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Skewsquares in quadratical quasigroups

Vladimir Volenec, Ružica Kolar–Šuper

Abstract. The concept of pseudosquare in a general quadratical quasigroup is introduced and connections to some other geometrical concepts are studied. The geometrical presentations of some proved statements are given in the quadratical quasigroup $\mathbb{C}(\frac{1+i}{2})$.

Keywords: quadratical quasigroup, skewsquare

Classification: 20N05

1. Introduction

The "geometrical" concept of skewsquare is defined and investigated in any quadratical quasigroup.

A groupoid $(Q,\cdot)$ is said to be *quadratical* if the identity

(1) $$ab\cdot a = ca\cdot bc$$

holds and the equation $ax=b$ has a unique solution $x\in Q$ for any $a,b\in Q$ (cf. [12] and [2]). Every quadratical groupoid $(Q,\cdot)$ is a quasigroup, i.e. the equations $xa=b$ and $ay=b$ have unique solutions for any $a,b\in Q$. In a quadratical quasigroup $(Q,\cdot)$ the identities

(2) $$aa=a,$$

(3) $$ab\cdot cd = ac\cdot bd,$$

(4) $$a\cdot ba = ab\cdot a,$$

(5) $$ab\cdot c = ac\cdot bc,$$

(6) $$a\cdot bc = ab\cdot ac,$$

and the equivalencies

(7) $$ab=cd \Leftrightarrow bc=da,$$

(8) $$ab=c \Leftrightarrow bc=ca$$

hold (cf. [12]).

If $\mathbb{C}$ is the set of all points of a Euclidean plane and if a groupoid $(\mathbb{C},\cdot)$ is defined so that $aa=a$ for any $a\in\mathbb{C}$ and for any two different points $a,b\in\mathbb{C}$ the point ab is the centre of the positively oriented square with two adjacent vertices a and b, then $(\mathbb{C},\cdot)$ is a quadratical quasigroup (cf. [12]). This quasigroup will be denoted by $\mathbb{C}(\frac{1+i}{2})$ because if $a=0$ and $b=1$ then $ab=\frac{1+i}{2}$. The figures in this quasigroup illustrate the "geometrical" relations in any quadratical quasigroup $(Q,\cdot)$.

From now on let $(Q,\cdot)$ be any quadratical quasigroup. The elements of Q are said to be *points*, the pairs of points are *segments*, the quadruples of points are *quadrangles* and an ordered quadruple of points is said to be an *oriented quadrangle*.

If an operation $\bullet$ is defined on the set Q by

$$a\bullet b=a\cdot ba=ab\cdot a=ca\cdot bc, \tag{9}$$

then $(Q,\bullet)$ is an idempotent medial commutative quasigroup (cf. [12]), i.e. the identities

$$a\bullet a=a, \tag{10}$$

$$(a\bullet b)\bullet(c\bullet d)=(a\bullet c)\bullet(b\bullet d), \tag{11}$$

$$a\bullet b=b\bullet a \tag{12}$$

hold and the operations $\cdot$ and $\bullet$ are mutually medial, i.e. the identity

$$ab\bullet cd=(a\bullet c)(b\bullet d) \tag{13}$$

holds. The point $a\bullet b$ is said to be the *midpoint* of two points a and b.
Because of

$$g(a,b,c,d)=(a\bullet c)\bullet(b\bullet d)\overset{(11)}{=}(a\bullet b)\bullet(c\bullet d)\overset{(12)}{=}(a\bullet b)\bullet(d\bullet c)\overset{(11)}{=}(a\bullet d)\bullet(b\bullet c)$$

the point $g(a,b,c,d)$ is said to be the *centroid* of the quadrangle $\{a,b,c,d\}$.

An oriented quadrangle (a,b,c,d) is said to be a *parallelogram* and we write $\mathrm{Par}(a,b,c,d)$ if $a\bullet c=b\bullet d$. If $a\bullet c=b\bullet d=o$, then we say that the point o is the *centre* of this parallelogram and we write $\mathrm{Par}_o(a,b,c,d)$. In [14] it is proved that (Q,Par) is a parallelogram space (cf. [8] and [11]) and the following statement which will be used later.

Lemma 1. *For any points a,b,c,d the statement $Par(a\bullet b,b\bullet c,c\bullet d,d\bullet a)$ is valid.*

An oriented quadrangle (a,b,c,d) is said to be a *square* and we write $\mathrm{S}(a,b,c,d)$ if $ab=bc=cd=da$. If $ab=bc=cd=da=o$, then we say that the point o is the *centre* of this square and we write $\mathrm{S}_o(a,b,c,d)$. Obviously $\mathrm{S}_o(a,b,c,d)\Rightarrow \mathrm{S}_o(e,f,g,h)$, where (e,f,g,h) is any cyclical permutation of (a,b,c,d).

In [13] it is proved:

Lemma 2. *The statement $S(a,b,c,d)$ is equivalent to any two of four (and then all four) equalities $ac=d$, $bd=a$, $ca=b$, $db=c$.*

In [14] the following statements are proved and they will be used later.

Lemma 3. *The statement $S_o(a,b,c,d)$ implies $Par_o(a,b,c,d)$.*

Lemma 4. $Par_o(a,b,c,d) \Leftrightarrow S_o(ba,cb,dc,ad)$.

2. The concept of skewsquare in quadratical quasigroup

In the set Q^2 a binary relation $\sim$ is defined by

$$(a,b) \sim (c,d) \Leftrightarrow \mathrm{Par}(a,b,d,c).$$

In [8] it is proved that $\sim$ is a relation of equivalence. The elements of the set $Q^2/_\sim$ are said to be the *vectors*. A vector with a representative (a,b) is denoted by $[a,b]$. Therefore, we have

$$[a,b]=[c,d] \Leftrightarrow \mathrm{Par}(a,b,d,c),$$

i.e.

$$[a,b]=[c,d] \Leftrightarrow a \bullet d = b \bullet c. \tag{14}$$

For any point a and any vector $\mathbf{v}$ there is one and only one point b such that $\mathbf{v}=[a,b]$.

A vector $\mathbf{u}$ is said to be *orthogonally equal* to a vector $\mathbf{v}$ and we write $\mathbf{u}\underline{\perp}\mathbf{v}$ if there are four points p,q,r,s such that

$$\mathbf{u}=[p,r], \quad \mathbf{v}=[q,s], \quad S(p,q,r,s)$$

(Figure 1).

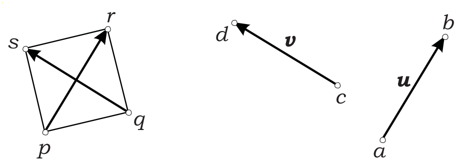

FIGURE 1

The properties of squares imply at once:

Theorem 1. *The statements $[a,b]\underline{\perp}[c,d]$, $[c,d]\underline{\perp}[b,a]$, $[b,a]\underline{\perp}[d,c]$ and $[d,c]\underline{\perp}[a,b]$ are mutually equivalent (Figure 1).*

The following theorem gives a simple characterization for orthogonally equal vectors.

Theorem 2. $[a, b] \underline{\perp} [c, d] \Leftrightarrow ac = bd.$

PROOF: Let $[a, b] = [p, r]$, $[c, d] = [q, s]$, $S_o(p, q, r, s)$ (Figure 1), i.e. $[a, b] \underline{\perp} [c, d]$. Then we have the equalities $pq = rs = o$ and by (14) the equalities $a \bullet r = b \bullet p$, $c \bullet s = d \bullet q$. Hence

$$ac \bullet o = ac \bullet rs \overset{(13)}{=} (a \bullet r)(c \bullet s) = (b \bullet p)(d \bullet q) \overset{(13)}{=} bd \bullet pq = bd \bullet o,$$

wherefrom $ac = bd$ follows. Conversely, let $ac = bd$ and let p be any point. There is a point r such that $[a, b] = [p, r]$. Let $q = rp$, $s = pr$, i.e. let $S(p, q, r, s)$ hold. There is a point d' such that $[q, s] = [c, d']$. Now we have $[a, b] \underline{\perp} [c, d']$ and the proved part of our theorem implies $ac = bd'$. Therefore we have $bd' = bd$, i.e. $d' = d$ and hence $[a, b] \underline{\perp} [c, d]$. □

Theorem 2 and the equivalence (7) give an alternative proof of Theorem 1.

The proof of Theorem 2 implies:

Corollary 1. *For any vector* $\mathbf{v}$ *and any point* c *there is one and only one point* d *such that* $\mathbf{v} \underline{\perp} [c, d]$ *holds.*

Because of Theorem 2 the equality (1) can be interpreted as the statement $[ca, ab] \underline{\perp} [bc, a]$.

Theorem 3. (i) $[a, b] \underline{\perp} [c, d]$, $[c, d] = [e, f] \Rightarrow [a, b] \underline{\perp} [e, f]$.
(ii) $[a, b] = [c, d]$, $[c, d] \underline{\perp} [e, f] \Rightarrow [a, b] \underline{\perp} [e, f]$.
(iii) $[a, b] \underline{\perp} [c, d]$, $[c, d] \underline{\perp} [e, f] \Rightarrow [a, b] = [f, e]$.
(iv) $[a, b] \underline{\perp} [d, e]$, $[b, c] \underline{\perp} [e, f] \Rightarrow [a, c] \underline{\perp} [d, f]$.

PROOF: (i) By Theorem 2 and by (14) we have the equalities $ac = bd$ and $c \bullet f = d \bullet e$. Therefore

$$ac \bullet ae = bd \bullet ae \overset{(13)}{=} (b \bullet a)(d \bullet e) \overset{(12)}{=} (a \bullet b)(c \bullet f) \overset{(13)}{=} ac \bullet bf,$$

wherefrom $ae = bf$ follows and by Theorem 2 we have the statement $[a, b] \underline{\perp} [e, f]$.
(ii) Now we have the equalities $a \bullet d = b \bullet c$ and $ce = df$ and we obtain

$$ae \bullet df \overset{(13)}{=} (a \bullet d)(e \bullet f) \overset{(12)}{=} (b \bullet c)(f \bullet e) \overset{(13)}{=} bf \bullet ce = bf \bullet df.$$

Therefore $ae = bf$, i.e. again $[a, b] \underline{\perp} [e, f]$.
(iii) We have the equalities $ac = bd$, $ce = df$, which imply

$$a \bullet e \overset{(12)}{=} e \bullet a \overset{(9)}{=} ce \cdot ac = df \cdot bd \overset{(9)}{=} f \bullet b \overset{(12)}{=} b \bullet f,$$

i.e. $[a, b] = [f, e]$ by (14).

(iv) By Theorem 2 we must prove the implication $ad = be$, $be = cf \Rightarrow ad = cf$. It is obvious. □

Because of (7) the following definition has a sense.

An oriented quadrangle (a, b, c, d) is a *skewsquare* and we write $SS(a, b, c, d)$ if $ab = cd$ and $bc = da$. It is sufficient to have only one of these two equalities (cf. [7] and [4]). If we have the equalities $ab = cd = p$ and $bc = da = q$, then the points p and q are said to be the *skewcenters* of the considered skewsquare and we write $SS_{p,q}(a, b, c, d)$ (Figure 2) (cf. [4], where p and q are said to be the *foci* of the skewsquare).

Obviously we get:

Theorem 4. *The statements* $SS_{p,q}(a, b, c, d)$, $SS_{q,p}(b, c, d, a)$, $SS_{p,q}(c, d, a, b)$ *and* $SS_{q,p}(d, a, b, c)$ *are mutually equivalent.*

According to Theorem 2 it follows.

Corollary 2. $SS(a, b, c, d) \Leftrightarrow [a, c] \perp [b, d]$ (*Figure* 2).

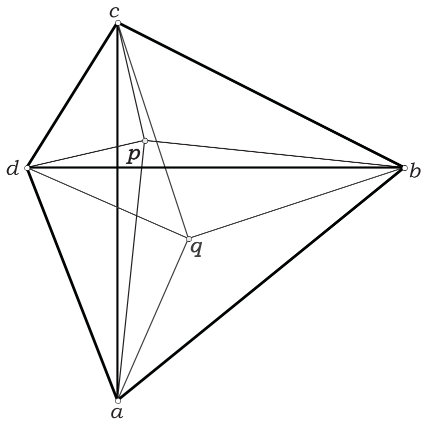

FIGURE 2

By Corollaries 1 and 2 we obtain the following statement.

Corollary 3. *For any points* a, b, c *there is one and only one point* d *such that* $SS(a, b, c, d)$ *holds.*

The equation $ax = b$ has a unique solution $x = (b \cdot ba) \cdot (b \cdot ba)(ba \cdot a)$ (cf. [12, Corollary]). Therefore the equality $ab = cd$ is equivalent to the equality

$$d = (ab)(ab \cdot c) \cdot [(ab)(ab \cdot c) \cdot (ab \cdot c)c], \tag{15}$$

i.e. we have the following theorem, which expresses the statement of Corollary 3 precisely.

Theorem 5. *The statement $SS(a,b,c,d)$ is equivalent to the equality* (15) (*Figure* 3)*.*

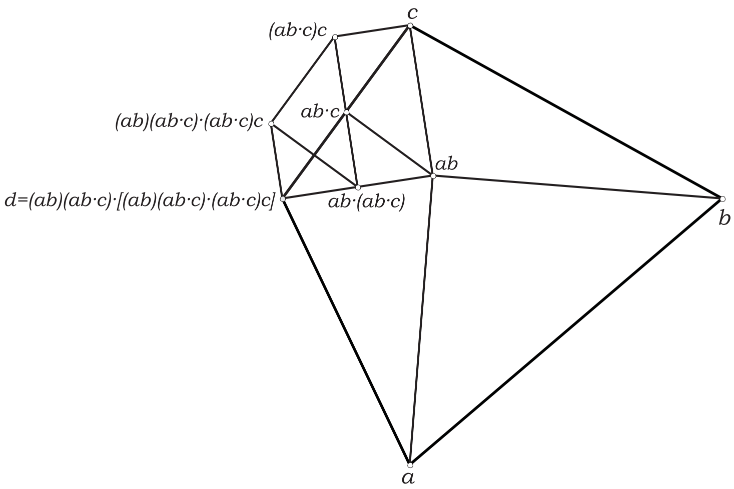

FIGURE 3

Obviously we obtain.

Theorem 6. $S_o(a,b,c,d) \Leftrightarrow SS_{o,o}(a,b,c,d)$.

Let us prove the following statement now.

Theorem 7. *The statement $SS_{p,q}(a,b,c,d)$ implies $S_o(p, a\bullet c, q, b\bullet d)$ (Figure 4) where*

$$o = a\bullet db = b\bullet ac = c\bullet bd = d\bullet ca = p\bullet q = g(a,b,c,d).$$

PROOF: Let $o = p\bullet q$. Because of $ab = p$, $da = q$ we get

$$o = p\bullet q = ab\bullet da \overset{(13)}{=} (a\bullet d)(b\bullet a) \overset{(12)}{=} (a\bullet d)(a\bullet b) \overset{(13)}{=} aa\bullet db \overset{(2)}{=} a\bullet db,$$

and similarly it can be obtained $o = b\bullet ac = c\bullet bd = d\bullet ca$. Further, we get

$$p(a\bullet c) = ab\cdot(a\bullet c) \overset{(9)}{=} ab\cdot(ac\cdot a) \overset{(1)}{=} (b\cdot ac)b \overset{(9)}{=} b\bullet ac = o,$$

$$(a\bullet c)q = (a\bullet c)\cdot da \overset{(9)}{=} (ac\cdot a)\cdot da \overset{(4)}{=} (a\cdot ca)\cdot da \overset{(1)}{=} (ca\cdot d)\cdot ca \overset{(9)}{=} ca\bullet d \overset{(12)}{=} d\bullet ca = o,$$

and similarly the following equalities $q(b\bullet d) = o$, $(b\bullet d)p = o$ can be proved, so it is valid $S_o(p, a\bullet c, q, b\bullet d)$, and then $\mathrm{Par}_o(p, a\bullet c, q, b\bullet d)$. Because of that we also get the equalities

$$p\bullet q = o = (a\bullet c)\bullet(b\bullet d) = g(a,b,c,d).$$

□

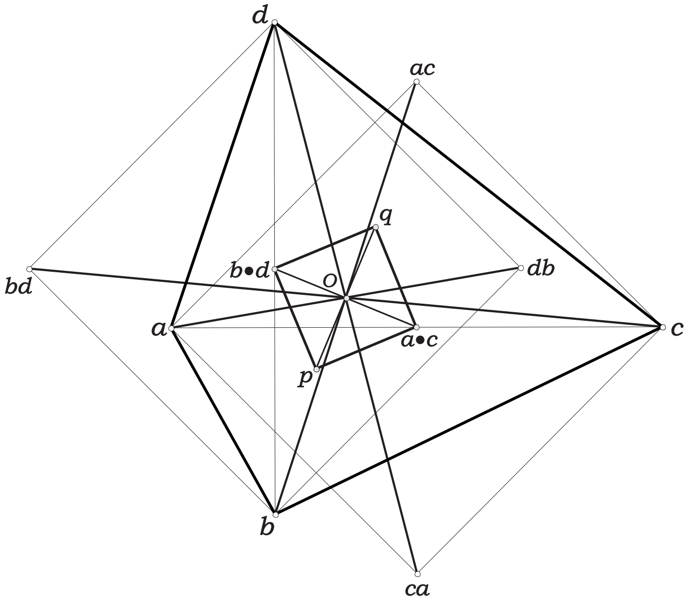

FIGURE 4

In the case of the quasigroup $\mathbb{C}(\frac{1+i}{2})$ Theorem 7 proves one statement from [4] and [7].

The point o from Theorem 7 will be called *centre* of the skewsquare (a, b, c, d).

Theorem 8. $SS(a, b, c, d) \Leftrightarrow S(a \bullet b, b \bullet c, c \bullet d, d \bullet a)$ (*Figure* 5).

PROOF: As we have

$$(a \bullet b)(b \bullet c) \stackrel{(13)}{=} ab \bullet bc,$$
$$(b \bullet c)(c \bullet d) \stackrel{(13)}{=} bc \bullet cd \stackrel{(12)}{=} cd \bullet bc,$$

the equalities $ab = cd$ and $(a \bullet b)(b \bullet c) = (b \bullet c)(c \bullet d)$ are equivalent. The equivalence of the remaining equalities can be proved in a similar way. □

One part of Theorem 8 can be stated more precisely in the form:

Theorem 9. $SS_{p,q}(a, b, c, d) \Rightarrow S_{p \bullet q}(a \bullet b, b \bullet c, c \bullet d, d \bullet a)$ (*Figure* 5).

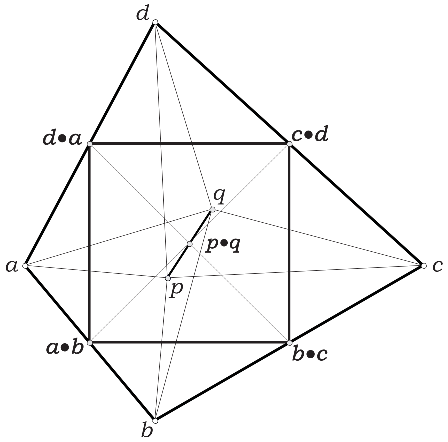

FIGURE 5

PROOF: We have for example

$$(a \bullet b)(b \bullet c) \overset{(13)}{=} ab \bullet bc = p \bullet q,$$

$$(b \bullet c)(c \bullet d) \overset{(13)}{=} bc \bullet cd = q \bullet p \overset{(12)}{=} p \bullet q$$

because of $p = ab = cd$, $q = bc$. □

In a case of the quasigroup $\mathbb{C}(\frac{i+1}{2})$ Theorem 9 proves one statement from [4].

Corollary 4. *The statement $SS(a,b,c,d)$ implies the equalities $(a \bullet b)(c \bullet d) = d \bullet a$, $(b \bullet c)(d \bullet a) = a \bullet b$, $(c \bullet d)(a \bullet b) = b \bullet c$, $(d \bullet a)(b \bullet c) = c \bullet d$ (Figure 5).*

Because of Lemma 3 and Theorem 6 the statement $S_o(a,b,c,d)$ implies $\mathrm{Par}_o(a,b,c,d)$ and $SS(a,b,c,d)$. However, the converse is also valid.

Theorem 10. *$\mathrm{Par}_o(a,b,c,d)$, $SS(a,b,c,d) \Rightarrow S_o(a,b,c,d)$.*

PROOF: Let $SS_{p,q}(a,b,c,d)$. Then according to Theorem 7 we get $S(p,o,q,o)$, since $\mathrm{Par}_o(a,b,c,d)$ implies $a \bullet c = b \bullet d = o$. Because of that we get $p = oo$, $q = oo$, i.e. because of (2) we obtain $p = q = o$, and then $SS_{o,o}(a,b,c,d)$, i.e. owing to Theorem 6 it follows $S_o(a,b,c,d)$. □

Theorem 11. *From statement $\mathrm{Par}_o(a,b,c,d)$ the statements $SS_{o,p}(ac,a,bd,b)$, $SS_{q,o}(ac,d,bd,c)$ follow where p and q are some points such that $qp = o$ (Figure 6).*

PROOF: Owing to (9) we have

$$ac \cdot a = a \bullet c = o, \quad bd \cdot b = b \bullet d = o, \quad d \cdot bd = d \bullet b = o, \quad c \cdot ac = c \bullet a = o,$$

and equalities $ac \cdot a = bd \cdot b$ and $d \cdot bd = c \cdot ac$ prove the first two statements of theorem. Because of that there are points p and q such that $a \cdot bd = b \cdot ac = p$ and $ac \cdot d = bd \cdot c = q$. Finally, we get

$$qp = (bd \cdot c)(a \cdot bd) \stackrel{(9)}{=} c \bullet a = o.$$

□

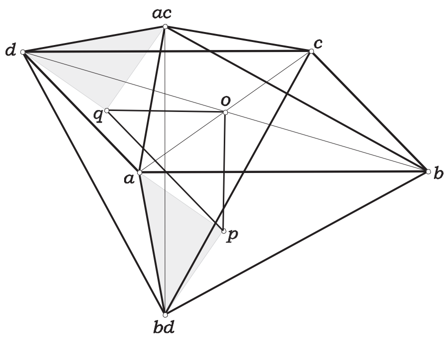

FIGURE 6

Theorem 12. *The validity of the statements $S(a,b,p,q)$, $S(c,a,s,r)$ and $o = cb$ imply the statements $SS_{o,a}(c,b,q,s)$ and $o = qs = p \bullet r$ (Figure 7).*

PROOF: On the basis of Lemma 2 we get equalities $pa = b, bq = a, ar = c, sc = a$. So we get $bq = sc$, wherefrom due to (7) it follows $qs = cb = o$. Besides that owing to (9) and (12) we obtain

$$p \bullet r = r \bullet p = ar \cdot pa = cb = o.$$

□

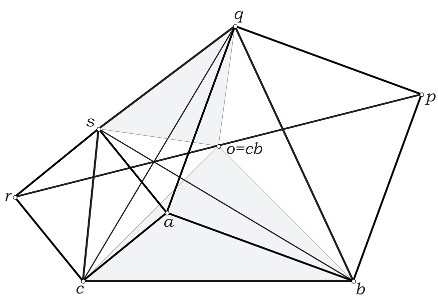

FIGURE 7

In the case of the quasigroup $\mathbb{C}(\frac{1+i}{2})$ Theorem 12 proves some results from [1].

Theorem 13. *The statements* $S_m(a,b,p,q)$ *and* $S_n(c,a,s,r)$ *imply* $S(m, q \bullet s, n, c \bullet b)$.

PROOF: According to Theorem 12 it follows $SS(b,q,s,c)$, and owing to Theorem 8 we get $S(b \bullet q, q \bullet s, s \bullet c, c \bullet b)$. However, because of Lemma 3 it follows $b \bullet q = m$, $s \bullet c = n$, so the statement we are looking for follows. □

Theorem 14. *The statements* $S(a,b,p,q)$, $S(b,a,q',p')$, $S(c,a,s,r)$, $S(a,c,r',s')$ *imply* $SS_{a,o}(p',p,r,r')$, *where* o *is some point* (*Figure* 8).

PROOF: According to Lemma 2 we get equalities $pa = b$, $ap' = b$, $ar = c$, $r'a = c$, so we have $ap' = pa$, $ar = r'a$, wherefrom owing to (8) the equalities $p'p = a$ and $rr' = a$ follow. □

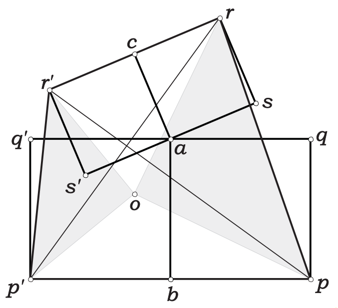

FIGURE 8

Theorem 15. *For any points* a,b,c,d *the statement* $SS_{a \bullet c, b \bullet d}(ba, cb, dc, ad)$ *is valid* (*Figure* 10) (*van Aubel's theorem*).

PROOF: Based on (9) and (12) we get

$$ba \cdot cb = a \bullet c = c \bullet a = dc \cdot ad,$$
$$cb \cdot dc = b \bullet d = d \bullet b = ad \cdot ba.$$

□

In the case of the quasigroup $\mathbb{C}(\frac{1+i}{2})$ Theorem 15 proves the well known statement from (cf. [3], [5], [10]).

With $d = a$ from Theorem 15 we obtain:

Corollary 5. *For any points* a,b,c *the statement* $SS_{a \bullet c, b \bullet a}(ba, cb, ac, a)$ *is valid.*

In the case of the quasigroup $\mathbb{C}(\frac{1+i}{2})$ Corollary 5 proves the known Belatti's result.

If we denote by $\cdot$ the mapping which maps the quadrangle (a,b,c,d) to the quadrangle (ba, cb, dc, ad), and if $\bullet$ denote the mapping which maps quadrangle

(a, b, c, d) to the quadrangle $(a \bullet b, b \bullet c, c \bullet d, d \bullet a)$, then on the basis of Lemma 1, Lemma 4, Theorem 8 and Theorem 15, we get following diagram (*Figure* 9).

In this diagram the operators $\cdot$ and $\bullet$ commute, it means: starting from the same quadrangle in two ways we get the same square. Really, on the basis of (13) we get for example

$$(b \bullet c)(a \bullet b) = ba \bullet cb.$$

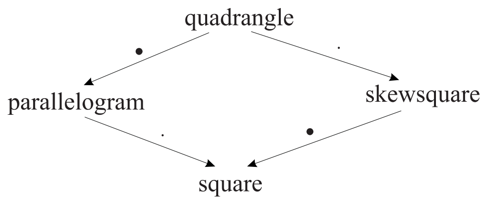

FIGURE 9

Theorem 16. *For any points a, b, c, d it is valid $S(a \bullet c, ba \bullet dc, b \bullet d, cb \bullet ad)$ (Figure 10).*

PROOF: On the basis of (12) and (13) we get

$$(b \bullet d)(a \bullet c) = ba \bullet dc,$$
$$(a \bullet c)(b \bullet d) = (c \bullet a)(b \bullet d) = cb \bullet ad,$$

so the statement follows according to Lemma 2. □

Theorem 17. *With the labels $e_1 = ba \cdot ad$, $e_2 = cb \cdot ba$, $e_3 = dc \cdot cb$, $e_4 = ad \cdot dc$ the statements $SS_{cb \bullet ad,\, ba \bullet dc}(e_1, e_2, e_3, e_4)$, $e_1 \bullet e_3 = a \bullet c$, $e_2 \bullet e_4 = b \bullet d$ hold (Figure 10).*

PROOF: If we apply Theorem 15 on the points ba, cb, dc, ad we will obtain the first statement. Since owing to Theorem 16 the equality $(ba \bullet dc)(cb \bullet ad) = a \bullet c$ holds, we get

$$e_1 \bullet e_3 = (ba \cdot ad) \bullet (dc \cdot cb) \overset{(13)}{=} (ba \bullet dc) \cdot (ad \bullet cb) = a \bullet c,$$

and similarly $e_2 \bullet e_4 = b \bullet d$. □

In the case of the quasigroup $\mathbb{C}(\frac{1+i}{2})$ Theorems 15, 16 and 17 prove results from [10] and [9].

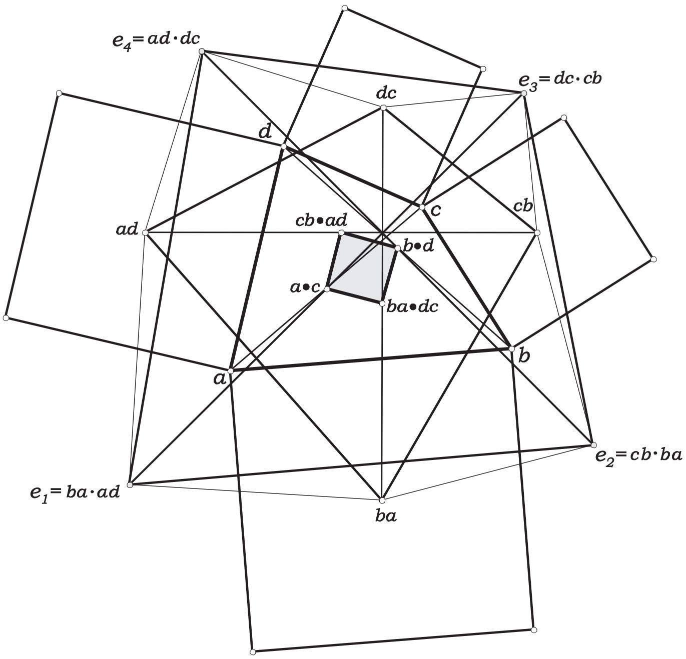

FIGURE 10

Theorem 18. *For any points $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8$ let us denote $a_{i,i+1} = a_i a_{i+1}$, $m_{i,i+1,i+4,i+5} = a_{i,i+1} \bullet a_{i+4,i+5}$, where indexes are taken modulo 8 from the set $\{1,2,3,4,5,6,7,8\}$. If $p = g(a_2, a_4, a_6, a_8)$, $q = g(a_1, a_3, a_5, a_7)$, then we get $SS_{p,q}(m_{1256}, m_{4581}, m_{7834}, m_{2367})$ (Figure 11).*

PROOF: On the bases of (9), (12) and (13) we get for example

$$\begin{aligned} m_{1256} m_{4581} &= (a_{12} \bullet a_{56})(a_{45} \bullet a_{81}) = (a_{12} \bullet a_{56})(a_{81} \bullet a_{45}) \\ &= a_{12} a_{81} \bullet a_{56} a_{45} = (a_1 a_2 \cdot a_8 a_1) \bullet (a_5 a_6 \cdot a_4 a_5) \\ &= (a_2 \bullet a_8) \bullet (a_6 \bullet a_4) = g(a_2, a_4, a_6, a_8) = p. \end{aligned}$$

□

In the case of the quasigroup $\mathbb{C}(\frac{1+i}{2})$ Theorem 18 proves the result stated in [3], [6] and [9]:

The centres of squares constructed on the sides of the given octagon determine new octagon, and the midpoints of the main diagonals of the obtained octagon determine an skewsquare (Figure 11).

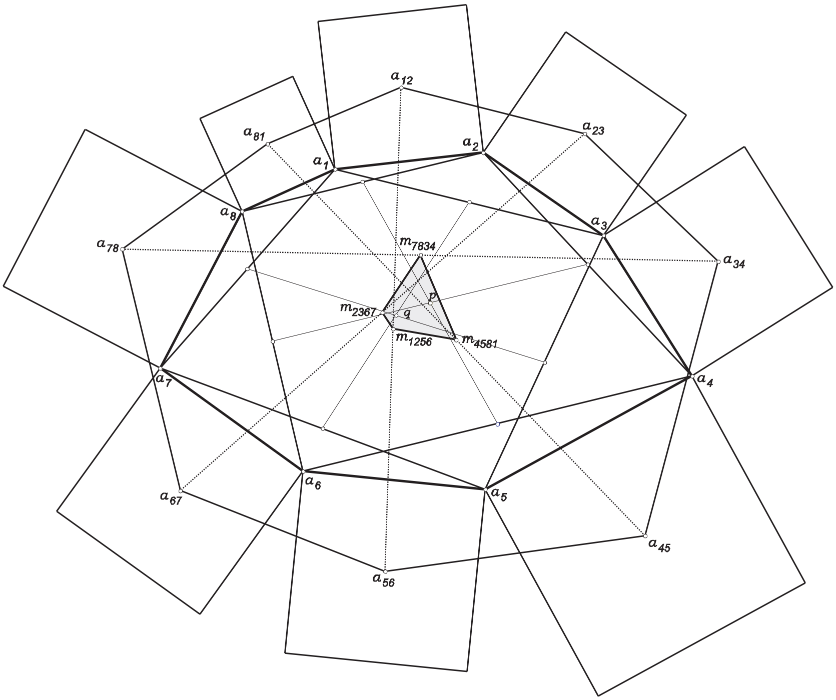

FIGURE 11

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