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Abstract. In this paper, it is proved that a first-countable paratopological group has a regular G_δ-diagonal, which gives an affirmative answer to Arhangel'skii and Burke's question [*Spaces with a regular G_δ-diagonal*, Topology Appl. **153** (2006), 1917–1929]. If G is a symmetrizable paratopological group, then G is a developable space. We also discuss copies of S_ω and of S_2 in paratopological groups and generalize some Nyikos [*Metrizability and the Fréchet-Urysohn property in topological groups*, Proc. Amer. Math. Soc. **83** (1981), no. 4, 793–801] and Svetlichnyi [*Intersection of topologies and metrizability in topological groups*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **4** (1989), 79–81] results.

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Classification: Primary 54H13, 54H99

1. Introduction

Recently, paratopological groups have been studied by many topologists ([3], [4], [19]). It is natural to ask what results on topological groups are valid on paratopological groups. In this paper, by discussing copies of S_ω and of S_2 on paratopological groups, we generalize some results from [14], [15] and [18]. We also discuss first-countable paratopological groups and prove that a first-countable paratopological group has a regular G_δ-diagonal, and give an affirmative answer to a question from [3].

Recall that a *paratopological group* is a group with a topology such that the multiplication is jointly continuous.

All spaces are regular T_1 unless stated otherwise. $\mathbb{N}$ denotes natural numbers and e denotes the neutral element of a group. We refer to [6] for notations and terminology not given explicitly.

2. Main results

A space X is said to have a *regular G_δ-diagonal* if the diagonal $\Delta = \{(x,x) : x \in X\}$ can be represented as the intersection of the closures of a countable family of open neighborhoods of Δ in $X \times X$. According to Zenor [21], a space X has a regular G_δ-diagonal if and only if there exists a sequence $\{\mathcal{G}_n : n \in \omega\}$ of open covers of X with the following property:

(*) For any two distinct points y and z in X, there are open neighborhoods O_y and O_z of y and z, respectively, and $k \in \omega$ such that no element of $\mathcal{G}_n$ intersects both O_x and O_y.

In [3], Arhangel'skii and Burke proved that every Hausdorff first countable Abelian paratopological group G has a regular G_δ-diagonal. We sharpen the result by showing the following

Theorem 2.1. *Let G be a Hausdorff first-countable paratopological group. Then G has a regular G_δ-diagonal.*

Proof: Fix a countable base $\{V_n : n \in \mathbb{N}\}$ at the neutral element e in G with $V_{n+1}^2 \subset V_n$. Let $x \in G$; then xV_n, V_nx are open for $n \in \mathbb{N}$ since G is a paratopological group. For $x \in G$, $n \in \mathbb{N}$, let $W_n(x) = xV_n \cap V_nx$. Then $W_n(x)$ is a neighborhood of x. Let $\mathcal{G}_n = \{W_n(x) : x \in G\}$ for $n \in \mathbb{N}$. Then $\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a sequence of open coverings of G.

By Zenor's characterization of regular G_δ-diagonal, we only prove the following

Claim: *For $y, z \in G$, $y \neq z$, there is $k \in \mathbb{N}$ such that no element of $\mathcal{G}_k$ intersects both yV_k and zV_k.*

Suppose not; for any $n \in \mathbb{N}$, there is an element $W_n(x_n) \in \mathcal{G}_n$ such that $yV_n \cap W_n(x_n) \neq \emptyset$ and $W_n(x_n) \cap zV_n \neq \emptyset$. Then there are a_n, b_n, c_n, d_n and f_n in V_n such that $ya_n = x_nb_n$, $x_nc_n = d_nx_n = zf_n$, $ya_n = d_n^{-1}d_nx_nb_n = d_n^{-1}zf_nb_n$. Since $a_n \to e$, we have $ya_n \to y$, hence $d_n^{-1}zf_nb_n \to y$. $d_n \to e$ since $d_n \in V_n$, G is a paratopological group, then $d_nd_n^{-1}zf_nb_n \to ey = y$, hence $zf_nb_n \to y$. Notice that $f_n, b_n \in V_n$, thus $f_nb_n \to e$, hence $zd_nb_n \to z$. G is Hausdorff, then $y = z$, this is a contradiction.

Therefore, G has a regular G_δ-diagonal. □

A subset A of a space X is said to be *bounded* [3] in X if every infinite family ξ of open subsets of X such that $V \cap A \neq \emptyset$, for every $V \in \xi$, has an accumulation point X. If X is bounded in itself, then we say that X is *pseudocompact*.

Notice that a pseudocompact or bounded subset of a regular space X is metrizable if X has a regular G_δ-diagonal [3]. We have the following

Corollary 2.1. *Let G be a regular first-countable paratopological group. Then every pseudocompact subspace of G is a metrizable compactum.*

Corollary 2.2. *Let G be a regular first-countable paratopological group. Then every bounded subspace of G is metrizable.*

The above theorem and corollaries give an affirmative answer to Arhangel'skii and Burke's question [3, Problem 25].

A space X is an $w\Delta$-*space* [8] if there exists a sequence $(\mathcal{G}_n)$ of open covers of X such that if $x_n \in \operatorname{st}(x, \mathcal{G}_n)$ for each $n \in \mathbb{N}$, then the set $\{x_n : n \in \mathbb{N}\}$ has a cluster point in X.

Since a space X with a regular G_δ-diagonal has a G^*_δ-diagonal, by [8, Theorem 3.3], we have the following

Corollary 2.3. *Let G be a first-countable paratopological group. Then G is a Moore space if G is an $w\Delta$-space.*

A space X is *quasi-developable* [8] if there exists a sequence $(\mathcal{G}_n)$ of families of subsets of X such that for each $x \in X$, $\{\mathrm{st}(x, \mathcal{G}_n) : n \in \mathbb{N}\}$ is a base at x. Recall that a topological space is said to be *symmetrizable* if its topology is generated by a *symmetric*, that is, by a distance function satisfying all the usual restrictions on a metric, except for the triangle inequality [1].

Theorem 2.2. *Every symmetrizable paratopological group G is a Moore space.*

PROOF: We fix a symmetric d on the paratopologcial group G generating the topology on G. Since G is weakly first-countable [1], by a result of Nyikos [15], G is first-countable. Put $B(x, 1/n) = \{y \in G : d(x, y) < 1/n\}$, and fix an open base $\{V_n : n \in \mathbb{N}\}$ at e with $V_n \subset \mathrm{int}(B(e, 1/n))$ and $V^2_{n+1} \subset V_n$. Let $A_{ij} = \{x \in G : V_i x \subset \mathrm{int}(B(x, 1/j))\}$ and $\mathcal{G}_{ij} = \{V_i x : x \in A_{ij}\}$ for $i, j \in \mathbb{N}$. Since $\{V_i x : i \in \mathbb{N}\}$ and $\{\mathrm{int}(B(x, 1/j)) : j \in \mathbb{N}\}$ are bases at x, $G = \bigcup\{A_{ij} : i, j \in \mathbb{N}\}$. We prove that $\{\mathrm{st}(x, \mathcal{G}_{ij}) : i, j \in \mathbb{N}\}$ is a base at $x \in G$. Let U be an open subset of X with $x \in U$. There exists $k \in \mathbb{N}$ such that $x \in \mathrm{int}(B(x, 1/k)) \subset U$ and pick $m, n \in \mathbb{N}$ such that $m < n$, $V_n x \subset V_m x \subset \mathrm{int}(B(x, 1/k))$. We choose k' such that $B(x, 1/k') \subset V_n x$ since $\{B(x, 1/i) : i \in \mathbb{N}\}$ is a weak base at x. For $x \in V_n y \in \mathcal{G}_{nk'}$, since $V_n y \subset B(y, 1/k')$, $d(x, y) = d(y, x) < 1/k'$, hence $y \in B(x, 1/k') \subset V_n x$. $V_n y \subset V_n V_n x \subset V_m x \subset \mathrm{int}(B(x, 1/k)) \subset U$, hence $x \in \mathrm{st}(x, \mathcal{G}_{nk'}) \subset U$. Therefore G is quasi-developable.

G is symmetrizable and first-countable, hence G is semi-stratifiable [8, Theorem 9.8], thus every closed subset of G is a G_δ-set. Therefore G is a developable space [8, Theorem 8.6]. □

We cannot replace "symmetrizable" with "first-countable" in Theorem 2.2, Sorgenfrey line is a first-countable paratopological group but not a Moore space.

Let S_κ be the quotient space obtained by identifying all limit points of the topological sum of κ many convergent sequences. S_ω is called sequential fan. The Arens' space $S_2 = \{\infty\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_n(m) : m, n \in \mathbb{N}\}$ is defined as follows: Each $x_n(m)$ is isolated; a basic neighborhood of x_n is $\{x_n\} \cup \{x_n(m) : m > k$, for some $k \in \mathbb{N}\}$; a basic neighborhood of ∞ is $\{\infty\} \cup (\bigcup\{V_n : n > k$ for some $k \in \mathbb{N}\})$, where V_n is a neighborhood of x_n.

In [14], it was proved that a topological group contains a (closed) copy of S_ω if and only if it contains a (closed) copy of S_2. We do not know if the result is still true for paratopological groups, but we have the following theorem by modifying Lemma 2.1 in [14].

Theorem 2.3. *Let G be a paratopological group. Then G contains a* (*closed*) *copy of S_ω if G has a* (*closed*) *copy of S_2.*

Proof: Let $A = \{e\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_n(m) : m, n \in \mathbb{N}\}$ be a closed copy of S_2, where e is the neutral element of G. For $n, m \in \mathbb{N}$, let $y_n(m) = x_n^{-1}x_n(m)$. Then $y_n(m) \to e$ as $m \to \infty$ for $n \in \mathbb{N}$. For each n, let $S_n = \{y_n(m) : m \in \mathbb{N}\}$. Then $F = \{n : S_m \cap S_n$ is infinite $\}$ is finite (otherwise, pick distinct $x_{n_i}^{-1}x_{n_i}(m_i) \in S_m \cap S_{n_i}$ for $n_i \in F$ with $n_i < n_{i+1}$, $x_{n_i}^{-1}x_{n_i}(m_i) \to e$, $x_{n_i} \to e$, hence $x_{n_i}(m_i) \to e$, a contradiction). Without loss of generality, we assume $S_i \cap S_j = \emptyset$ if $i \neq j$. Let $B = \{e\} \cup \{y_n(m) : n, m \in \mathbb{N}\}$.

Claim: *B is a closed copy of S_ω.*

Suppose B is not closed. Then there is $x \in X \backslash B$ with $x \in \overline{B}$. Since A is closed, there exists an open neighborhood V of the neutral element e such that Vx meets $\{x_n(m) : m \in \mathbb{N}\}$ for at most one n. Let U be open neighborhood of e with $U^2 \subset V$; Ux contains an infinite subset $\{y_{n_i}(m_i) : i \in \mathbb{N}\}$ of B. Since $x_n \to e$, without loss of generality, $\{x_{n_i} : i \in \mathbb{N}\} \subset U$. $\{x_{n_i}y_{n_i}(m_i) : i \in \mathbb{N}\} \subset UUx \subset Vx$, it means $\{x_{n_i}(m_i) : i \in \mathbb{N}\} \subset Vx$, a contradiction.

If $f : \omega \to \omega$, then $C = \bigcup\{y_n(m) : m \leq f(n), n \in \mathbb{N}\}$ does not have a cluster point. Otherwise, there exists $x \in \overline{C \backslash \{x\}}$. Let V be an open neighborhood V of the neutral element e such that Vx meets $|Vx \cap \{x_n(m) : m \leq f(n), n \in \mathbb{N}\}| \leq 1$. Let U be open neighborhood of e with $U^2 \subset V$, Ux contains an infinite subset $\{y_{n_i}(m_i) : i \in \mathbb{N}\} \subset C$, hence $x_{n_i}(m_i) = x_{n_i}y_{n_i}(m_i) \in UUx \subset Vx$ for each $i \in \mathbb{N}$, which is a contradiction. Hence B is a copy of S_ω. □

Nogura, Shakhmatov and Tanaka proved the following corollary as G is a topological group [14]. By Theorem 2.3, we can see the following corollary is still true for a paratopological group G.

Note that a sequential space is an A-space[1] if and only if it contains no closed copy of S_ω [20]. By Theorem 2.3, a paratopological group contains no closed copy of S_2 if it is an A-space. A sequential space that each point is a G_δ-set or is hereditarily normal is strongly Fréchet if it contains no closed copy of S_ω and S_2 [20, Theorem 3.1]. A strongly Fréchet space is an α_4-space[2] [2, Theorem 5.26].

Corollary 2.4. *Suppose that G is a sequential paratopological group such that either* (a) *$e \in G$ is a G_δ-set, or* (b) *G is hereditarily normal. Then the following*

[1]A space X is an A-space if, whenever $\{A_n : n \in \mathbb{N}\}$ is a decreasing sequence of subsets of X, and $x \in X$ is a point with $x \in \bigcap\{\overline{A_n \backslash \{x\}} : n \in \mathbb{N}\}$, then for every $n \in \mathbb{N}$ one can find a (possibly empty) set $B_n \subset A_n$ such that $\bigcup\{\overline{B_n} : n \in \mathbb{N}\}$ is not closed in X.

[2]A countable collection $\{S_n : n \in \mathbb{N}\}$ of convergent sequences in a space X is called a *sheaf* (*with a vertex x*) if each sequence S_n converges to the same point $x \in X$. A space is called α_4-space, if for every point $x \in X$ and each sheaf $\{S_n : n \in \mathbb{N}\}$ with the vertex x, there exists a sequence converging to x which meets infinitely many sequences S_n.

are equivalent:

(1) *G is an α_4-space;*
(2) *G is an A-space, and*
(3) *G is strongly Fréchet.*

A paratopological group G is said to have the property (**), if there exists a sequence $\{x_n : n \in \mathbb{N}\} \subset G$ such that $x_n \to e$ and $x_n^{-1} \to e$. Obviously, every topological group has the property (**). Not every paratopological group has the property (**), for instance, Sorgenfrey line $\mathbb{S}$ does not have the property (**). A paratopological group having the property (**) need not be a topological group: for instance, if $(\mathbb{R}, +)$ is the real line with the usual topology, then $\mathbb{S} \times \mathbb{R}$ is a paratopological group having the property (**) but not a topological group.

Theorem 2.4. *Let G be a paratopological group having the property* (**). *Then G has a* (*closed*) *copy of S_2 if it has a* (*closed*) *copy of S_ω.*

PROOF: Let $A = \{e\} \cup \{y_n(m) : m, n \in \mathbb{N}\}$ be a closed copy of S_ω, for each n, $y_n(m) \to e$ as $m \to \infty$. Since G has the property (**), there is a sequence $\{x_n : n \in \mathbb{N}\}$ such that $x_n \to e$ and $x_n^{-1} \to e$. Let U_n be an open neighborhood of x_n for each n with $\overline{U_i} \cap \overline{U_j} = \emptyset$ if $i \neq j$. Let $x_n(m) = x_n y_n(m)$ for $n, m \in \mathbb{N}$. For any $n \in \mathbb{N}$, we have $x_n(m) \to x_n$ as $m \to \infty$. Without loss of generality, we assume $\{x_n(m) : m \in \mathbb{N}\} \subset U_n$. Let $B = \{e\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_n(m) : n, m \in \mathbb{N}\}$.

Claim: *B is a closed copy of S_2.*

Suppose B is not closed. Then there exists $x \notin B, e \neq x \in \overline{B \backslash \{x\}}$. Since A is closed, there is a neighborhood of e such that $Vx \cap (A \backslash \{x\}) = \emptyset$. Let U be a neighborhood of e with $U^2 \subset V$ and Ux contains at most one x_n. Ux contains infinitely many elements of B, since U contains infinitely many x_n^{-1}'s, UUx contains infinitely many $y_n(m)$. Hence Vx contains infinitely many elements of A, this is a contradiction.

If $f : \omega \to \omega$, similarly as in the proof of Theorem 2.3, $\{x_n(m) : n \geq k$ for some $k, m \leq f(n)\}$ is closed. Hence B is a closed copy of S_2. □

Note that a Fréchet-Urysohn space contains no closed copy of S_2, then a Fréchet-Urysohn paratopological group having the property (**) contains no closed copy of S_ω by Theorem 2.4, hence it is a strongly Fréchet space [20] (or countably bisequential space [13]), therefore it is an α_4-space [2, Theorem 5.23].

Corollary 2.5. *Let G be a paratopological group with the property* (**). *If G is a Fréchet-Urysohn space, then G is a α_4-space.*

Corollary 2.5 gives a partial answer to Nyikos' question [15, Problem 3]: "Is a Fréchet-Urysohn paratopological group an α_4-space?".

Question 2.1. Can we omit the property (**) in Theorem 2.4 or in Corollary 2.5?

A space X is called *weakly quasi-first countable* or $\aleph_0$-*weakly first-countable* ([17], [18]) if for each $i \in \mathbb{N}$, there exists a mapping $B^i : \mathbb{N} \times X \to \mathcal{P}(X)$, where $\mathcal{P}(X)$ denotes the power set of X, such that the following (1) and (2) hold:

(1) for $i \in \mathbb{N}$, for each $n \in \mathbb{N}$ and $x \in X$, $B^i(n+1, x) \subset B^i(n, x)$, and $\{x\} = \bigcap\{B^i(n, x) : n \in \mathbb{N}\}$; and
(2) a subset V of X is open if and only if for each $y \in V$ and for each $i \in \mathbb{N}$ there exists $n(i)$ with $B^i(n(i), y) \subset V$.

If $B^i = B$ for $i \in \mathbb{N}$, then X is called *weakly first countable* or *g-first countable*. Obviously, a weakly first countable space is weakly quasi-first countable.

Corollary 2.6. *Let G be a Fréchet-Urysohn paratopological group with the property (**). If G is $\aleph_0$-weakly first-countable, then G is first-countable.*

PROOF: By Corollary 2.5, G is an α_4-space, hence G is weakly first-countable [10], thus G is first-countable [15, Theorem 2]. □

By Corollary 2.5, we have the following:

Corollary 2.7 ([18]). *A Fréchet-Urysohn, $\aleph_0$-weakly first-countable topological group is metrizable.*

Next, we discuss when we cannot embed a copy of S_{ω_1} to some paratopological group.

A family $\{B_\alpha : \alpha \in I\}$ of subsets of a space X is *hereditarily closure-preserving* (*weakly hereditarily closure-preserving* [5]) (simply, HCP (wHCP)) if

$$\bigcup\{\overline{C_\alpha} : \alpha \in J\} = \overline{(\bigcup\{C_\alpha : \alpha \in J\}})(\{x_\alpha : \alpha \in J\} \text{ is closed discrete}),$$

whenever $J \subset I$ and $C_\alpha \subset B_\alpha (x_\alpha \in B_\alpha)$ for each $\alpha \in J$. Obviously, a HCP family is wHCP. Spaces with a σ-wHCP weak base (base) were discussed in [11], [12]. Let $\mathcal{P}$ be a cover of a space X. Then $\mathcal{P}$ is a *k-network* for X if whenever $K \subset U$ with K compact and U open in X, $K \subset \bigcup \mathcal{P}' \subset U$ for some finite $\mathcal{P}' \subset \mathcal{P}$. A k-network is a network. A space with a σ-locally finite k-network is an $\aleph$-space [16]. S_{ω_1} is a closed image of a metric space, hence it has a σ-HCP closed k-network [7] but it is not an $\aleph$-space [9].

Theorem 2.5. *Let G be a paratopological topological group with the property (**). If G has a σ-wHCP closed k-network, then G contains no closed copy of S_{ω_1}.*

PROOF: Suppose G contains a closed copy of $S_{\omega_1} = \{e\} \cup \{x_n(\alpha) : \alpha < \omega_1, n \in \mathbb{N}\}$, where e is the neutral element of G and $x_n(\alpha) \to e$ as $n \to \infty$. Since G has the property (**), there exists a sequence $\{x_n : n \in \mathbb{N}\} \subset G$ such that $x_n \to e$, $x_n^{-1} \to e$. G is regular, we take open subsets U_n of G such that $x_n \in U_n$, $\overline{U_n} \cap \overline{U_m} = \emptyset$ $(n \neq m)$ and $\overline{U_n} \cap \{x_n : n \in \mathbb{N}\} = \{x_n\}$. For each $m \in \mathbb{N}$, $x_m x_n(\alpha) \to x_m (n \to \infty)$, $\{x_m x_n(\alpha) : n \in \mathbb{N}\}$ is eventually in U_m for $\alpha < \omega_1$. Without loss of generality, we assume $\{x_m x_n(\alpha) : n \in \mathbb{N}\} \subset U_m$.

Claim: $B = \{x_{n(\alpha)}x_{m(\alpha)}(\alpha) : \alpha < \omega_1\}$ *is a discrete subset of* G *for* $n(\alpha)$, $m(\alpha) \in \mathbb{N}$.

Case 1: $\{n(\alpha) : \alpha < \omega_1\}$ is finite.

We rewrite $\{n(\alpha) : \alpha < \omega_1\} = \{r_1, \dots r_k\}$. Since $\{x_{g(\alpha)}(\alpha) : \alpha < \omega_1\}$ is discrete for every $g : \omega_1 \to \mathbb{N}$, then $\{x_{r_i}x_{g(\alpha)}(\alpha) : \alpha < \omega_1\}$ is discrete for each $i \leq k$, hence B is discrete.

Case 2: $\{n(\alpha) : \alpha < \omega_1\}$ is infinite.

Suppose B is not discrete and let x be the cluster point of B. For every $g : \omega_1 \to \mathbb{N}$, there exists an open neighborhood V of e such that $|Vx \cap \{x_{g(\alpha)}(\alpha) : \alpha < \omega_1\}| \leq 1$. Let U be an open neighborhood of e with $U^2 \subset V$. Then $C = Ux \cap \{x_{n(\alpha)}x_{m(\alpha)}(\alpha) : \alpha < \omega_1\} \neq \emptyset$ for infinitely many $n(\alpha)$. Since $x_n^{-1} \to e$, $\{x_n : n \in \mathbb{N}\}$ is eventually in U, $\{x_n^{-1} : n \geq k\}C \subset UUx \subset Vx$. Then $|Vx \cap \{x_{g(\alpha)}(\alpha) : \alpha < \omega_1\}| \geq \omega$, a contradiction.

For $\alpha < \omega_1$, let $C_\alpha = \{e\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_n x_i(\alpha) : n \in \mathbb{N}, i \geq f_n(\alpha)\}$. Note that $x_n x_{j_n}(\alpha) \to e(n \to \infty)$, where $j_m \geq f_m(\alpha)$. Since every infinite subset of C_α has a cluster point in it, C_α is a countably compact. Since every countably compact space with a σ-wHCP network has a countable network [12, Proposition 6], C_α is compact [11].

Let $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a σ-wHCP k-network consisting of closed subsets. Then there is a finite $\mathcal{P}' \subset \mathcal{P}$ such that $C_0 \subset \bigcup \mathcal{P}'$. Pick $P_0 \in \mathcal{P}'$ so that P_0 contains $k_0 = x_{n(0)}x_{m(0)}(0)$ and infinitely many x_n's. We assume that for each $\alpha < \beta$, there exists $P_\alpha \in \mathcal{P}$ such that P_α contains infinitely many x_n's and a point $k_\alpha = x_{n(\alpha)}x_{m(\alpha)}(\alpha)$. We have $C_\beta \subset G \backslash \{k_\alpha : \alpha < \beta\}$, which is open in G by the Claim. There is a finite $\mathcal{P}'' \subset \mathcal{P}$ such that $C_\beta \subset \bigcup \mathcal{P}'' \subset G \backslash \{k_\alpha : \alpha < \beta\}$, pick $P_\beta \in \mathcal{P}''$ so that P_β contains infinitely many x_n and $k_\beta = x_{n(\beta)}x_{m(\beta)}(\beta)$. By induction, we obtain $\{P_\alpha : \alpha < \omega_1\} \subset \mathcal{P}$ such that $P_\alpha \neq P_\beta$ if $\alpha \neq \beta$ and each P_α contains infinitely many x_n's, hence there are uncountably many $P_\alpha \in \mathcal{P}_n$ for some $n \in \mathbb{N}$. Note that $\mathcal{P}_n$ is wHCP and there is a subsequence L of $\{x_n : n \in \mathbb{N}\}$ such that L is discrete, which is a contradiction. □

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