-resolvable* (*resp., almost-resolvable, almost-*ω*-resolvable*)*" is equivalent to "Every crowded topological space with* $\mathcal{P}$ *contains a non empty* κ*-resolvable* (*resp., almost-resolvable, almost-*ω*-resolvable*) *subspace". When "open" is substituted by "regular closed" and the topological spaces are regular in this proposition, then also we obtain a true claim.*

PROOF: Assume that every space with property $\mathcal{P}$ contains a non empty κ-resolvable (resp., almost-resolvable, almost-ω-resolvable) subspace. Let X be any crowded space with property $\mathcal{P}$. Let Y be the union of every κ-resolvable (resp., almost-resolvable, almost-ω-resolvable) subspace of X. The subspace Y is closed in X and it is κ-resolvable (resp., almost-resolvable, almost-ω-resolvable) (see Proposition 2.3 and Remark 2.4.4). Then $X = Y \cup A$, where A contains no κ-resolvable (resp., almost-resolvable, almost-ω-resolvable) subspace. If A is non-empty, then A contains an open (a regular closed) subset which is crowded and does not contain a κ-resolvable (resp., almost-resolvable, almost-ω-resolvable) subspace. Then A must be empty and X is κ-resolvable (resp., almost-resolvable, almost-ω-resolvable). □

The following results are easy to prove and well known.

2.6 Proposition. *Let* Y *be a* κ*-resolvable* (*resp., almost-resolvable, almost-*ω*-resolvable*) *space. If* $f : X \to Y$ *is a continuous and onto function, and for each open subset* A *of* X *the interior of* $f[A]$ *is not empty, then* X *is* κ*-resolvable* (*resp., almost-resolvable, almost-*ω*-resolvable*).

2.7 Corollary. *If* X *is* κ*-resolvable* (*resp., almost resolvable, almost-*ω*-resolvable*) *and* Y *is an arbitrary topological space, then* $X \times Y$ *is* κ*-resolvable* (*resp., almost resolvable, almost-*ω*-resolvable*).

2.8 Proposition. *Let $f : X \to Y$ be continuous and bijective. If X is κ-resolvable (resp., almost-resolvable, almost-ω-resolvable), so is Y.*

Let X be a κ-resolvable (resp., almost-resolvable, almost-ω-resolvable) space. A *resolution* (resp., an *almost resolution*, an *almost-ω-resolution*) for X is a partition $\{V_\lambda : \lambda < \kappa\}$ (resp., a partition $\{V_n : n < \omega\}$) of X such that each V_λ is a dense subset of X (resp., $\mathrm{int}(V_n) = \emptyset$ for every $n < \omega$, $\mathrm{int}(\bigcup_{i=0}^{n} V_i) = \emptyset$ for every $n < \omega$).

2.9 Proposition. *Let Y be a crowded space. Then the following propositions are equivalent:*

(1) *Y is almost-ω-resolvable;*
(2) *there exist an almost-ω-resolvable space X, an almost-ω-resolution $\mathcal{V} = \{V_n : n < \omega\}$ on X and an onto continuous function $f : X \to Y$ such that, for every $y \in Y$,*

$$|\{V \in \mathcal{V} : V \cap f^{-1}[\{y\}] \neq \emptyset\}| < \aleph_0.$$

Proof: The implication (1) $\Rightarrow$ (2) is trivial; we only have to put $X = Y$, $\mathcal{V}$ an arbitrary almost-ω-resolution on Y and the identity function f.

(2) $\Rightarrow$ (1): For each $y \in Y$, we define $n(y) = \max\{k < \omega : f^{-1}[\{y\}] \cap V_k \neq \emptyset\}$. Now, for each $t < \omega$ we define $W_t = \{y \in Y : n(y) = t\}$. The collection $\{W_t : t < \omega\}$ is an almost-ω- resolution for Y. □

The following can be proved in a similar way.

2.10 Proposition. *Let Y be a crowded space. Then the following propositions are equivalent:*

(1) *Y is almost resolvable;*
(2) *there exist an almost resolvable space X, an almost resolution $\mathcal{V} = \{V_n : n < \omega\}$ on X and an onto continuous function $f : X \to Y$ such that, for every $y \in Y$, $f^{-1}[\{y\}]$ has a nonempty intersection with only one element of $\mathcal{V}$.*

2.11 Corollary. *Let X and Y be crowded spaces. If X is almost-ω-resolvable and $u : X \to Y$ is an onto continuous function such that each fiber of u is finite, then Y is almost-ω-resolvable.*

2.12 Remarks. *The last result cannot be ameliorated by putting "with countable fibers" or "with compact fibers". In fact, assume that X is a non-almost-ω-resolvable space. Now, take the space $Y = \bigoplus_{x \in X} Y_x$ where Y_x is homeomorphic to $[0, 1]$ (resp., homeomorphic to $\mathbb{Q}$). Now, let $f : Y \to X$ be defined as $f(r) = x$ if $r \in Y_x$. The function f is continuous and onto, Y is almost-ω-resolvable, each fiber is compact (resp., countable) and, nevertheless, X in not almost-ω-resolvable.*

Let κ be an infinite cardinal number. A space X is called *S_κ-space* if it has the form $X = \bigcup_{\lambda<\kappa} X_\lambda$ where: (1) $X_\lambda \neq \emptyset$ for each $\lambda < \kappa$, (2) for $\lambda \neq \gamma$, $X_\lambda \cap X_\gamma = \emptyset$, (3) each $x \in X_\lambda$ belongs to $\mathrm{cl}_X X_{\lambda+1}$, and (4) if λ is an ordinal limit, each $x \in \bigcup_{\gamma<\lambda} X_\gamma$ is an accumulation point of the set X_λ. (Observe that (3) and (4) are equivalent to (3'): for each $\lambda < \kappa$ with $0 < \lambda$, each $x \in \bigcup_{\gamma<\lambda} X_\gamma$ is an accumulation point of the set X_λ.)

2.13 Proposition. *Every S_κ-space is κ-resolvable.*

Proof: Let $X = \bigcup_{\lambda<\kappa} X_\lambda$ be an S_κ-space. Of course, X does not have any isolated points. Let $\{A_\lambda : \lambda < \kappa\}$ be a partition of κ constituted by subsets of cardinality κ. For each $\lambda \in \kappa$, let Y_λ be equal to $\bigcup_{\gamma\in A_\lambda} X_\gamma$. It happens that the collection $\{Y_\lambda : \lambda < \kappa\}$ is a family of disjoint dense subsets of X. □

3. Tightness, π-weight and almost-ω-resolvability

It was noted in [TV] that every crowded space of the first category is almost-ω-resolvable. It was also proved that every crowded T_1 separable space is almost-ω-resolvable. In our first two results of this section we will generalize this last proposition in two different directions. First, a definition:

A collection of spaces $\mathcal{P}$ is called an *r-ideal* if it satisfies:

(1) if E is crowded and $E \in \mathcal{P}$, then E is almost-ω-resolvable,
(2) if A, B are subspaces of a space X and both belong to $\mathcal{P}$, then $A \cup B \in \mathcal{P}$,
(3) if $E \in \mathcal{P}$ and F is a closed subset of E, then $F \in \mathcal{P}$.

Naturally, r-ideals can contain very large classes of spaces as can be appreciated in Theorem 2.2.

3.1 Proposition. *Let $\mathcal{P}$ be an r-ideal. If X is a crowded space which is the union of a countable collection of closed subsets of X each of them belonging to $\mathcal{P}$, then X is almost-ω-resolvable.*

Proof: Let X be equal to $\bigcup_{n<\omega} K_n$ where $K_n \in \mathcal{P}$ and is closed for each $n < \omega$.

First case: For every $n < \omega$, $\mathrm{int}(\bigcup_{i=0}^{n} K_n) = \emptyset$.

In this case $\{K_n : n < \omega\}$ is an almost-ω-resolution for X, so X is almost-ω-resolvable.

Second case: There is a natural number n_0 such that $\mathrm{int}(\bigcup_{i=0}^{n_0} K_i) \neq \emptyset$, and for all $n > n_0$,

$$(*) \qquad \mathrm{int}\Big(\bigcup_{i=0}^{n} K_i\Big) = \mathrm{int}\Big(\bigcup_{i=0}^{n_0} K_i\Big).$$

Since $\mathrm{int}(\bigcup_{i=0}^{n_0} K_i)$ is an open subset of a crowded space, it is crowded. Thus, $Y = \mathrm{cl}(\mathrm{int}(\bigcup_{i=0}^{n_0} K_i)$ is crowded and belongs to $\mathcal{P}$ because it is a closed subset of $\bigcup_{i=0}^{n_0} K_i$. If $Y = X$ we have finished our proof.

Assume that $Z = X \setminus Y$ is not empty. The space Z is crowded and can be written as $Z = \bigcup_{n<\omega} Z_n$, where $Z_n = K_n \setminus Y$. If $T = \{m < \omega : Z_m \neq \emptyset\}$ is finite, then Z would be a non empty open set of X contained in $\bigcup_{i=0}^{k_0} K_i \setminus Y$ for a $k_0 < \omega$. But, this contradicts either the definition of n_0 or (*). Hence, without loss of generality, we can suppose that Z_n is not void for every n.

Now, for every $n < \omega$, $\mathrm{int}_Z \bigcup_{i=0}^{n} Z_i$ is empty because, since Z is open in X, $\mathrm{int}_Z \bigcup_{i=0}^{n} Z_i = \mathrm{int}_X \bigcup_{i=0}^{n} Z_i$, so if $\mathrm{int}_Z \bigcup_{i=0}^{n} Z_i$ is not empty we again arrive to a contradiction with respect to either the definition of n_0 or (*). Therefore, Z is almost-ω-resolvable.

Third case: There is a sequence $n_0 < n_1 < \cdots < n_k < \dots$ of natural numbers such that $\mathrm{int}(\bigcup_{i=0}^{n_j} K_i)$ is a proper subset of $\mathrm{int}(\bigcup_{i=0}^{n_{j+1}} K_i)$ for every $j < \omega$.

In this case, each $\mathrm{cl}(\mathrm{int}(\bigcup_{i=0}^{n_k} K_i))$ is a closed and crowded subset of $\bigcup_{i=0}^{n_{k+1}} K_i$. So, it is almost-$\omega$-resolvable. Therefore,

$$P = \mathrm{cl}\Big[\bigcup_{k<\omega} \mathrm{cl}\Big(\mathrm{int}\Big(\bigcup_{i=0}^{n_k} K_i\Big)\Big)\Big]$$

is almost-ω-resolvable.

Now $X \setminus P$ is open, so, if it is not void it is crowded and it is equal to $\bigcup_{i<\omega}(K_i \setminus P))$. Moreover, following a similar argumentation to that given for the second case, for every $n < \omega$, $\mathrm{int}_P(\bigcup_{i=0}^{n}(K_i \setminus P)) = \emptyset$. Hence, $\{K_i \setminus P : i < \omega\}$ is an almost-ω-resolution for $X \setminus P$. That is, $X \setminus P$ is almost-ω-resolvable. We conclude that, in this case too, X is almost-ω-resolvable. □

An evident consequence of Theorem 2.2 and Proposition 3.1 is:

3.2 Corollary. *Let X be a crowded space. If X is T_2 and σ-k-space* (*in particular, σ-compact or σ-metrizable*), *then X is almost-ω-resolvable.*

Recall that there are countable irresolvable spaces (see Examples 2.1), so there are σ-compact spaces which are not resolvable.

3.3 Problem. *Is every crowded T_2 Lindelöf space almost-ω-resolvable in ZFC?*

3.4 Proposition. *Let X be a crowded space. If the tightness of X is countable, then each point $x \in X$ is contained in a countable crowded subspace of X.*

Proof: Let $x_0 \in X$ be an arbitrary fixed point. Since X is crowded, $x_0 \in \mathrm{cl}_X[X \setminus \{x_0\}]$; so there is a countable subset $F_1 \subset X \setminus \{x_0\}$ such that $x_0 \in \mathrm{cl}_X F_1$. If $F_0 \cup F_1$ is crowded, where $F_0 = \{x_0\}$, then, being countable, it is an almost-ω-resolvable space containing x_0. Otherwise, for each isolated point x of $F_0 \cup F_1$, there is a countable subset $F_x^2 \subset X \setminus (\{x_0\} \cup F_1)$ such that $x \in \mathrm{cl}_X F_x^2$. Let $F_2 = \bigcup_{x\in G_1} F_x^2$ where G_1 is the set of isolated points of $F_0 \cup F_1$. Again, there are two possible situations: either $F_0 \cup F_1 \cup F_2$ is a countable crowded subspace containing x_0, or $G_2 = \{x \in F_2 : x \text{ is an isolated point of } F_0 \cup F_1 \cup F_2\}$ is not empty. In this last case, for each $x \in G_2$ we take a countable subset $F_x^3 \subset X \setminus (F_0 \cup F_1 \cup F_2)$ for which $x \in \mathrm{cl}_X F_x^3$. We write $F_3 = \bigcup_{x\in G_2} F_x^3$. Continuing this process if necessary, we obtain either a finite sequence $F_0, \dots, F_n$ of subsets of X such that $x_0 \in F = \bigcup_{0\le i\le n} F_n$ and F is countable and crowded, or we have to go further: $x_0 \in F = \bigcup_{n<\omega} F_n$. In this last case too, F is countable and crowded. □

3.5 Corollary. *Let X be a crowded space. If the tightness of X is countable, then X is hereditarily almost-ω-resolvable.*

Proof: Since the tightness is a monotone cardinal topological function, it is sufficient to prove that a crowded space with countable tightness is almost-ω-resolvable. By Proposition 3.4, each point of X is contained in a countable crowded subspace Y_x of X. Since each Y_x is almost-ω-resolvable, X is almost-ω-resolvable being the union of almost-ω-resolvable subspaces (Proposition 2.3). □

This last theorem cannot be improved by putting "tightness $\le \aleph_1$". In fact, as we have mentioned in Examples 2.1, it was proved in Theorem 4.4 of [KST] the existence of a T_2 Baire open-hereditarily irresolvable topology $\mathcal{T}$ on ω_1. This space $(\omega_1, \mathcal{T})$ is not almost-ω-resolvable (see Proposition 5.10 below).

Let X be a topological space. By $\pi cnw(X)$ we will denote the smallest cardinal number of a collection $\mathcal{N}$ of subsets of X with the property: each element in $\mathcal{N}$ is crowded and $\mathcal{N}$ is a π-network of X. Such a family will be called *π-crowded network* of X. The number $\pi cnw(X)$ is called the π-crowded netweight of X.

3.6 Proposition. *Let X be a crowded T_1 space. If the π-crowded netweight of X is less or equal to $\aleph_1$, then X can be written as the union of two disjoint subspaces Y and Z, where Y is a closed almost-ω-resolvable subspace of X, and Z is an open maximal resolvable subspace of X.*

Proof: If X is a crowded T_1 space with a countable π-network, then it is almost-ω-resolvable. In fact, let $R_0, \dots, R_n, \dots$ be a π-network in X. Take $x_n \in R_n$ for each $n < \omega$. So, $D = \{x_0, x_1, \dots, x_n, \dots\}$ is a dense set. Since X is T_1 and crowded, D is also crowded. Hence, X is an almost-ω-resolvable space.

Now assume that X is a T_1 crowded space with $\pi cnw(X) = \aleph_1$. Let $\mathcal{N}$ be a π-crowded network of X of cardinality $\aleph_1$. Let $\mathcal{B}_0 = \{U \subset X : U \text{ is open and } |\{N \in \mathcal{N} : N \subset U\}| \le \aleph_0\}$. Since $\{N \in \mathcal{N} : N \subset U\}$ is a π-network of U, each $U \in \mathcal{B}_0$ has countable π-network; thus it is almost-ω-resolvable. Therefore $Y_0 = \mathrm{cl}_X(\bigcup \mathcal{B}_0)$ is almost-ω-resolvable. Let Z_0 be equal to $X \setminus Y_0$ and assume that $Z_0 \ne \emptyset$.

Let $\mathcal{B}_1 = \{N \in \mathcal{N} : N \subset Z_0 \text{ and } |N| \leq \aleph_0\}$. Since each $N \in \mathcal{B}_1$ is crowded, it is almost-ω-resolvable. Thus, $Y_1 = \mathrm{cl}_X(\bigcup \mathcal{B}_1)$ is almost-ω-resolvable. Now, put $Z = X \setminus (Y_0 \cup Y_1)$, and let $\mathcal{B}_2 = \{N \in \mathcal{N} : N \subset Z\}$. Then $\mathcal{B}_2$ is a π-crowded network of Z, each $N \in \mathcal{B}_2$ has cardinality $\aleph_1$ and $|\mathcal{B}_2| \leq \aleph_1$. Using Theorem 2.2.1, we conclude that Z is maximally resolvable. □

3.7 Corollary. *Every crowded T_1 space with $\pi cnw(X) \leq \aleph_1$ is an almost-ω-resolvable space.*

PROOF: In fact, because of Proposition 3.6 X can be written as $Y \cup Z$ where Y is almost-ω-resolvable and Z is maximally resolvable. Every crowded T_1 space is infinite. So, if Z is not empty, Z is crowded T_1 and at least ω-resolvable, then it is almost-ω-resolvable. Therefore, because of Theorem 2.2, each crowded T_1 space with $\pi cnw(X) \leq \aleph_1$ is almost-ω-resolvable. □

Since the π-weight is monotone and $\pi cnw(X) \leq \pi w(X)$, we have:

3.8 Corollary. *Every crowded T_1 space with $\pi w(X) \leq \aleph_1$ is hereditarily almost-ω-resolvable.*

As we have already said (see the paragraph that follows the proof of Corollary 3.5), because of Theorem 4.4 in [KST] (see Examples 2.1, above), T_2 crowded spaces with density $\leq \aleph_1$ or even with cardinality $\aleph_1$ are not necessarily almost-ω-resolvable.

3.9 Problem. *Is every T_1 crowded space with π-crowded netweight $\leq \aleph_2$ almost-ω-resolvable?*

Recall that a space X is *collectionwise Hausdorff* if for each relatively discrete subspace Y of X there is a collection $\{A_y : y \in Y\}$ of open sets such that $y \in A_y$ for each $y \in Y$ and $A_y \cap A_z = \emptyset$ if y, z are two different points in Y.

3.10 Proposition. *If each point x of a crowded space X is an accumulation point of a relatively discrete subspace of $X \setminus \{x\}$, and X is collectionwise Hausdorff, then each point $x \in X$ is contained in an S_ω subspace of X.*

PROOF: Assume that X is collectionwise Hausdorff. Let x_0 be an element in X and put $F_0 = \{x_0\}$. By hypothesis, there is a discrete subspace F_1 of $X \setminus F_0$ such that $x_0 \in \mathrm{cl}_X F_1$. Since F_1 is discrete and X is collectionwise Hausdorff there is, for each $y \in F_1$, an open set V_2^y such that $y \in V_2^y$, $V_2^y \cap F_1 = \{y\}$, $x_0 \notin V_2^y$ and $V_2^y \cap V_2^z = \emptyset$ if $y, z \in F_1$ and $y \neq z$. Now, for each $y \in F_1$, we take a discrete subspace F_2^y contained in $V_2^y \setminus \{y\}$ in such a way that $y \in \mathrm{cl}_X F_2^y$. We set $F_2 = \bigcup_{y \in F_1} F_2^y$. Observe that F_2 is a discrete subspace of X. We continue in the same way: for each $y \in F_2$ we select an open subset of X, V_3^y, such that $V_3^y \subseteq V_2^z$ if $y \in F_2^z$, $V_3^y \cap F_2^z = \{y\}$ and $z \notin V_3^y$. Furthermore, $V_3^y \cap V_3^x = \emptyset$ if $y, x \in F_2$ and $x \neq y$. Let F_3 be equal to $\bigcup_{y \in F_2} F_3^y$.

Following this process, we obtain a sequence $F_0, F_1, \dots, F_n, \dots$ of disjoint discrete subspaces such that $F = \bigcup_{n<\omega} F_n$ is an S_ω-subspace of X which contains x_0. □

3.11 Corollary. *If each point x of a crowded space X is an accumulation point of a relatively discrete subspace of $X \setminus \{x\}$, and X is collectionwise Hausdorff, then X is ω-resolvable.*

Proof: Because of Proposition 3.10, each point x in X belongs to an ω-resolvable subspace Y_x of X (see Proposition 2.13). Therefore, X is ω-resolvable (Proposition 2.3). □

In [Pa], it is noted that, applying Ulam matrices to Malykhin's method used in [M1] in order to prove that $V = L$ implies that every space is almost resolvable, we can prove that every crowded space with countable cellularity and having cardinality < the first inaccessible non-countable cardinal is almost resolvable. Note that the topology constructed in [KST] which is Baire open-hereditarily irresolvable on a set of cardinality $\aleph_1$ is not almost resolvable (see Examples 2.1), so its cellularity must be equal to $\aleph_1$.

3.12 Problem. (1) *Is every space of cardinality $\aleph_1$ and countable cellularity almost-ω-resolvable?*
(2) *Is every space with countable cellularity and cardinality less than the first inaccesible non-countable cardinal almost-ω-resolvable?*

4. Some remarks on generalized metric spaces

A natural question is: what subclasses, or similar classes, of the class of paracompact crowded spaces are ω-resolvable (resp., almost-ω-resolvable, almost resolvable)? We can enumerate classes as submetrizable spaces, σ-spaces, M-spaces and, in general, many of those studied in [Gr].

One of the more general classes of spaces studied in [Gr] is that of σ-spaces: X is a σ-space if it is regular T_1 and has a σ-discrete network (or, equivalently, a σ-locally finite network).

We are going to give here a more inclusive definition of σ-space. A space X will be called *σ-space* if X contains a σ-discrete network.

The proof of the following result is equivalent to that given for Theorem 4.8 in [Gr].

4.1 Lemma. *If X is a T_i σ-space, with $i \in \{0, 1, 2\}$, then there is a T_i space Y with a σ-discrete base and a continuous and bijective function $f : Y \to X$. If, additionally, X is regular and T_1, Y is metrizable.*

Every crowded countable space is a crowded σ-space. For example, $\mathbb{Q}$ with its Euclidean topology $\mathcal{T}$ is a σ-space. Because of Theorem 4.3 in [CGF] (see Examples 2.1, above), there is a topology $\mathcal{T}_0$ for $\mathbb{Q}$ such that $(\mathbb{Q}, \mathcal{T}_0)$ is Tychonoff and irresolvable. Of course, $(\mathbb{Q}, \mathcal{T}_0)$ is a σ-space because it is still countable. So, there are Tychonoff σ-spaces, even countable, which are not resolvable.

Recall that a space X is *σ-discrete* if it is the countable union of relatively discrete spaces, and X is *strongly σ-discrete* if it is the union of a countable family of closed and discrete subspaces. Observe that every strongly σ-discrete space is σ-discrete and it is a σ-space. It was proved in [TV] that every crowded σ-discrete space is almost-ω-resolvable. We can also define the following: A space X is *σ-locally finite* if $X = \bigcup_{n<\omega} X_n$ and, for each $n < \omega$, $\{\{x\} : x \in X_n\}$ is a relatively locally finite collection, and $X = \bigcup_{n<\omega} X_n$ is a *strongly σ-locally finite* space if for each $n < \omega$, $\{\{x\} : x \in X_n\}$ is locally finite in X. It can be proved (see the proof of Theorem 3.5 in [TV]) that a crowded σ-locally finite space is almost-ω-resolvable too.

4.2 Definition. A space X is a *strictly-σ-space* if it is a σ-space and it is the condensation (the image under a continuous and bijective function) of a crowded space with a σ-discrete base.

Observe that there are strictly-σ-spaces which are not first countable: in fact, the netweight of the space $C_p([0,1])$ of the real-valued continuous functions defined on $[0,1]$ and with its pointwise convergence topology, is equal to $nw([0,1]) = \aleph_0$, and so $C_p([0,1])$ is a Tychonoff σ-space with character equal to $|[0,1]| = 2^\omega$ (see [Ar]). Moreover, $C_p([0,1])$ is the condensation of $C([0,1])$ with its topology of uniform convergence. Observe also that each strictly σ-space is crowded.

As a consequence of Proposition 2.8 and Theorem 2.2.3, we have:

4.3 Proposition. *Each T_2 strictly-σ-space is ω-resolvable.*

It is easy to prove the following observation.

4.4 Remark. *Let X be a T_2 space. If X has a σ-discrete network $\mathcal{N}$ such that for $n \in \mathbb{N}$ and $N_0, \dots, N_n \in \mathcal{N}$, $N_0 \cap \dots \cap N_n$ is empty or infinite, then X is a strictly-σ-space.*

The example given after Lemma 4.1 tells us that not every crowded σ-space is a strictly σ-space.

4.5 Proposition. *Every crowded σ-space is hereditarily almost-ω-resolvable.*

Proof: The property of beeing a σ-space is herditarily; so, it is enought to prove that every σ-space is almost-ω-resolvable. Let $\mathcal{R} = \bigcup_{n<\omega} \mathcal{R}_n$ be a σ-discrete network of a crowded σ-space X where each $\mathcal{R}_n$ is discrete. Take $\mathcal{S} = \{\bigcap \mathcal{L} : \mathcal{L} \subset \mathcal{R}$ and $|\mathcal{L}| < \aleph_0\}$. For each $F \in [\omega]^{<\omega}$ let $\mathcal{B}_F = \{\bigcap \mathcal{L} \in \mathcal{S} : \mathcal{L} \subset \bigcup_{n\in F} \mathcal{R}_n$ and $\mathcal{L} \cap \mathcal{R}_n \neq \emptyset \ \forall\, n \in F\}$. It happens that $\mathcal{S}$ is equal to $\bigcup_{F\in[\omega]^{<\omega}} \mathcal{B}_F$ and each $\mathcal{B}_F$ is discrete because it is a refinement of $\mathcal{R}_n$ for $n \in F$. Then, $\mathcal{S}$ is a σ-discrete network of X because $\mathcal{S}$ refines $\mathcal{R}$. Now we modify the collection $\mathcal{S}$ as follows: Take $\mathcal{S}_0 = \{R \in \mathcal{S} : |R| \geq \aleph_0\}$, $\mathcal{S}_1 = \{\{x\} : \exists\, R \in \mathcal{S}$ such that $x \in R$ and $|R| < \aleph_0\}$, and $\mathcal{S}_2 = \mathcal{S}_0 \cup \mathcal{S}_1$.

The collection $\mathcal{S}_0$ is still a σ-discrete collection and $\mathcal{S}_1$ is σ-locally finite.

Assume that $\mathcal{S}_1$ is not empty and take $Y = \{x \in X : \{x\} \in \mathcal{S}_1\}$. If Y is crowded, then it is a crowded σ-locally finite space. Hence, Y is almost-ω-resolvable. If Y is not crowded, we can find an ordinal number $\alpha > 0$, and for each $\lambda < \alpha$, an ω-resolvable subspace M_λ of X such that $X_1 = \mathrm{cl}_X(Y \cup \mathrm{cl}_X(\bigcup_{\lambda<\alpha} M_\lambda))$ is almost-ω-resolvable. In fact, let D_0 be the set of isolated points in $Y_0 = Y$. For each $x \in D_0$, there is an open set A_x in X such that $A_x \cap Y_0 = \{x\}$. Observe that $A_x \setminus \{x\}$ is a dense subset of A_x and it is ω-resolvable because of Remark 4.4. Thus, $M_0 = \mathrm{cl}_X(\bigcup_{x\in D_0} A_x)$ is an ω-resolvable space. Assume that we have already constructed ω-resolvable subspaces M_λ of X with $\lambda < \gamma$. Put $Y_\gamma = Y \setminus \mathrm{cl}_X(\bigcup_{\lambda<\gamma} M_\lambda)$. If Y_γ is empty or crowded, we take $\alpha = \gamma$, and in this case $\mathrm{cl}_X(Y \cup \mathrm{cl}_X(\bigcup_{\lambda<\gamma} M_\lambda))$ is almost-ω-resolvable, because Y_γ is empty or crowded and σ-locally finite. If Y_γ is not empty and is not crowded, let D_γ be the set of isolated points in Y_γ. For each $x \in D_\gamma$ there is an open set A_x in X such that $A_x \cap Y_\gamma = \{x\}$ and $A_x \cap \mathrm{cl}_X(\bigcup_{\lambda<\gamma} M_\lambda) = \emptyset$. Observe that $A_x \setminus \{x\}$ is a dense subset of A_x and it is ω-resolvable because of Remark 4.4. Thus, $M_\gamma = \mathrm{cl}_X(\bigcup_{x\in D_\gamma} A_x)$ is an ω-resolvable space. Continuing with this process we have to find an ordinal number α for which $X_1 = \mathrm{cl}_X(Y \cup \mathrm{cl}_X(\bigcup_{\lambda<\alpha} M_\lambda))$ must be almost-ω-resolvable.

Now, if $X_0 = X \setminus X_1$ is not empty, then it is a crowded space and $\mathcal{N} = \{N \in \mathcal{S}_0 : N \subset X_0\}$ is a σ-discrete net in X_0 such that the intersection of the members of each finite subcollection of $\mathcal{N}$ is empty or infinite. Then, again by Remark 4.4, X_0 is ω-resolvable. Therefore, $X = X_0 \cup X_1$ is almost-ω-resolvable. $\square$

The Lasnev spaces (which are the closed and continuous images of metrizable spaces) are M_1 spaces (Theorem 5.5 in [Gr]) which are defined as the regular T_1 spaces containing a σ-closure preserving base, and these are M_3 spaces, also called stratifiable, which are contained in the class of σ-spaces. The countable hereditarily irresolvable space constructed in Example 3.3 of [vD] (see Examples 2.1, above) is an M_3-space. Finally, the regular T_1 σ-spaces are semi-stratifiable.

4.6 Problem. *Is every crowded semi-stratifiable space almost-ω-resolvable in ZFC?*

Recall that an onto continuous function $f : X \to Y$ is *irreducible* if $f[A] \neq Y$ for every proper closed subset $A \subset X$. A particular case of Proposition 2.6 is the following:

4.7 Lemma. *The crowded irreducible closed preimage of a κ-resolvable* (*resp., almost resolvable, almost-ω-resolvable*) *space has the same property.*

Proof: Let $f : X \to Y$ be a continuous, irreducible, closed and onto function. Assume that Y is κ-resolvable (resp., almost resolvable, almost-ω-resolvable). If A is a non-empty open subset of X, $F = X \setminus A$ is a closed proper subset of X.

Therefore, $f[F]$ is a closed proper subset of Y. That is, $\operatorname{int} f[A] \neq \emptyset$. Applying Proposition 2.6, we obtain what we wanted. □

The following result is due to Lasnev:

4.8 Lemma. *For every continuous and closed function $f : X \to Y$ from a paracompact T_2 space X onto a Frechét space Y, there is a closed subset X_0 of X such that $f[X_0] = Y$ and $f \restriction X_0 : X_0 \to Y$ is irreducible.*

4.9 Lemma. *If $f : X \to Y$ is onto, closed and irreducible, and if Y is crowded, then X is crowded.*

PROOF: If for a $x \in X$ the set $\{x\}$ is open in X, then $f[X \setminus \{x\}]$ is closed in Y. Then, either $f[X \setminus \{x\}] = Y$ and so f is not irreducible, or $f[X \setminus \{x\}] = Y \setminus \{f(x)\}$ and $f(x)$ is an isolated point in Y, again a contradiction. □

4.10 Proposition. *Every paracompact T_2 space which is the continuous closed preimage of a crowded Frechét T_2 space, contains a closed ω-resolvable subset.*

PROOF: Let $f : X \to Y$ be a continuous, closed and onto function. Assume that X is paracompact T_2 and Y is crowded, Frechét and T_2. Because of Lemma 4.8, there is a closed subset X_0 of X such that $f[X_0] = Y$ and $f \restriction X_0 : X_0 \to Y$ is irreducible. The relation $f \restriction X_0$ is an onto continuous closed and irreducible function. By Lemma 4.9, X_0 is crowded, and by Lemma 4.7 and Theorem 2.2.4, X_0 is ω-resolvable. □

It is not difficult to construct examples of crowded irresolvable paracompact T_2 spaces which are the closed inverse image of a crowded Fréchet T_2 space. For instance, let X_0 be the quotient obtained by identifying in a point p all the 0-points of a countable collection of disjoint copies of $[0, 1]$ (this is the so called non-metrizable hedgehog). Let X_1 be a Tychonoff crowded irresolvable countable space, and let $\phi : X_0 \oplus X_1 \to X_0$ be defined by $\phi(x) = x$ if $x \in X_0$ and $\phi(x) = p$ if $x \in X_1$. Then ϕ, $X_0 \oplus X_1$ and X_0 constitute our example.

A consequence of Theorem 4.10 and Corollary 2.5 is:

4.11 Corollary. *Every crowded space for which every open set contains a crowded paracompact T_2 space which is the continuous closed preimage of a crowded Frechét T_2 space is ω-resolvable.*

Recall that a space X is *scattered* if every non empty subset Y of X has an isolated point with respect to the subspace topology in Y. For each non-empty topological space Z there is an ordinal number $\alpha \geq 0$ and, if $\alpha > 0$, non-empty subspaces Z_λ for each $\lambda < \alpha$, such that Z_0 is the set of isolated points in Z and, for each $\lambda < \alpha$, Z_λ is the set of isolated points belonging to the subspace $Z \setminus \bigcup_{\gamma<\lambda} Z_\gamma$, Z_α does not have any isolated point, and $Z = \bigcup_{\lambda\leq\alpha} Z_\lambda$. Observe that if $\bigcup_{\lambda<\alpha} Z_\lambda$ is not empty then it is scattered, and Z_α is either empty or crowded and closed. We will denote in the following theorem this Z_α by $Z^\star$.

4.12 Theorem. *Let X be a crowded paracompact T_2 space, Y a crowded Fréchet T_2 space and $f : X \to Y$ a continuous, closed and onto function. Assume that for each $y \in Y$ $(f^{-1}[\{y\}])^\star$ is empty or ω-resolvable (resp., almost-resolvable, almost-ω-resolvable). Then X is ω-resolvable (resp., almost-resolvable, almost-ω-resolvable).*

Proof: We give the proof for the ω-resolvable case. The proof of the other cases is similar. Let X_0 be the closure of $\bigcup\{E \subset X : E$ is ω-resolvable$\}$. By Proposition 2.3 and Remark 2.4.4, $W = \mathrm{cl}_X(\bigcup(f^{-1}[\{y\}])^\star)$ is contained in X_0. Let X_1 be the difference $X \setminus X_0$. The subspace X_1 is open and if it is not void, we can choose a non-empty open set A such that $V = \mathrm{cl}_X A \subset X_1$. Observe that $f \restriction V : V \to f[V]$ is a continuous, closed and onto function, V is paracompact T_2 and $f[V]$ is a Fréchet T_2 space. If $f[V]$ were crowded, then V would contain an ω-resolvable subspace (Proposition 4.10) which is not possible. So, $f[V]$ contains an isolated point z. This means that there is an open subset B of Y such that $B \cap f[V] = \{z\}$. The set $f^{-1}[B]$ is open in X, so $f^{-1}[B] \cap X_1$ is open in X too, and contains a point $x \in V$. Thus, there is a $x_0 \in A \cap f^{-1}[B]$. But $A \cap f^{-1}[B]$ is equal to $A \cap f^{-1}[\{z\}]$ (because, if the contrary, B would share with $f[V]$ two different points). Therefore, $A \cap f^{-1}[\{z\}]$ is a non-empty open subset of the crowded space X contained in $f^{-1}[\{z\}] \setminus [f^{-1}[\{z\}]]^\star$. So, $A \cap f^{-1}[\{z\}]$ has an isolated point x in it. But $A \cap f^{-1}[\{z\}]$ is open in X. So $\{x\}$ is open in X. This contradiction implies that X_1 must be empty and X must be ω-resolvable. □

Observe that, in the last three results, if the Fréchet T_2 space Y is κ-resolvable ($\kappa \geq \omega$), so is X.

A particular case of the previous result is that when the paracompact T_2 space X is the perfect preimage of a Fréchet T_2 space or, even more, of a metrizable space. In this last case X is called *paracompact M-space*. All these kind of spaces are T_2 k-spaces and, therefore, maximally resolvable.

Recall that a regular T_1 space X is an *M-space* if it is the quasi-perfect preimage of a metrizable space. Every countably compact regular T_1 space is an M-space, and in Example 3.23 in [Gr] is presented a countably compact subset X of $\beta\mathbb{N}$ which is not a k-space. Nevertheless, this space X is an ω-resolvable M-space because it is a countably compact regular T_1 space (see Theorem 2.2.5). So, it is natural to ask:

4.13 Problem. *Is every M-space an ω-resolvable (almost-ω-resolvable) space?*

5. Baire property and almost-ω-resolvable spaces

The following claims are obvious.

5.1 Lemma. (1) *A space X is a Baire space iff each open subset of X is a Baire space.*

(2) *If X is of the first category and A its open subset, then A is of the first category with respect to its relative topology.*

The following two lemmas were proved in [FL].

5.2 Lemma. *Suppose X is a topological space. Then X can be written as the union of mutually disjoint subsets F, B and N (resp., R, I, and M) such that F, B, R and I are open, N and M are nowhere dense (in fact, N is the boundary of F and M is the boundary of R), F is of first category, R is resolvable, B is Baire or $B = \emptyset$, and I is irresolvable and does not contain an open resolvable subset, or $I = \emptyset$.*

5.3 Lemma. *A crowded space X is almost resolvable if and only if $X = X_0 \cup X_1$ where X_0 is closed (with a non empty interior if it is not void) and it is resolvable, and X_1 is an open set of the first category.*

As a consequence of this last result we have (see [FL]):

5.4 Corollary. *In the class of Baire spaces, resolvability and almost-resolvability are the same concept.*

As every Tychonoff pseudocompact space is a Baire space, we obtain a partial answer to Problem 8.12 posed in [CGF]:

5.5 Corollary. *Every Tychonoff pseudocompact almost-resolvable space is resolvable.*

By Corollary 5.4, every almost-ω-resolvable Baire space is resolvable. We are going to see below that every Baire dense-hereditarily almost-ω-resolvable space is ω-resolvable. (It was constructed in [TV], using a measurable cardinal, a (T_0) Baire resolvable space which is not almost-ω-resolvable, as it was already mentioned in Examples 2.1.)

5.6 Problem. *Is every Baire almost-ω-resolvable space 3-resolvable?*

For a cardinal number $\kappa \geq 1$, we will say that X is *exactly κ-resolvable*, in symbols $E_\kappa R$, if X is κ-resolvable but is not κ^+-resolvable. The space X is said to be $OE_\kappa R$ if every nonempty open set in X is $E_\kappa R$. The concept and examples of $E_n R$ spaces for $n \in \omega$ have existed in the literature for some time (see, for example, [E3] and [vD]). It is clear that the $OE_\kappa R$ spaces are $E_\kappa R$. The above definitions can be viewed as natural generalizations of the concepts of irresolvable and open-hereditarily irresolvable spaces since $E_1 R$ and irresolvability are the same concept and $OE_1 R$ and open-hereditarily irresolvability coincide.

5.7 Proposition. *Let X be an $OE_n R$ space for a $n \in \mathbb{N}$ with $n \geq 2$ and such that every crowded dense subspace of X is almost-ω-resolvable. Then, X is of the first category.*

Proof: Let $D_1, \dots, D_n$ be disjoint and dense subsets of X whose union is equal to X. Since $n \geq 2$ and $D_i \cap D_j = \emptyset$ for different $i, j \in \{1, \dots, n\}$, each D_i is crowded. So, each D_i is almost-ω-resolvable, and it can be expressed as

$$D_i = \bigcup_{n<\omega} T_n^i$$

where $\{T_n^i : n < \omega\}$ is an almost-ω-resolution. For each $k < \omega$, let M_k be the set $\bigcup_{i=1}^n T_k^i$.

Claim 1: $\{M_n : n < \omega\}$ is an almost-ω-resolution of X.

Indeed, assume that there is a nonempty open set A of X, and $A \subset M_0 \cup \dots \cup M_k$. It happens that $A \cap D_0 \neq \emptyset$ because D_0 is dense. Moreover, $A \cap D_0 \subset (M_0 \cup \dots \cup M_k) \cap D_0 = T_0^0 \cup \dots \cup T_k^0$. But this contradicts the hypothesis that we made about $\{T_n^0 : n < \omega\}$.

Assume now that X is not of the first category. Then, for a $k < \omega$, $W = \operatorname{int}_X \operatorname{cl}_X(M_0 \cup \dots \cup M_k) \neq \emptyset$. Put $M = M_0 \cup \dots \cup M_k$.

Claim 2: $M \cap W$ and $W \setminus M$ are dense subsets of W.

In fact, $\operatorname{cl}_X M = M \cup \operatorname{Fr}_X M$ and $W \subset M \cup \operatorname{Fr}_X M$. Let A be an open and nonempty subset of W and let $a \in A$. If $a \in M$, then $A \cap M \neq \emptyset$. If $a \in \operatorname{Fr}_X M$, then, since A is open in X, A must intersect M.

On the other hand, if $A \subset M$, then $\operatorname{int}_X(M_0, \cup \dots \cup M_k) \neq \emptyset$ contradicting Claim 1. So, $A \cap (W \setminus M) \neq \emptyset$.

Claim 3: $M \cap W$, $(W \setminus M) \cap D_0$, ... , $(W \setminus M) \cap D_n$ are $(n+1)$ disjoint dense subsets of W.

In fact, by Claim 2, $M \cap W$ is dense in W. Now, assume that A is a nonempty open subset of W such that $A \cap [(W \setminus M) \cap D_i] = \emptyset$ for a $i \in \{1, \dots n\}$. Then A is open in X and $A \cap D_i$ is an open set of D_i contained in $M \cap D_i = (M_0 \cup \dots \cup M_k) \cap D_i = T_0^i \cup T_1^i \cup \dots \cup T_k^i$, but this assertion contradicts the nature of $\{T_n^i : n < \omega\}$.

Therefore, X must be of the first category. □

The following Theorem is due to Li Feng and O. Masaveu [FM].

5.8 Theorem. *Let X be a crowded topological space. Then X can be written as*

$$X = \Omega \cup \operatorname{cl}_X \Big(\bigcup_{n=1}^{\infty} O_n \Big),$$

where

(1) *for each n, O_n is an open, possibly empty, subset of X;*

(2) *for each* n, *if* $O_n \neq \emptyset$, *then it is* OE_nR;
(3) *for* $n \neq m$, $O_n \cap O_m = \emptyset$; *and*
(4) Ω *is an open, possibly empty,* ω*-resolvable subset of* X.

Now, we present the main result of this section.

5.9 Theorem. *Let* X *be a crowded Baire space such that every crowded dense subset of it is almost-*ω*-resolvable. Then* X *is* ω*-resolvable.*

PROOF: Observe that since every crowded dense subset of X is almost-ω-resolvable, then every crowded dense subset D of each non-empty open subset A of X is almost-ω-resolvable too because D is an open and crowded subset of the crowded and dense subset $D \cup (X \setminus A)$ of X. The space X can be written as $\Omega \cup \mathrm{cl}_X(\bigcup_{n=1}^{\infty} O_n)$ where Ω and each O_n satisfy the properties listed in Theorem 5.8. For each $n \in \mathbb{N}$, $n \geq 2$, O_n must be empty, because if we had the contrary, every crowded dense subset of O_n would be almost-ω-resolvable and so O_n would be of the first category (Proposition 5.7), but this is not possible (Lemma 5.1.1). Moreover, O_1 is empty because if not then O_1 would be a Baire irresolvable subspace of X; but this class of spaces is not almost-ω-resolvable (see Corollary 4.9 in [TV]), contrary to our hypothesis. So, X must be equal to Ω which is ω-resolvable. □

The proof of the following propositions can be found in [TV].

5.10 Proposition. (1) *If* X *is a Baire irresolvable space, then* X *is not almost resolvable.*
(2) *If we assume* $V = L$, *then every Baire crowded space is resolvable.*
(3) $V = L$ *implies that every crowded space is almost resolvable.*

(Proposition 5.11.3 is due to Malykhin [M1].) Moreover, we can deduce from Section 4 in [TV] that $V = L$ implies that every crowded space is almost-ω-resolvable. Next, we are going to make the proof of this last result explicit.

5.11 Theorem. $V = L$ *implies that every crowded space is almost-*ω*-resolvable.*

PROOF: In Theorem 4.14 of [TV] it was proved that if every Baire space without isolated points is resolvable, then every maximal space is almost-ω-resolvable. By Proposition 2.8, every space is almost-ω-resolvable. □

Also, by using Proposition 2.8 and other results we obtain:

5.12 Proposition. *The following assertions are equivalent:*

(1) *every crowded Baire space is resolvable;*
(2) *every crowded space is almost-resolvable;*
(3) *every crowded space is almost-*ω*-resolvable.*

Proof: If every Baire space is resolvable, then, by Lemma 4.14 in [TV], every maximal space is almost resolvable, and so, every crowded space is almost resolvable (Proposition 2.8). The implication (2) $\Rightarrow$ (3) is trivial, and Corollary 4.9 in [TV] gives us (3) $\Rightarrow$ (1). $\square$

5.13 Question. *Is every pseudocompact Tychonoff space almost-ω-resolvable in ZFC?*

As a consequence of Theorems 5.9 and 5.11 we obtain the following results which were already presented by O. Pavlov in [Pa] and proved using different tools.

5.14 Corollary. (1) ($V = L$) *Every crowded Baire space is ω-resolvable.*
(2) ($V = L$) *Every crowded Tychonoff pseudocompact space is ω-resolvable.*

Because of Corollary 3.5, Corollary 3.7, Proposition 4.5 and Theorem 5.9, we conclude:

5.15 Corollary. *Every crowded Baire space with countable tightness, every crowded Baire T_1 space with π-weight $\leq \aleph_1$ and every crowded Baire σ-space is ω-resolvable.*

So, every crowded pseudocompact Tychonoff space with either π-weight $\leq \aleph_1$ or with countable tightness or σ is ω-resolvable. With respect to the tightness, in [BM] even more was proved: every crowded pseudocompact Tychonoff space with countable tightness is maximally resolvable.

5.16 Corollary. *Every dense-hereditarily almost-ω-resolvable space X can be written as*

$$X = X_0 \cup X_1 = Y_0 \cup Y_1$$

where X_0 (resp., Y_0) is empty or open (resp., regular closed) and ω-resolvable and X_1 (resp., Y_1) is empty or regular closed (resp., open) of the first category. Moreover, $X_0 \cap X_1 = \emptyset = Y_0 \cap Y_1$.

Proof: Because of Theorem 5.8,

$$X = \Omega \cup \mathrm{cl}_X \Big(\bigcup_{n=1}^{\infty} O_n \Big),$$

where

(1) for each n, O_n is an open, possibly empty, subset of X;
(2) for each n, if $O_n \neq \emptyset$, then O_n is an OE_nR space;
(3) for $n \neq m$, $O_n \cap O_m = \emptyset$; and
(4) Ω is an open, possibly empty, ω-resolvable subset of X.

By Theorem 5.9 each non-empty O_n is of the first category. Then $\bigcup_{n<\omega} O_n$ is empty or of the first category.

Moreover, if $\bigcup_{n<\omega} O_n$ is not empty, $\mathrm{Fr}(\bigcup_{n<\omega} O_n)$ is nowhere dense because it is the boundary of an open set. Take $X_0 = \Omega$, $X_1 = \mathrm{cl}_X(\bigcup_{n=1}^{\infty} O_n)$, $Y_0 = \mathrm{cl}_X \Omega$ and $Y_1 = \bigcup_{n=1}^{\infty} O_n$. □

5.17 Problem. *Is every crowded Baire (respectively, Tychonoff pseudocompact) hereditarilly almost-ω-resolvable space maximally resolvable?*

5.18 Proposition. (1) *Each topological non-Baire space X can be represented as the union of three mutually disjoint subsets O, R and I, where O is a closed and almost-ω-resolvable subspace and $\mathrm{int}(O) \neq \emptyset$, R is resolvable, Baire, $\mathrm{int}(R) \neq \emptyset$ and no subspace of R is almost-ω-resolvable or $R = \emptyset$, and I is open, irresolvable and Baire and no subspace of I is almost-ω-resolvable or $I = \emptyset$.*

(2) *Each Baire space X can be represented as the union of three mutually disjoint subsets O, R and I, where O is a closed and almost-ω-resolvable subspace, R is resolvable, Baire, $\mathrm{int}(R) \neq \emptyset$ and no subspace of R is almost-ω-resolvable or $R = \emptyset$, and I is open, irresolvable and Baire and no subspace of I is almost-ω-resolvable or $I = \emptyset$.*

(3) *Each almost resolvable space X is the union of two disjoint subsets O and R, where O is a closed and almost-ω-resolvable (and $\mathrm{int}(O) \neq \emptyset$ if X is not Baire) and R is open, resolvable and Baire, and no subspace of R is almost-ω-resolvable.*

Proof: 1. By Lemma 5.2, X can be represented as $F \cup B \cup N$ where F is of the first category and open, B is Baire or empty and N is nowhere dense. In the proof of Lemma 6.3 in [FL], N is the boundary of F. Since X is not Baire, F is not empty. Let O be equal to the union of all almost-ω-resolvable subspaces of X. Observe that O must be closed and $F \cup N \subset O$. Since the open set F is not empty, $\mathrm{int}(O) \neq \emptyset$. It remains to consider the open set $A = X \setminus O \subset B$ (possibly empty). Assume that A is not an empty set. Since A is an open subset of the Baire space B, A is a Baire space. By Lemma 5.2, A can be written as $R' \cup I \cup M$ where R' and I are open in A, R' is resolvable, I is irresolvable or empty and M is the boundary of R'. The sets O, $R = R' \cup M$, and I satisfy the requirements. Observe, in particular that if R is not empty, it is a Baire space being a regular closed subset of a Baire space. The same for I: if I is not empty, then I is a Baire space because it is open in A.

2. We obtain the proof of this proposition in the same way that we proved (1). In this case, we cannot guarantee that $\mathrm{int}(O) \neq \emptyset$ because F can be void.

3. We write X as $X_0 \cup X_1$ as Lemma 5.3 says. Because of (1) and (2), $X = O \cup R \cup I$ with the characteristics explained in these points. Assume that X is not Baire. Since X_1 is of the first category and open, $X_1 \subset O$ and $\mathrm{int}(O) \neq \emptyset$. Then $X \setminus O$

is a subset of X_0, and it is open; so I must be void because every open subset of a resolvable space is resolvable.

Assume now that X is a Baire space. If I is not empty, then it must contain a non-empty open-hereditarily irresolvable subspace W (see Theorem 28 in [H]). The set $W \cap X_1$ is void because, on the contrary, it would be Baire and of the first category with respect to its relative topology because it is an open subset of both X and X_1, which is a contradiction. So W is an open subset of the resolvable space X_0 which is again a contradiction. Thus, we must have $I = \emptyset$. □

An immediate consequence of Theorem 5.18.1 (or 5.18.2) is:

5.19 Corollary. *For crowded spaces, the following assertions are equivalent:*

(1) *every Baire space is almost-ω-resolvable;*
(2) *every Baire space contains an almost-ω-resolvable subspace;*
(3) *every space is almost-ω-resolvable.*

An immediate consequence of Theorem 5.18.3 is:

5.20 Corollary. *For crowded spaces, the following assertions are equivalent:*

(1) *every almost resolvable space is almost-ω-resolvable;*
(2) *every resolvable space is an almost-ω-resolvable space;*
(3) *every resolvable space contains an almost-ω-resolvable subspace.*

5.21 Corollary. *Every crowded space X which does not contain an open Baire subspace, is almost-ω-resolvable.*

Proof: In fact, because of Theorem 5.18.1 X can be represented as $O \cup R \cup I$ where O is closed and almost-ω-resolvable, R is resolvable, is Baire and with non-empty interior if it is non void, and I is open Baire irresolvable. By hypothesis, R and I must be empty and, so, X is almost-ω-resolvable. □

6. Product of almost-ω-resolvable spaces

It is proved in [M1] that $V = L$ implies that the product of two crowded spaces is always resolvable. We will prove in this section that, in ZFC, the product of two almost resolvable spaces is resolvable, and the product of an infinite collection of almost resolvable spaces is ω-resolvable.

6.1 Proposition. *If X and Y are almost-ω-resolvable, then $X \times Y$ is resolvable.*

Proof: Let $\mathcal{D} = \{D_n : n < \omega\}$ and $\mathcal{F} = \{F_n : n < \omega\}$ be almost-ω-resolutions of X and Y, respectively. Define

$$P_0 = \{(x, y) : x \in D_n, y \in F_m \ \text{ and } \ n < m\}$$

and

$$P_1 = \{(x, y) : x \in D_n, y \in F_m \ \text{ and } \ n > m\}.$$

Of course $P_0 \cap P_1 = \emptyset$. We are going to prove that P_0 and P_1 are dense subsets in $X \times Y$. Let $A \times B$ be a canonical open set in $X \times Y$, and let $D_{k_0}, D_{k_1}, \dots, D_{k_n}, \dots$ and $F_{s_0}, F_{s_1}, \dots, F_{s_m}, \dots$ the sequences of elements in $\mathcal{D}$ and $\mathcal{F}$, respectively, which have a non-empty intersection with A and B, respectively, with $k_0 < k_1 < \dots < k_n < \dots$ and $s_0 < s_1 < s_2 < \dots < s_m < \dots$. So, there is $x \in A \cap D_{k_0}$. There is s_m satisfying $s_m > k_0$. Take $y \in B \cap F_{s_m}$. Then $(x, y) \in P_0 \cap (A \times B)$. In a similar way we can prove that there is an element in $P_1 \cap (A \times B)$. □

6.2 Corollary. *The product of two almost resolvable spaces is resolvable.*

Proof: Let X and Y be two almost resolvable spaces. By Lemma 5.3, $X = X_0 \cup X_1$ and $Y = Y_0 \cup Y_1$ where X_0 and Y_0 are closed and resolvable, X_1 and Y_1 are crowded, open and of the first category. (Of course, some of the spaces X_0, X_1, Y_0, Y_1 could be empty, but this situation does not modify our argumentation.) So, $X \times Y = (X_0 \times Y_0) \cup (X_0 \times Y_1) \cup (X_1 \times Y_0) \cup (X_1 \times Y_1)$. By virtue of Theorem 3.5 in [TV], Corollary 2.7 and Proposition 6.1, each $X_i \times Y_j$ $(i, j \in \{0, 1\})$ is resolvable. Thus, $X \times Y$ is resolvable. □

As a consequence, we obtain Malykhin's result:

6.3 Corollary. $[V = L]$ *The product $X \times Y$ is resolvable for every crowded spaces X and Y.*

Proof: In fact, $V = L$ implies that every crowded space X is almost-ω-resolvable (see Theorem 5.11). Now, applying Corollary 6.2 we get our result. □

6.4 Lemma. *The product of two spaces X and Y such that X is k-resolvable and Y is m-resolvable $(k, m < \omega)$ is $(k \cdot m)$-resolvable.*

Proof: In fact, let $D_1, D_2, \dots, D_k$ be disjoint dense subsets of X and let $F_1, F_2, \dots, F_m$ be disjoint dense subsets of Y. Then, for each $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, m\}$, $D_i \times F_j$ is dense in $X \times Y$. Moreover, if $(i, j) \neq (k, l)$ with $i, k \in \{1, \dots, k\}$ and $j, l \in \{1, \dots, m\}$, then $(D_i \times F_j) \cap (D_k \times F_l) = \emptyset$. □

6.5 Corollary. *Let n be a natural number bigger than 1. The product of n almost resolvable spaces $X_1, X_2, \dots, X_n$, $\prod_{i=1}^{n} X_i$, is 2^m-resolvable where either $2m = n$ or $2m + 1 = n$.*

As a consequence of the previous result and using the so well known fact that every space X which is n-resolvable for all $n \in \mathbb{N}$ is ω-resolvable [I], we conclude that the product of an infinite family of almost resolvable spaces is ω-resolvable. However, as was pointed out by W.W. Comfort to the authors, a more general fact was already proved by O. Masaveu: If $X = \prod_{\lambda < \kappa} X_\lambda$ where $\kappa \geq \omega$ and each space X_λ has more than one point, then X is 2^κ-resolvable. In fact, it can be proved that there are 2^κ pairwise disjoint σ-products in X.

Theorem 7 in [M2] states that under CH there is an ultrafilter a on ω such that the filter base $a \times a$ is contained in exactly three ultrafilters. As a conclusion, under CH there is a crowded T_1 countable space X such that X^2 is at most 3-resolvable.

6.6 Problems. (1) [Pa1, Question 24] *Is there a Hausdorff (regular) crowded space X such that X^2 is not ω-resolvable?*

(2) *Is the product of two almost-ω-resolvable Hausdorff (regular, Tychonoff) crowded spaces ω-resolvable?*

(3) *Is X almost-ω-resolvable if X^2 satisfies this property?*

(4) *Is every ω-resolvable space hereditarily almost-ω-resolvable?*

(5) *Is every dense subspace of an ω-resolvable space ω-resolvable (almost-ω-resolvable)?*

(6) *Is the square X^2 almost-ω-resolvable for every crowded space X?*

(7) *Is the product of two hereditarily almost-ω-resolvable spaces almost-ω-resolvable (ω-resolvable)?*

(8) *Let X and Y be hereditarily almost-ω-resolvable (resp., hereditarily almost resolvable, hereditarily κ-resolvable) spaces, can we, then, imply that $X \times Y$ is hereditarily almost-ω-resolvable (resp., hereditarily almost resolvable, hereditarily κ-resolvable) too?*

(9) *Let X be hereditarily almost-ω-resolvable (resp., hereditarily almost resolvable, hereditarily ω-resolvable), and let Y be closed-hereditarily almost-ω-resolvable (resp., closed-hereditarily almost resolvable, closed-hereditarily ω-resolvable), can we deduce from this that $X \times Y$ is closed-hereditarily almost-ω-resolvable (resp., closed-hereditarily almost resolvable, closed-hereditarily ω-resolvable)?*

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