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Locally soluble-by-finite groups with small deviation for non-subnormal subgroups

LEONID A. KURDACHENKO, HOWARD SMITH

Abstract. A group G has subnormal deviation at most 1 if, for every descending chain $H_0 > H_1 > \dots$ of non-subnormal subgroups of G , for all but finitely many i there is no infinite descending chain of non-subnormal subgroups of G that contain H_{i+1} and are contained in H_i . This property \mathfrak{P} , say, was investigated in a previous paper by the authors, where soluble groups with \mathfrak{P} and locally nilpotent groups with \mathfrak{P} were effectively classified. The present article affirms a conjecture from that article by showing that locally soluble-by-finite groups with \mathfrak{P} are soluble-by-finite and are therefore classified.

Keywords: subnormal subgroups, locally soluble-by-finite groups

Classification: 20E15, 20F19

This article continues the study of groups with “small deviation for non-subnormal subgroups”; more precisely, of groups G for which this deviation is at most 1, where we write $\text{dev}_{\text{non-sn}}(G) \leq 1$. The discussion of groups with this property was begun in [6], following some general results on groups G for which the non-subnormal deviation exists at all. We refer the reader to [6] for the more general case and for precisely what it means to say that the non-subnormal deviation of a group G *exists*, but we recall here that a group G has non-subnormal deviation at most 1 if, for every descending chain $H_0 > H_1 > H_2 > \dots$ of non-subnormal subgroups of G , almost all of the “intervals” $I[H_{i+1}, H_i]$ satisfy the minimal condition for non-subnormal subgroups of G , that is, for all but finitely many i there is no infinite descending chain of non-subnormal subgroups of G all of which contain H_{i+1} and are contained in H_i . It was observed in [6] that a group with the weak minimal condition for non-subnormal subgroups has non-subnormal deviation at most 1 but that the converse is false — groups with this weak minimal condition were the topic of discussion in [4] and [5]. The main results of [6] were concerned with soluble groups G satisfying $\text{dev}_{\text{non-sn}}(G) \leq 1$ and with locally nilpotent groups G with this property. We summarize the results on soluble groups as follows.

Theorem 1. *Let G be a soluble group such that $\text{dev}_{\text{non-sn}}(G) \leq 1$, and let T be the torsion subgroup of the Baer radical B of G .*

- (i) *If B is minimax then G too is minimax.*

- (ii) If B/T is not minimax then $G = B$ and every subgroup of G is subnormal.
- (iii) If B/T is minimax but T is not minimax then T contains all elements of G of finite order, G/T is minimax and T satisfies $\min - \langle g \rangle$ for each element $g \in G \setminus B$. Furthermore, every subgroup of B is subnormal.

We were also able to show that a locally nilpotent group G with $\text{dev}_{\text{non-sn}}(G) \leq 1$ is soluble and of a certain restricted structure. It is clear from the above theorem that the structure of a soluble-by-finite group with non-subnormal deviation at most 1 has also been determined, and we conjectured in [6] that a locally soluble-by-finite group with deviation at most 1 is soluble-by-finite. Verification of this conjecture is the main purpose of the present article.

Theorem 2. *Let G be a locally soluble-by-finite group such that $\text{dev}_{\text{non-sn}}(G) \leq 1$. Then G is soluble-by-finite.*

We approach the proof of this theorem by means of a sequence of lemmas.

Lemma 1. *Let G be a locally finite group satisfying $\text{dev}_{\text{non-sn}}(G) \leq 1$. Then G is soluble-by-finite.*

PROOF: If G is Chernikov then the result is clear, while if G is not Chernikov then it is a Baer group, by Lemma 2.8 of [6], hence locally nilpotent and hence soluble, as pointed out above. \square

Recall next that a group G is *radical* if it is the union of an ascending series of normal subgroups with successive factor groups locally nilpotent.

Lemma 2. *Let G be a radical group satisfying $\text{dev}_{\text{non-sn}}(G) \leq 1$. Then G is soluble.*

PROOF: By Theorem 5.7 of [6] every locally nilpotent section of G is soluble. Let H_1 be the locally nilpotent radical of G , H_2/H_1 the locally nilpotent radical of G/H_1 , and let T denote the maximal normal torsion subgroup of G . By Lemma 1 T is soluble and, factoring, we may assume that $T = 1$. Let $B = B(G)$ denote the Baer radical of G ; then $B = B(H_2)$. If B is not minimax then $H_2 = B$, by Theorem 1, and so $G = H_1$, which is soluble. But if B is minimax then every abelian subgroup of G is minimax, by Corollary 2.3 of [6], and then G is minimax, by Theorem 10.35 of [8], and hence soluble. This establishes the result. \square

Lemma 3. *Let G be a finitely generated soluble-by-finite group satisfying $\text{dev}_{\text{non-sn}}(G) \leq 1$. Then G is minimax.*

PROOF: We may suppose that G is soluble. If G is not minimax then it has infinite (Prüfer) rank, by Theorem 10.38 of [8], and hence a section K isomorphic to the wreath product of a cyclic group of prime order by an infinite cyclic group, by [3]. But the Baer radical of K is abelian of prime exponent and clearly does not satisfy $\min - K$, and Theorem 1(iii) gives a contradiction. \square

Our next result reduces the proof of Theorem 2 to that of the locally soluble case. This is followed by two straightforward lemmas, the first of which is well-known. These will assist in establishing a Proposition that, in turn, provides a key step in the final stage of the proof of the theorem. In the statement of Lemma 6 (and elsewhere) we employ for convenience the term “periodic index” to indicate that a subgroup H of a group G has the property that every element of G has a non-zero power in H .

Lemma 4. *Suppose that every locally soluble group with non-subnormal deviation at most 1 is soluble. Then every locally soluble-by-finite group with non-subnormal deviation at most 1 is soluble-by-finite.*

PROOF: Let G be a locally soluble-by-finite group with $\text{dev}_{\text{non-sn}}(G) \leq 1$ and suppose that G is not soluble-by-finite. If there are integers d and k such that every finitely generated subgroup of G has its soluble radical of derived length at most d and index at most k then G too has this property, by Proposition 1.K.2 of [2]. Thus we may assume that G is countable and, by Lemma 2, we may further assume (by factoring if necessary) that G has trivial Hirsch-Plotkin radical. G is an ascending union of finitely generated subgroups $F_1 < F_2 < \dots$; for each i the soluble radical S_i of F_i has finite index, and it is easy to see that the subgroup S generated by all the S_i is locally soluble and hence, by hypothesis, soluble. Since G has trivial Baer radical we deduce from Corollary 2.3 of [6] that every abelian subgroup of G is minimax and hence that S is minimax, by Theorem 10.35 of [8]. It follows that there is a finite upper bound r for the torsion-free ranks of the subgroups F_i . By Lemma 3 each F_i is minimax and hence nilpotent-by-abelian-by-finite (by Theorem 3.25 of [8]), and we may apply Proposition 1 of [1] to deduce that G has a finite ascending normal series whose factors are soluble or locally finite. But each locally finite factor of this series is soluble-by-finite, by Lemma 1, and since each finite G -invariant factor of G has its centralizer of finite index in G we easily obtain the contradiction that G is soluble-by-finite. \square

Lemma 5. *Let X be a finitely generated nilpotent-by-finite group, Y a subgroup of X , and let U be a normal subgroup of finite index in Y . Then there is a normal subgroup N of finite index in X such that $N \cap Y \leq U$.*

PROOF: There is a normal nilpotent subgroup V of finite index in X , and if $M \cap (Y \cap V) \leq U \cap V$ for some normal subgroup M of finite index in V then, with $N = \text{Core}_X(M)$, we have X/N finite and $N \cap Y = N \cap (Y \cap V) \leq (U \cap V) \leq U$, and the result follows. Thus we may suppose that $X = V$ and in particular that Y is subnormal in X . If Y is normal in X then $Y/\text{Core}_X(U)$ is finite, and since $X/\text{Core}_X(U)$ is residually finite the result follows in this case. By induction on the subnormal defect of Y in X we may assume that there is a normal subgroup W of X containing Y , and a normal subgroup K of finite index in W , such that $K \cap Y \leq U$; replacing K by its core if necessary we may assume that K is normal in X . Since X/K is residually finite there is a normal subgroup N of finite index

in X such that $N \cap W = K$, and then $N \cap Y = N \cap W \cap Y = K \cap Y \leq U$, and the lemma is proved. \square

Lemma 6. *Let G be a locally soluble group with $\text{dev}_{\text{non-sn}}(G) \leq 1$ and suppose that H is a subnormal subgroup of G that has periodic index in G . Then there is a positive integer d such that $G^{(d)} \leq H$ (where $G^{(d)}$ denotes the d th term of the derived series of G).*

PROOF: There is a finite subnormal series from H to G , and each factor of this series is locally finite and hence soluble, by Lemma 1. \square

The following Proposition, with its rather technical hypotheses, will allow us to deal, in particular, with the case where G is locally nilpotent-by-finite, and its generality will also allow us to reduce to that case. Much of the proof of the Proposition is similar to that of Theorem 4 of [1], though there are significant differences.

Proposition. *Let G be a locally soluble group that is the ascending union of a chain of finitely generated subgroups $F_1 < F_2 < \dots$ and, for each i , let K_i be a normal subgroup of F_i such that F_i/K_i is nilpotent-by-finite, F_1/K_1 and each index $[F_{i+1} : F_i K_{i+1}]$ is infinite, and $K_{i+1} \cap F_i \leq K_i$. Suppose also that F_i has derived length exactly d_i and that F_{i+1}/K_{i+1} has derived length at least $2d_i$ (for each i). Then there are subgroups M and L of G such that $L \leq M$, L has periodic index in G and $G^{(d)} \not\leq M$ for any positive integer d , and there is an infinite descending chain of subgroups of M that contain L .*

PROOF: There is a nilpotent normal subgroup M_1/K_1 of finite index in F_1/K_1 and a proper F_1 -invariant subgroup L_1/K_1 of finite index in M_1/K_1 . By Lemma 5, applied to the group F_2/K_2 , there is a normal subgroup M_2/K_2 of finite index in F_2/K_2 such that $M_2 \cap F_1 \leq K_2 L_1$ and so $M_2 \cap F_1 \leq K_2 L_1 \cap F_1 = L_1 (K_2 \cap F_1) = L_1$. Clearly we may choose M_2 such that M_2/K_2 is nilpotent and F_2/M_2 has derived length equal to that of F_2/K_2 (using the residual finiteness of F_2/K_2). Since F_2/K_2 is polycyclic, the intersection of all subgroups U of finite index in F_2 that contain $K_2 F_1$ is $K_2 F_1$, which has infinite index in F_2 . So we may choose such a U that does not contain M_2 , and then $L_2 := \text{Core}_{F_2}(U \cap M_2)$ contains K_2 , has finite index in M_2 , and is such that M_2 is not contained in $F_1 L_2$. We continue in this manner: for each i there is a normal nilpotent subgroup M_{i+1}/K_{i+1} of finite index in F_{i+1}/K_{i+1} such that $M_{i+1} \cap F_i \leq L_i$ and F_{i+1}/M_{i+1} has derived length equal to that of F_{i+1}/K_{i+1} , and an F_{i+1} -invariant subgroup L_{i+1}/K_{i+1} of finite index in M_{i+1}/K_{i+1} such that $M_{i+1} \not\leq F_i L_{i+1}$. Let $L = \langle L_i : i \geq 1 \rangle$, $M = \langle M_i : i \geq 1 \rangle$; clearly L has periodic index in G and $L \leq M$.

Suppose next that $G^{(d)} \leq M$ for some integer d , and choose $i > 1$ such that $d_{i-1} > d$. We have $F_i^{(d)} \leq M \cap F_i$, and since $M_{j+1} \cap F_j \leq M_j$ for all j it follows that $M \cap F_i = M_1 \dots M_i$ (see Lemma 2 of [1]) and hence that $F_i^{(d)} \leq F_{i-1} M_i$.

But this gives $F_i^{(d+d_{i-1})} \leq M_i$, so $d_i \leq (d + d_{i-1}) < 2d_{i-1}$, a contradiction.

Now let $H_0 = M$ and, for $i \geq 1$, let $H_i = (L_1 \dots L_i)(M_{i+1}M_{i+2} \dots)$; thus $L \leq H_i \leq M$ for all i . Assume for a contradiction that the chain $H_0 \geq H_1 \geq H_2 \geq \dots$ terminates after finitely many steps, so that $M_t \leq (L_1 \dots L_t)(M_{t+1}M_{t+2} \dots)$ for some t . Now if $x \in (M_{t+1} \dots M_{t+k}) \cap F_t$ for some $k > 1$ then

$$x \in (M_{t+1} \dots M_{t+k-1})(M_{t+k} \cap F_{t+k-1}) \cap F_t = (M_{t+1} \dots M_{t+k-1}) \cap F_t,$$

and an easy induction on k shows that $x \in M_{t+1} \cap F_t \leq L_t$. Thus

$$\begin{aligned} M_t \leq (L_1 \dots L_t)(M_{t+1}M_{t+2} \dots) \cap F_t &= (L_1 \dots L_t)((M_{t+1}M_{t+2} \dots) \cap F_t) \\ &= (L_1 \dots L_t) \leq F_{t-1}L_t, \end{aligned}$$

contradicting the choice of L_t . Thus the H_i form an infinite descending chain, and the Proposition is proved. \square

PROOF OF THEOREM 2: By Lemma 4 we may assume that G is locally soluble, and our aim is to show that G is soluble, so we may also assume that G is countable. Suppose the result false. The iterated Hirsch-Plotkin radical of G is soluble, by Lemma 2, so we may factor and assume that G has no nontrivial subnormal soluble subgroups; as in the proof of Lemma 4, every soluble subgroup of G is therefore minimax. If every insoluble subgroup of G is subnormal then G satisfies the weak minimal condition for non-subnormal subgroups (since soluble minimax groups satisfy the weak minimal condition for *all* subgroups), and Theorem B of [5] implies that either G is soluble or G has all subgroups subnormal, in which case G is again soluble [7], a contradiction. Thus we see that every insoluble subgroup of G contains a non-subnormal insoluble subgroup, and since $\text{dev}_{\text{non-sn}}(G) \leq 1$ there exists an insoluble subgroup G_0 of G such that, if $X < Y$ are any non-subnormal insoluble subgroups of G_0 then the interval $I[X, Y]$ satisfies the minimal condition for non-subnormal subgroups of G . We may assume that $G = G_0$ and, again factoring if necessary, we may retain the condition that the Hirsch-Plotkin radical of G is trivial and hence that every soluble subgroup is minimax.

By Lemma 3, G is the ascending union of finitely generated subgroups $F_1 < F_2 < \dots$, each of which is minimax. For each i , let K_i denote the locally nilpotent radical of F_i ; then K_i is nilpotent and F_i/K_i is abelian-by-finite, by Theorem 3.25 of [8]. Certainly $F_i \cap K_{i+1} \leq K_i$ for each i . If there is an upper bound b for the derived lengths of the factors F_i/K_i then $G^{(b)}$ is locally nilpotent and therefore trivial, and we have the contradiction that G is soluble. Passing to an appropriate subsequence of the F_i if necessary, we may therefore assume that the derived length of F_{i+1}/K_{i+1} is at least twice that of F_i , for each i .

If there is no upper bound for the torsion-free ranks (that is, Hirsch lengths) of the F_i/K_i then we may again re-label the F_i and assume that the torsion-free

rank of F_{i+1}/K_{i+1} is greater than that of F_i for each i and that F_1/K_1 is infinite. But now G satisfies the hypotheses and hence the conclusion of the Proposition and, with the notation of the Proposition, we may apply Lemma 6 to deduce that neither L nor any of the subgroups H_i is subnormal in G , so the interval $I[L, M]$ does not satisfy min for non-subnormal subgroups of G , so that L must be soluble and hence minimax, which gives, since L has periodic index in G , the contradiction that the torsion-free ranks of the F_i are bounded. It follows that the torsion-free rank of each F_{i+1}/K_{i+1} is at most r , for some fixed positive integer r .

For each i there is a normal torsion-free abelian subgroup U_i/K_i of finite index in F_i/K_i , and the rank of each U_i/K_i is at most r . By Theorem 3.23 of [8], there is an integer $d = d(r)$ such that (for all i), if C_i denotes the centralizer in F_i of U_i/K_i , then F_i/C_i has derived length at most d . As C_i/K_i is centre-by-finite it has finite derived group, and so $F_i^{(d+1)}$ is nilpotent-by-finite, which gives $G^{(d+1)}$ locally nilpotent-by-finite, so there is no loss in assuming that G is locally nilpotent-by-finite, that is, F_i/K_i is finite for each i . Observe that, by factoring by the iterated Hirsch-Plotkin radical if necessary, we may again retain the condition that all soluble subgroups of G are minimax.

If the torsion-free ranks of the F_i are bounded above then, as in the proof of Lemma 4, we may apply Proposition 1 of [1] to deduce that G has a finite ascending normal series whose factors are soluble or locally finite and hence to obtain the contradiction that G is soluble. Thus we may assume that the torsion-free ranks of the F_i increase with i . But now G satisfies the hypotheses of the Proposition, with each “ K_i ” of those hypotheses trivial, and as in the earlier part of the present proof we have subgroups L and M and an infinite descending chain of subgroups H_i of M , each containing L , such that no H_i is subnormal in G . Thus L is soluble and hence minimax and, since L has periodic index in G , we obtain the contradiction that the torsion-free ranks of the F_i are bounded. This completes the proof of Theorem 2. \square

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