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Spaces of continuous functions, Σ-products and Box Topology

J. Angoa, Á. Tamariz-Mascarúa

Abstract. For a Tychonoff space X, we will denote by X_0 the set of its isolated points and X_1 will be equal to $X \setminus X_0$. The symbol $C(X)$ denotes the space of real-valued continuous functions defined on X. $\square\mathbb{R}^\kappa$ is the Cartesian product $\mathbb{R}^\kappa$ with its box topology, and $C_\square(X)$ is $C(X)$ with the topology inherited from $\square\mathbb{R}^X$. By $\widehat{C}(X_1)$ we denote the set $\{f \in C(X_1) : f \text{ can be continuously extended to all of } X\}$. A space X is almost-ω-resolvable if it can be partitioned by a countable family of subsets in such a way that every non-empty open subset of X has a non-empty intersection with the elements of an infinite subcollection of the given partition. We analyze $C_\square(X)$ when X_0 is F_σ and prove: (1) for every topological space X, if X_0 is F_σ in X, and $\emptyset \neq X_1 \subset \mathrm{cl}_X X_0$, then $C_\square(X) \cong \square\mathbb{R}^{X_0}$; (2) for every space X such that X_0 is F_σ, $\mathrm{cl}_X X_0 \cap X_1 \neq \emptyset$, and $X_1 \setminus \mathrm{cl}_X X_0$ is almost-ω-resolvable, then $C_\square(X)$ is homeomorphic to a free topological sum of $\leq |\widehat{C}(X_1)|$ copies of $\square\mathbb{R}^{X_0}$, and, in this case, $C_\square(X) \cong \square\mathbb{R}^{X_0}$ if and only if $|\widehat{C}(X_1)| \leq 2^{|X_0|}$. We conclude that for a space X such that X_0 is F_σ, $C_\square(X)$ is never normal if $|X_0| > \aleph_0$ [La], and, assuming CH, $C_\square(X)$ is paracompact if $|X_0| = \aleph_0$ [Ru2]. We also analyze $C_\square(X)$ when $|X_1| = 1$ and when X is countably compact, and we scrutinize under what conditions $\square\mathbb{R}^\kappa$ is homeomorphic to some of its "Σ-products"; in particular, we prove that $\square\mathbb{R}^\omega$ is homeomorphic to each of its subspaces $\{f \in \square\mathbb{R}^\omega : \{n \in \omega : f(n) = 0\} \in p\}$ for every $p \in \omega^*$, and it is homeomorphic to $\{f \in \square\mathbb{R}^\omega : \forall\, \epsilon > 0\ \{n \in \omega : |f(n)| < \epsilon\} \in \mathcal{F}_0\}$ where $\mathcal{F}_0$ is the Fréchet filter on ω.

Keywords: spaces of real-valued continuous functions, box topology, Σ-product, almost-ω-resolvable space

Classification: 54C35, 54B10, 54D15

0. Introduction

All topological spaces considered in this article will be Tychonoff.

The spaces of continuous functions defined on a topological space X and with values in $\mathbb{R}$, $C(X)$, have been widely studied as a purely algebraic structure ([GJ]), and with a topological-algebraic structure ([BNS], [DH]).

One of the natural topologies associated with $C(X)$ is the pointwise convergence topology, which is the topology in $C(X)$ inherited from the Tychonoff topology of $\mathbb{R}^X$. This space is usually denoted by $C_p(X)$. A classical general problem

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on C_p-spaces consists of determining the relations between the topological properties of space X with the topological properties of $C_p(X)$ ([Ar]).

A generalization of the Tychonoff topology for a product of topological spaces, is the box topology (see definition in Section 1) which was introduced by Tietze in [Ti]. The study of the box product of an infinite family of topological spaces has been a very useful source to construct some interesting topological spaces ([Ru1], [V]). With respect to paracompactness of box products, in 1963, in [Kn], the question, due to A.H. Stone, whether the cartesian product of an infinite collection of copies of the real line with its box topology is a normal space, was posed. In [Ru2], M.E. Rudin proved under CH that the box product of countably many σ-compact locally compact metrizable spaces is paracompact; and K. Kunen [Ku] showed, also using CH, that the box product of a countable family $\{K_n : n < \omega\}$ of compact spaces is paracompact if and only if the Lindelöf degree of $\square_{n<\omega} K_n$ is equal to ω_1. Moreover, E.K. van Douwen [vD] showed that a box product of a countable collection of metrizable separable spaces need not be normal. Finally, L.B. Lawrence [La] proved in ZFC that the product of an uncountable family of copies of the real line is not normal.

So, it seems natural to ask about the relations between the topological properties of a space X and those of $C(X)$ with its box topology, which we denote by $C_\square(X)$. In particular, it is natural to ask under what conditions on X, $C_\square(X)$ is normal or paracompact (the set $C_\square(X)$ being a closed subset of $\square\mathbb{R}^X$). In [TV], A. Tamariz-Mascarúa and H. Villegas-Rodríguez analyzed the space $C_\square(X)$ when X is a topological space without isolated points. They obtained the following results (see the definition of an almost-ω-resolvable space in Section 1):

0.1 Theorem. *Let X be a dense-in-itself space. Then,*

(1) *X is an almost-ω-resolvable space iff $C_\square(X)$ is a discrete space;*
(2) *Con(ZFC) implies Con(ZFC+ for every space X, $C_\square(X)$ is a discrete subspace of $\square\mathbb{R}^X$);*
(3) *if X is a Baire irresolvable space, then $C_\square(X)$ is not a discrete space.*

We recall here that a topological space X is *irresolvable* if it is dense-in-itself and it is not the union of two disjoint dense subsets; and, of course, a space X is *Baire irresolvable* if it is irresolvable and satisfies the Baire property. In [KST], it was proved that the existence of a Baire irresolvable space is equiconsistent with the existence of a measurable cardinal; then, the existence of a dense-in-itself space for which $C_\square(X)$ is not discrete, is equiconsistent with the existence of a measurable cardinal (see [TV, Theorem 4.16]). In particular, if X is a dense-in-itself almost-ω-resolvable space, $C_\square(X)$ is more than a paracompact space.

The purpose of this article is to analyze spaces $C_\square(X)$ when the subset X_0 of isolated points of X is not empty, and to show the topological relations between $C_\square(X)$ and $\square\mathbb{R}^{X_0}$ and how the former can be expressed in terms of the latter. One of our main results states that for every space X for which $Z = X \setminus \mathrm{cl}_X X_0$ is almost-ω-resolvable and X_0 is an F_σ-subset of X, $C_\square(X)$ is a free topological sum of copies of $\square\mathbb{R}^{X_0}$. Concluding that, by the results of Rudin and Lawrence, if X has the properties mentioned, $C_\square(X)$ is not normal if $|X_0| > \aleph_0$, and CH implies $C_\square(X)$ is paracompact when $|X_0| = \aleph_0$. We also obtain sufficient and necessary conditions on X under which $C_\square(X)$ is homeomorphic to $\square\mathbb{R}^{X_0}$. One immediate conclusion which comes after these results is the fact that for spaces X and Y even with opposite topological properties, $C_\square(X) \cong C_\square(Y)$ can happen. This fact will allow us to decide when $\square\mathbb{R}^\kappa$ is or is not homeomorphic to some of its "Σ-products". So, we can say that for a wide class of topological spaces X, $C_\square(X)$ is completely determined by some weak topological properties (of a set-theoretical type) of X and X_0.

In Section 1 we give some basic definitions and preliminary results. In Section 2 we discuss spaces $C_\square(X)$ when X_0 is an F_σ-subset of X and $\emptyset \neq X_1 = X \setminus X_0 \subset \operatorname{cl}_X X_0$, and we prove that, in this case, $C_\square(X) \cong \square\mathbb{R}^{X_0}$. Section 3 is devoted to analyzing $C_\square(X)$ in a more general situation: X_0 is an F_σ-subset of X, $X^b = X_1 \cap \operatorname{cl}_X X_0 \neq \emptyset$ and $Z = X \setminus \operatorname{cl}_X X_0$ is an almost-ω-resolvable space; we obtain that, in this case, $C_\square(X)$ is homeomorphic to a free topological sum of copies of $\square\mathbb{R}^{X_0}$. In Sections 4 and 5 we study $C_\square(X)$ when $|X_1| = 1$ and when X is countably compact, and we scrutinize under what conditions $\square\mathbb{R}^\kappa$ is homeomorphic to some of its "Σ-products"; in particular, we prove that $\square\mathbb{R}^\omega$ is homeomorphic to its subspaces $\Sigma_p^\square\mathbb{R}^\omega = \{f \in \square\mathbb{R}^\omega : \{n \in \omega : f(n) = 0\} \in p\}$ for every $p \in \omega^*$, and it is homeomorphic to $\Sigma_{*,\mathcal{F}_0}^\square\mathbb{R}^\omega = \{f \in \square\mathbb{R}^\omega : \ \forall\, \epsilon > 0\ \{n \in \omega : |f(n)| < \epsilon\} \in \mathcal{F}_0\}$ where $\mathcal{F}_0$ is the Fréchet filter on ω.

1. Basic definitions and preliminaries

For a set X and a cardinal number κ, $\mathcal{P}(X)$ will be the collection of subsets of X, $[X]^\kappa$ is the collection of elements in $\mathcal{P}(X)$ having cardinality κ, and $[X]^{<\kappa}$ is the collection of elements in $\mathcal{P}(X)$ having cardinality $< \kappa$. For a function $f : X \to Y$ and a subset B of X, $f \upharpoonright B$ is the restriction of f to B. As we have already said, *every topological space X considered in this article will be completely regular and T_2; that is, Tychonoff*. For a space X, βX is the Stone-Čech compactification of X.

Let $\mathcal{F} = \{X_\alpha : \alpha \in A\}$ be a collection of topological spaces. By $\square_{\alpha\in A} X_\alpha$, we will represent the Cartesian product $X = \prod_{\alpha\in A} X_\alpha$ of the family $\mathcal{F}$ endowed with the box topology. The box topology is that generated in X by the open boxes; that is, by the subsets of the form $\prod_{\alpha\in A} O_\alpha$ where O_α is an open subset of X_α for each $\alpha \in A$. Recall that the Tychonoff topology in X is generated by sets of the form $\prod_{\alpha\in A} O_\alpha$ where each O_α is open in X_α and the set $\{\alpha \in A : O_\alpha \neq X_\alpha\}$ is finite. It is obvious that the Tychonoff topology in X is contained in the box topology, and that they coincide iff $|A| < \aleph_0$.

It is well known that, for an infinite family $\{X_\alpha : \alpha \in A\}$ of non-trivial topological spaces, $\square_{\alpha \in A} X_\alpha$ is neither first countable nor locally compact, and it is never a topological vector space, but it is a topological group if each of the spaces X_α is a topological group, with group operation $+_\alpha$, when we consider the sum of two elements $(x_\alpha)_{\alpha \in A}$ and $(b_\alpha)_{\alpha \in A}$ in $\square_{\alpha \in A} X_\alpha$ to be $(a_\alpha +_\alpha b_\alpha)_{\alpha \in A}$. A good survey of the characteristics of the box topology can be found in [Wi].

For a topological space X we denote by $\mathbb{R}^X$ the Cartesian product of $|X|$ copies of the real line $\mathbb{R}$ which can be considered as the set of functions from X to $\mathbb{R}$. The subset of $\mathbb{R}^X$ whose elements are the continuous functions is denoted by $C(X)$. The space $\square\mathbb{R}^X$ (resp., $T\mathbb{R}^X$) will be the set $\mathbb{R}^X$ with the box topology (resp., the Tychonoff topology), and $C_\square(X)$ (resp., $C_p(X)$) is the set $C(X)$ considered as a subspace of $\square\mathbb{R}^X$ (resp., $T\mathbb{R}^X$).

A space X is *almost-ω-resolvable* if there is a partition $\{F_n : n < \omega\}$ of X such that every non-empty open subset V of X has a non-empty intersection with each F_n for every $n \in J_V$, where J_V is an infinite subset of ω. It will be useful to consider the empty space $\emptyset$ included in the class of almost-ω-resolvable spaces.

For a space X, we will denote by X_0 the set of isolated points in X, and X_1 is its complement $X \setminus X_0$. The symbol X^b represents the set $X_1 \cap \operatorname{cl}_X X_0$ and Z will denote the set $X \setminus (X^b \cup X_0)$.

Observe that, by Theorem 0.1, if Z is an almost-ω-resolvable space and X^b is empty, then $C_\square(X) = C_\square(Z \oplus X_0) \cong C_\square(Z) \times C_\square(X_0)$ is the free topological sum of $|C(X_1)|$ copies of the space $\square\mathbb{R}^\kappa$ where $\kappa = |X_0|$. So, in this case, we have already obtained a clear relation between $C_\square(X)$ and $\square\mathbb{R}^{X_0}$. *Hence, from now on we will assume that every space X satisfies $X^b \neq \emptyset$.*

The symbol $\widehat{C}(X_1)$ stands for the set $\{f \in C(X_1) : f$ has a continuous extension to all of $X\}$. For each $\widehat{x} \in \widehat{C}(X_1)$, we take $A_{\widehat{x}}(X) = \{f \in C(X) : f \upharpoonright_{X_1} = \widehat{x}\}$. We will denote by $\widehat{0}$ the function in $C(X_1)$ which is equal to 0 everywhere. For $f, g \in \mathbb{R}^X$, the function $f + g \in \mathbb{R}^X$ is defined as $(f+g)(x) = f(x) + g(x)$ for each $x \in X$. Every mention to an algebraic structure on $\mathbb{R}^X$ will refer to this operation. For two topological spaces X and Y, we will write $X \cong Y$ if they are homeomorphic, and for topological groups G and H, the symbol $H \simeq G$ will signify that H and G are topologically isomorphic. Finally, for an element x of a topological space X, $\mathcal{N}(x)$ will denote the system of neighborhoods of x in X. It is easy to prove the following results.

1.1 Proposition. *For a topological space X we have:*

(1) *$\square\mathbb{R}^X$ is a topological group;*
(2) *$C_\square(X)$ is a closed topological subgroup of $\square\mathbb{R}^X$;*
(3) *$A_{\widehat{x}}(X)$ is a closed topological subgroup of $C_\square(X)$ for every $\widehat{x} \in \widehat{C}(X_1)$;*
(4) *for $\widehat{x} \in \widehat{C}(X_1)$, $A_{\widehat{x}}(X)$ and $A_{\widehat{0}}(X)$ are topologically isomorphic;*
(5) *the family $\{A_{\widehat{x}}(X) : \widehat{x} \in \widehat{C}(X_1)\}$ is a partition of $C(X)$.*

(The referee pointed out to the authors that Erik van Douwen was probably the first to observe, in 1975, that the box product of topological groups is a topological group.)

For a subset Y of X, the symbol π_Y will represent the natural projection from $\square\mathbb{R}^X$ to $\square\mathbb{R}^Y$; that is, π_Y is the function defined by $\pi_Y(f) = f \restriction Y$. If Y is the one-point set $\{y\}$, we will write π_y instead of $\pi_{\{y\}}$. The following lemma is very useful.

1.2 Lemma. *Let X be a topological space and let Y be a subset of X containing X_0. Then, the function $\phi = \pi_Y \restriction A_{\widehat{0}}(X) : A_{\widehat{0}}(X) \to \square\mathbb{R}^Y$ is an isomorphic embedding.*

Proof: It is trivial that ϕ is one-to-one and furthermore $\phi(f-g) = (f-g)\restriction Y = (f\restriction Y) - (g \restriction Y) = \phi(f) - \phi(g)$. Besides, if for each $x \in Y$ we take an open subset G_x of $\mathbb{R}$ which has 0 as one of its elements, then $\phi^{-1}[\prod_{x\in Y} G_x] = A_{\widehat{0}}(X) \cap \prod_{x\in X} H_x$ where $H_x = G_x$ if $x \in Y$ and $H_x = \mathbb{R}$ if $x \notin Y$. So, ϕ is a continuous function. Finally, if for each $x \in X$, H_x is an open subset of $\mathbb{R}$ containing 0, then $\phi[\prod_{x\in X} H_x \cap A_{\widehat{0}}(X)] = \prod_{x\in Y} H_x \cap \phi[A_{\widehat{0}}(X)]$. $\square$

Let Y be a set, $S \subset Y$, $T = Y \setminus S$ and let $\mathcal{F} = \{F_n : n < \omega\}$ be a partition of S (that is, $\bigcup_{n<\omega} F_n = S$, $F_n \neq \emptyset$ for each $n < \omega$, and if $n \neq m$, then $F_n \cap F_m = \emptyset$). We define $E(\mathcal{F}) \subset \mathbb{R}^Y$ as $E(\mathcal{F}) = \bigcap_{k<\omega} E_k(\mathcal{F})$, and $E_k(\mathcal{F}) = \bigcup_{m<\omega} E_{k,m}(\mathcal{F})$, where

$$E_{k,m}(\mathcal{F})(x) = \begin{cases} \mathbb{R} & \text{if } x \in F_i \text{ and } i \leq m, \\ \left[-\frac{1}{2^{i+k}}, \frac{1}{2^{i+k}}\right] & \text{if } x \in F_i \text{ and } m < i, \\ \mathbb{R} & \text{if } x \in T. \end{cases}$$

Let us obtain some properties of the sets just defined (see [Ru3]).

1.3 Proposition. *Let Y be a topological space and let $\mathcal{F}$ be a partition of $S \subset Y$. Then, $E(\mathcal{F})$ is a clopen topological subgroup of $\square\mathbb{R}^Y$.*

Proof: Let $\mathcal{F} = \{F_n : n < \omega\}$ be a partition of S. If $z \in E(\mathcal{F})$, there is a strictly increasing sequence $\{m_k : k < \omega\}$ such that $z \in E_{k,m_k}(\mathcal{F})$ for each $k < \omega$. We define the following open box W which contains z:

$$W(x) = \begin{cases} \mathbb{R} & \text{if } x \in F_i,\ i \leq m_1, \\ \left(-\frac{1}{2^{i+k-1}}, \frac{1}{2^{i+k-1}}\right) & \text{if } x \in F_i,\ m_k < i \leq m_{k+1} \text{ and } k \geq 1, \\ \mathbb{R} & \text{if } x \notin S. \end{cases}$$

Let $h \in W$. We take $t_{k-1} = m_k$ for $k \geq 1$. Trivially, the sequence $\{t_{k-1} : 1 \leq k < \omega\}$ is strictly increasing and $h \in E_{k-1,t_{k-1}}(\mathcal{F})$ for all $k \geq 1$. Thus, $W \subset E(\mathcal{F})$; that is, $E(\mathcal{F})$ is open.

Now, let $w \notin E(\mathcal{F})$. There exists $k_0 < \omega$ such that for every m there are $i_m > m$ and $x_{i_m} \in F_{i_m}$ such that $w(x_{i_m}) \notin [-\frac{1}{2^{i_m+k_0}}, \frac{1}{2^{i_m+k_0}}]$. We take $A \subset \omega$

for which $\{i_m : m \in A\} = \{i_m : m < \omega\}$ and for all $n, m \in A$ with $n \neq m$, $i_m \neq i_n$. For each $m \in A$, let V_{i_m} be an open subset of $\mathbb{R}$ such that $w(x_{i_m}) \in V_{i_m}$ and $V_{i_m} \cap [-\frac{1}{2^{i_m+k_0}}, \frac{1}{2^{i_m+k_0}}] = \emptyset$. Take O as the open box defined by $O(x_{i_m}) = V_{i_m}$ for each $m \in A$, and $O(x) = \mathbb{R}$ if $x \notin \{x_{i_m} : m \in A\}$. It happens that $w \in O$ and $O \subset \square\mathbb{R}^Y \setminus E(\mathcal{F})$; so, $E(\mathcal{F})$ is closed.

Now, let $f, g \in E(\mathcal{F})$. Take two sequences $\{m_k : k < \omega\}$ and $\{l_k : k < \omega\}$ satisfying: for all $k < \omega$ and for all $i > m_k$, if $x \in F_i$ then $f(x) \in \left[-\frac{1}{2^{i+k}}, \frac{1}{2^{i+k}}\right]$, and for all $j > l_k$, if $y \in F_j$ then $g(y) \in \left[-\frac{1}{2^{j+k}}, \frac{1}{2^{j+k}}\right]$. We take $t_{k-1} = \max\{m_k, l_k\}$ for $k \geq 1$. We have that $f - g \in E_{k-1,t_{k-1}}(\mathcal{F})$ for all $k \geq 1$. Therefore, $f - g \in E(\mathcal{F})$ and, since $\widehat{0} \in E(\mathcal{F})$, we conclude that $E(\mathcal{F})$ is a topological subgroup of $\square\mathbb{R}^Y$. $\square$

We need to introduce the following definition which relativizes the concept of almost-ω-resolvability.

1.4 Definition. Let X be a topological space, and let A and B be subsets of X. We say that A is *almost-ω-resolvable with respect to B* (briefly: *A is a-ω-rwrt$B$*), if there is a partition $\{F_n : n < \omega\}$ of A, such that for every open subset O of X which has a non-empty intersection with B, $|\{n : F_n \cap O \neq \emptyset\}| = \aleph_0$. Such a partition is called *a resolution of A with respect to B*.

In the following proposition we emphasize the relation between the concepts just defined and the structure of $C_\square(X)$. Recall that $A_{\widehat{x}}(X)$ is closed in $C_\square(X)$ for all $\widehat{x} \in \widehat{C}(X_1)$. First, a technical result.

1.5 Lemma. *Let S and T be two subsets of a topological space Y. If S is a-ω-rwrt$T$, $\{F_n : n < \omega\}$ is a resolution of S with respect to T, $g \in C(Y)$ and O_g is the open box constituted by those elements f in $C(Y)$ such that $f(x) \in (g(x) - \frac{1}{2^n}, g(x) + \frac{1}{2^n})$ if $x \in F_n$, then $g \restriction T = h \restriction T$ holds for every $h \in O_g$.*

Proof: Assume that there are $h \in O_g$ and $z \in T$ such that $0 < |h(z) - g(z)| = \epsilon$. Since g and h are continuous, we can take an element V in $\mathcal{N}(z)$, the system of neighborhoods of z, such that $g(V) \subset (g(z) - \frac{\epsilon}{3}, g(z) + \frac{\epsilon}{3})$ and $h(V) \subset (h(z) - \frac{\epsilon}{3}, h(z) + \frac{\epsilon}{3})$. There is $x \in S$ such that $x \in F_n \cap V$ for an $n \in \omega$ such that $\frac{1}{2^n} < \frac{\epsilon}{3}$. For this x, $|h(x) - g(x)| < \frac{\epsilon}{3}$, $|h(z) - h(x)| < \frac{\epsilon}{3}$ and $|g(x) - g(z)| < \frac{\epsilon}{3}$; so, $|h(z) - g(z)| < \epsilon$, a contradiction. Then, $h(z) = g(z)$ for every $z \in T$. $\square$

1.6 Proposition. *A space X is a-ω-rwrtX_1 if and only if $A_{\widehat{0}}(X)$ is an open subset of $C_\square(X)$.*

Proof: That $A_{\widehat{0}}(X)$ is open in $C_\square(X)$ is a consequence of Lemma 1.5; we just have to take $X = S$ and $X_1 = T$.

Now, let us assume that $A_{\widehat{0}}(X)$ is an open subset of $C_\square(X)$. Since the function $\widehat{0}$, which is equal to 0 everywhere, belongs to $A_{\widehat{0}}(X)$, for each $x \in X$ there exists an open subset G_x of $\mathbb{R}$ such that $\widehat{0} \in \left(\prod_{x\in X} G_x\right) \cap C(X) \subset A_{\widehat{0}}(X)$. We define $d(x) = \min\{n < \omega : \left(-\frac{1}{2^n}, \frac{1}{2^n}\right) \subset G_x\}$ and $F_n = \{x \in X : d(x) = n\}$. It is clear that $\{F_n : n < \omega\}$ is a partition of X.

We will prove that $\{F_n : n < \omega\}$ is a resolution for X with respect to X_1. Assume the contrary; that is, there are $z \in X_1$, an open $V \in \mathcal{N}(z)$ and $n_0 < \omega$ satisfying $V \cap F_n = \emptyset$ for every $n > n_0$. Let $H : X \longrightarrow \left[0, \frac{1}{2^{n_0+1}}\right]$ be a continuous function for which $H(X \setminus V) \subset \{0\}$ and $H(z) = \frac{1}{2^{n_0+1}}$. If $x \in V$, then $d(x) \leq n_0$. So, for every $x \in V$ we have $\frac{1}{2^{n_0+1}} < \frac{1}{2^{d(x)}}$, and $H(x) \leq \frac{1}{2^{n_0+1}}$. Thus, for every $x \in V$, $H(x) \in \left(-\frac{1}{2^{d(x)}}, \frac{1}{2^{d(x)}}\right) \subset G_x$. On the other hand, if $x \in X \setminus V$, $H(x) = 0 \in G_x$. So $H \in \left(\prod_{x\in X} G_x\right) \cap C(X) \subset A_{\widehat{0}}(X)$. Hence, $H(z) = 0$ which is a contradiction. We conclude that $\{F_n : n < \omega\}$ is a resolution of X with respect to X_1. □

As a consequence of Propositions 1.1 and 1.6, we obtain:

1.7 Corollary. *Let X be a-ω-rwrtX_1. Then,*

(1) *for each $\widehat{x} \in \widehat{C}(X_1)$, $A_{\widehat{x}}(X)$ is a clopen subset of $C_\Box(X)$, and*
(2) *$C_\Box(X) = \oplus_{\widehat{x}\in\widehat{C}(X_1)} A_{\widehat{x}}(X) \simeq \oplus_{\widehat{x}\in\widehat{C}(X_1)} (A_{\widehat{0}}(X))_{\widehat{x}}$ where each $(A_{\widehat{0}}(X))_{\widehat{x}}$ is a copy of $A_{\widehat{0}}(X)$.*

If X_0 is a-ω-rwrtX_1, then X is a-ω-rwrtX_1, and there exists a space X which is a-ω-rwrtX_1 and X_0 is not a-ω-rwrtX_1 (see an example in the paragraph before Problem 2.8). From now on, under a *complete minimal system of representatives of the cosets belonging to a quotient space* $X/\!\sim$ we will understand a subset J of X such that $X/\!\sim\, = \bigcup_{x\in J}[x]$ and $[x] \neq [y]$ for each pair x, y of different elements in J, where $[x]$ is the class of equivalence of x related to the equivalence relation $\sim$. It is not difficult to prove the following result.

1.8 Proposition. *If X_1 is almost-ω-resolvable, then X is a-ω-rwrtX_1.*

1.9 Remark. For a partition $\mathcal{F}$ of the set X_0 of isolated points of a space X, we can consider the clopen topological subgroup $E(\mathcal{F})$ of $\Box\mathbb{R}^X$, as was defined before Proposition 1.3. For each $f, g \in A_{\widehat{0}}(X)$, $\left(E(\mathcal{F}) \cap A_{\widehat{0}}(X)\right) + f$ and $\left(E(\mathcal{F}) \cap A_{\widehat{0}}(X)\right) + g$ are clopen topologically isomorphic subgroups of $A_{\widehat{0}}(X)$. So, for a complete minimal system D_1 of representatives of the cosets belonging to the quotient group $A_{\widehat{0}}(X)/[E(\mathcal{F}) \cap A_{\widehat{0}}(X)]$, we have

$$A_{\widehat{0}}(X) = \bigoplus_{f\in D_1} \left[E(\mathcal{F}) \cap A_{\widehat{0}(X)}\right] + f.$$

So, we obtain:

1.10 Proposition. *If X is a-ω-rwrtX_1, $\mathcal{F}$ is a resolution of X with respect to X_1 and D_1 is a complete minimal system of representatives of the cosets belonging* *to the quotient* $A_{\widehat{0}}(X)/[E(\mathcal{F}) \cap A_{\widehat{0}}(X)]$, *then*

$$C_{\square}(X) \simeq \oplus_{\widehat{x} \in \widehat{C}(X_1), f \in D_1} \left[E(\mathcal{F}) \cap A_{\widehat{0}}(X)\right]_{\widehat{x},f},$$

where each $(E(\mathcal{F}) \cap A_{\widehat{0}})_{\widehat{x},f}$ *is a copy of* $E(\mathcal{F}) \cap A_{\widehat{0}}$.

Now we are going to give some results about box products and their σ-products which will be useful for our purposes. The important role which the σ-products play in the general study of box products was emphasized in [NyP].

As usual, for a topological space X, $l(X)$, $d(X)$, $c(X)$ and $e(X)$ denote the Lindelöf number, density, cellularity and extent of X, respectively (see [H] for definitions).

For a family $\{X_\alpha : \alpha \in A\}$ of topological spaces and $x \in \prod_{\alpha \in A} X_\alpha = X$, let σ_x be the σ-product of X; that is, $\sigma_x = \{y \in X : |\{\alpha \in A : y(\alpha) \neq x(\alpha)\}| < \aleph_0\}$. We denote by $\sigma_x^{\square}(\prod_{\alpha \in A} X_\alpha)$ (or simply, $\sigma_x^{\square}$) the set σ_x with the topology inherited from $\square_{\alpha \in A} X_\alpha$.

The following result is due to M.E. Rudin ([Ru3, p. 55]).

1.11 Proposition. *Let κ be an infinite cardinal number, and let $\{X_\alpha : \alpha < \kappa\}$ be a family of connected Tychonoff spaces. If $x \in \square_{\alpha<\kappa} X_\alpha$ and C_x is the connected component of x in $\square_{\alpha<\kappa} X_\alpha$, then $C_x = \sigma_x$.*

1.12 Proposition. *For each infinite cardinal number κ and every $x \in \square\mathbb{R}^\kappa$,*

$$l(\sigma_x^{\square}\mathbb{R}^\kappa) = e(\sigma_x^{\square}\mathbb{R}^\kappa) = d(\sigma_x^{\square}\mathbb{R}^\kappa) = c(\sigma_x^{\square}\mathbb{R}^\kappa) = \kappa.$$

PROOF: For each $J \in [\kappa]^{<\aleph_0} = \{A \subset \kappa : |A| < \aleph_0\}$, let $H_J \subset \square\mathbb{R}^\kappa$ be the box defined by $H_J(\alpha) = \mathbb{R}$ if $\alpha \in J$ and $H_J(\alpha) = \{x(\alpha)\}$ if $\alpha \notin J$. The set H_J with the box topology is homeomorphic to $T\mathbb{R}^J$, which is a Lindelöf space. Thus, for each open cover $\mathcal{C}$ of $\sigma_x^{\square}\mathbb{R}^\kappa$, and for each $J \in [\kappa]^{<\aleph_0}$, we can select a countable subfamily $\mathcal{C}_J$ of $\mathcal{C}$ such that $H_J \subset \bigcup \mathcal{C}_J$. Since $\sigma_x^{\square}\mathbb{R}^\kappa = \bigcup_{J \in [\kappa]^{<\aleph_0}} H_J$, $\mathcal{D} = \bigcup_{J \in [\kappa]^{<\aleph_0}} \mathcal{C}_J$ is a subcollection of $\mathcal{C}$ which covers $\sigma_x^{\square}\mathbb{R}^\kappa$ and has cardinality $\leq \kappa$. So, $l\left(\sigma_x^{\square}\mathbb{R}^\kappa\right) \leq \kappa$.

Now, for each $\delta < \kappa$, we take $z_\delta \in \mathbb{R} \setminus \{x(\delta)\}$. We define for each $\alpha < \kappa$

$$t_\alpha(\delta) = \begin{cases} z_\delta & \text{if } \alpha = \delta, \\ x(\delta) & \text{if } \alpha \neq \delta. \end{cases}$$

The subset $D = \{t_\alpha : \alpha < \kappa\}$ of $\sigma_x^{\square}\mathbb{R}^\kappa$ is closed and discrete. We conclude that $\kappa \leq e(\sigma_x^{\square}\mathbb{R}^\kappa) \leq l(\sigma_x^{\square}\mathbb{R}^\kappa) \leq \kappa$.

Now, we are going to make some calculations in order to obtain the density of $\sigma_x^{\square}\mathbb{R}^\kappa$. For each $J \in [\kappa]^{<\aleph_0}$, we have that $d\left(T\mathbb{R}^J\right) = \aleph_0$. So, for each

$J \in [\kappa]^{<\aleph_0}$, there exists $D_J \subset H_J$ which is countable and dense in H_J. Thus, the set $D = \bigcup_{J\in[\kappa]^{<\aleph_0}} D_J$ is dense in $\sigma_x^{\Box}\mathbb{R}^\kappa$. Since $|D| \le \kappa$, $d(\sigma_x^{\Box}\mathbb{R}^\kappa) \le \kappa$.

Let A be a subset of $\sigma_x^{\Box}\mathbb{R}^\kappa$ with cardinality $< \kappa$. For each $a \in A$, let $J_a \in [\kappa]^{<\aleph_0}$ be such that $a \in H_{J_a}$. We take $T = \bigcup_{a\in A} J_a$. We have that $|T| \le |A| < \kappa$. Let α_0 be an element of $\kappa \setminus T$ and $O = \prod_{\alpha<\kappa} O_\alpha$ where $O_\alpha = \mathbb{R}$ if $\alpha \ne \alpha_0$, and O_{α_0} is an open subset of $\mathbb{R}$ which does not contain $x(\alpha_0)$. It is clear that $A \cap O = \emptyset$; then A cannot be dense in $\sigma_x^{\Box}\mathbb{R}^\kappa$. So, we can conclude that $d(\sigma_x^{\Box}\mathbb{R}^\kappa) = \kappa$. $\square$

1.13 Corollary. *Let κ and τ be infinite cardinal numbers. Then, $\Box\mathbb{R}^\kappa \simeq \Box\mathbb{R}^\tau$ if and only if $\Box\mathbb{R}^\kappa \cong \Box\mathbb{R}^\tau$, if and only if $\kappa = \tau$.*

PROOF: Let $\psi : \Box\mathbb{R}^\kappa \longrightarrow \Box\mathbb{R}^\tau$ be a homeomorphism, and let $x \in \Box\mathbb{R}^\kappa$. By Proposition 1.11, we must have $\sigma_x^{\Box}\mathbb{R}^\kappa \cong \sigma_{\psi(x)}^{\Box}\mathbb{R}^\tau$. Now, Proposition 1.12 produces $\kappa = \tau$. $\square$

We will prove something more general than this corollary in Section 3 (see Corollary 3.4).

In order to describe $C_\Box(X)$, it will be convenient to keep in mind some results concerning the cardinality of $C(X)$. The following result was proved by W.W. Comfort and A.W. Hager in [CH].

1.14 Proposition. *For a space X, $|C(X)| = w(\beta X)^\omega$.*

2. Spaces $C_\Box(X)$ when X_0 is an F_σ-set and $Z = \emptyset$

Recall that for a space X we are denoting by X_0 its subset of isolated points, and X_1 is equal to $X \setminus X_0$. In this section we will analyze spaces $C_\Box(X)$ when X_0 is an F_σ-subset of X and $\emptyset \ne X_1 \subset \mathrm{cl}_X X_0$. Some examples of spaces having these characteristics are: the convergent sequence $\{0\} \cup \{1/n : n \in \mathbb{N}\} \subset \mathbb{R}$, the Stone-Čech compactification $\beta\omega$ of the natural numbers, every Mrówka-Isbell space (or Ψ-space, see [GJ]) $\Psi(\mathcal{A})$, and the countable Fréchet-Urysohn fan $V(\aleph_0)$ ([Ar]). The main result of this section (Theorem 2.4) proclaims that for every space X satisfying these properties, its related space of real-valued continuous functions $C_\Box(X)$ is a box product of real lines.

Let $\mathcal{F} = \{F_n : n < \omega\}$ be a partition of X_0. We denote simply by $E(\mathcal{F})$ the subset of $\Box\mathbb{R}^X$ as was constructed before Proposition 1.3, when $Y = X$ and $S = X_0$; and we denote by $E_0(\mathcal{F})$ the subset of $\Box\mathbb{R}^{X_0}$ obtained in a similar way but when $Y = S = X_0$.

The following lemma describes the relation between $E(\mathcal{F})$ and $E_0(\mathcal{F})$ for a partition $\mathcal{F}$ of X_0.

2.1 Lemma. *Let $\mathcal{F} = \{F_n : n < \omega\}$ be a partition of X_0, where each F_n is a closed subset of X. Then, $\phi' = \pi_{X_0} \restriction (E(\mathcal{F}) \cap A_{\widehat{0}}(X)) : E(\mathcal{F}) \cap A_{\widehat{0}}(X) \longrightarrow E_0(\mathcal{F})$ is a topological isomorphism.*

Proof: Because of Lemma 1.2 and the fact that $E(\mathcal{F})$ is open, we conclude that ϕ' is an embedding, and obviously $\phi'[E(\mathcal{F}) \cap A_{\widehat{0}}(X)] \subset E_0(\mathcal{F})$. The only part of the proof not entirely trivial refers to the surjection of ϕ'. Let $h \in E_0(\mathcal{F})$, and let $h' \in \mathbb{R}^X$ with $h' \restriction X_0 = h$ and $h'(z) = 0$ for every $z \in X_1$; then, $h' \in A_{\widehat{0}}(X)$. In fact, let $z \in X_1$ and $\epsilon > 0$, and let $\{m_k < \omega : k < \omega\}$ be a strictly increasing sequence of natural numbers such that $h \in E_0^{k,m_k}$ for all $k < \omega$. Let us take $k > 0$ such that $\frac{1}{2^{m_k}} < \epsilon$. The set $X \setminus \big(\bigcup_{i \leq m_k} F_i\big)$ is a neighborhood of z, $h'\big(X \setminus \big(\bigcup_{i \leq m_k} F_i\big)\big) \subset (-\epsilon, \epsilon)$ and $h'(z) = 0 \in (-\epsilon, \epsilon)$. We conclude that $h' \in A_{\widehat{0}}(X)$. It is clear that $h' \in E(\mathcal{F})$ and $\phi'(h') = h$. $\square$

We define a relation of equivalence $\sim$ on $\mathbb{R}^{X_0}$: for $f, g \in \mathbb{R}^{X_0}$, $f \sim g$ if and only if for each $\epsilon > 0$ and each $z \in X_1$, there exists $V \in \mathcal{N}(z)$ such that $(f-g)\,[V \cap X_0] \subset (-\epsilon, \epsilon)$. For each $f \in \mathbb{R}^{X_0}$, $[f]$ is the $\sim$-class of equivalence of f.

2.2 Lemma. *The relation $\sim$ is a relation of equivalence, and for $f, g \in \mathbb{R}^{X_0}$, if $f - g \in E_0(\mathcal{F})$, then $f \sim g$.*

Proof: It is easy to confirm that $\sim$ is of equivalence. Now, assume that there are $z_0 \in X_1$ and $\epsilon_0 > 0$ such that for each $V \in \mathcal{N}(z_0)$, there is $x_V \in V \cap X_0$ such that $(f-g)(x_V) \notin (-\epsilon_0, \epsilon_0)$. Let $k < \omega$ be such that $\frac{1}{2^k} < \epsilon_0$. Fix a $m < \omega$. It is clear that $V_1 = \big(X \setminus \bigcup_{i \leq m} F_i\big) \in \mathcal{N}(z_0)$. Let $x_{V_1} \in V_1 \cap X_0$ satisfying $(f-g)(x_{V_1}) \notin (-\epsilon_0, \epsilon_0)$. In particular, there exist $i_m > m$, such that $x_{V_1} \in F_{i_m}$. Hence, $(f-g)(x_{V_1}) \notin \big[-\frac{1}{2^{i_m+k}}, \frac{1}{2^{i_m+k}}\big]$; that is, $f - g \notin E_0(\mathcal{F})$. $\square$

In order to prove the main result of this section (Theorem 2.4) we are next going to prove a proposition which is apparently less general.

2.3 Proposition. *Assume that a space X satisfies:*

(1) $\emptyset \neq X_1 \subset \mathrm{cl}_X X_0$,
(2) *there is a partition $\{F_n : n < \omega\}$ of X_0 constituted by closed subsets of X, and*
(3) *there is a partition $\{C_\alpha \subset X_0 : \alpha < |X_0|\}$ of X_0 such that*
    (a) *$|C_\alpha \cap F_n| \leq 1$ for each $\alpha < |X_0|$ and each $n < \omega$, and*
    (b) *$J_\alpha = \{n < \omega : C_\alpha \cap F_n \neq \emptyset\}$ is infinite for each $\alpha < |X_0|$.*

Then, $A_{\widehat{0}}(X) \cong \square\mathbb{R}^{X_0} \cong C_\square(X)$.

Proof: Let $\mathcal{F} = \{F_n : n < \omega\}$ be a partition of X_0 which testifies (2) and (3) in this proposition. Let $D_0 \subset \mathbb{R}^{X_0}$ be a minimal complete system of representatives of the cosets in $\square\mathbb{R}^{X_0}/E_0(\mathcal{F})$, and let $D_1 \subset A_{\widehat{0}}(X)$ be a complete minimal system of representatives of $A_{\widehat{0}}(X)/[A_{\widehat{0}}(X) \cap E(\mathcal{F})]$. By Proposition 1.3, Remark 1.9 and Lemma 2.1, we have that (a) $\square\mathbb{R}^{X_0} = \bigoplus_{f \in D_0}(E_0(\mathcal{F}) + f)$ and (b) $C_\square(X) \simeq \bigoplus_{\widehat{x} \in \widehat{C}(X_1), f \in D_1} \big(\big[A_{\widehat{0}}(X) \cap E(\mathcal{F})\big] + f\big)_{\widehat{x}}$, where each term that appears in the sum in (a) is topologically isomorphic to each term that appears in the sum in (b).

Because of the definitions of $E_0(\mathcal{F})$ and $E(\mathcal{F})$, we have that for $f, g \in A_{\widehat{0}}(X)$, $f - g \in E(\mathcal{F})$ if and only if $(f \restriction X_0) - (g \restriction X_0) \in E_0(\mathcal{F})$. For this reason $|D_1| \leq |D_0|$.

We are going to prove now that $|\widehat{C}(X_1)| \leq |D_0|$. Let $\mathbb{R}^{X_0}/\!\sim$ be the quotient set determined by the relation $\sim$ defined before Lemma 2.2. The relation $H : \widehat{C}(X_1) \longrightarrow \mathbb{R}^{X_0}/\!\sim$ which sends each $\widehat{x}$ to $[f_{\widehat{x}} \restriction X_0]$ (where $f_{\widehat{x}}$ is an element of $A_{\widehat{x}}(X)$) is a well defined injective function. So, $\left|\widehat{C}(X_1)\right| \leq \left|\mathbb{R}^{X_0}/\!\sim\right|$. Now, using Lemma 2.2 we obtain $\left|\widehat{C}(X_1)\right| \leq |D_0|$.

Hence, we have already proved that $|\widehat{C}(X_1)| \cdot |D_1| \leq |D_0|$. Now we are going to prove that in fact $|\widehat{C}(X_1)| \cdot |D_1|$ is equal to $|D_0|$.

Recall that we are assuming that X has properties (1), (2) and (3) listed in our proposition. For each $\alpha < |X_0|$, we can enumerate C_α as $\{x_{\alpha,n} : n \in J_\alpha\}$ in such a way that for every $n \in J_\alpha$, $x_{\alpha,n} \in F_n$. So, for each $A \in \mathcal{P}(|X_0|)$, we define $f_A \in \mathbb{R}^X$ as:

$$f_A(y) = \begin{cases} \frac{1}{2^n} & \text{if } y = x_{\alpha,n},\ \alpha \in A \text{ and } n \in J_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $f_A \in A_{\widehat{0}}(X)$. Moreover, for every two different subsets $A, B \in \mathcal{P}(|X_0|)$, $f_A - f_B \notin E(\mathcal{F})$ because of condition (3). Therefore, the function

$$T : \mathcal{P}(|X_0|) \longrightarrow A_{\widehat{0}}(X)/A_{\widehat{0}}(X) \cap E(\mathcal{F})$$

where $T(A) = A_{\widehat{0}}(X) \cap E(\mathcal{F}) + f_A$, is injective. We conclude that $2^{|X_0|} \leq |D_1|$. Since $|D_0| \leq 2^{|X_0|}$, $|D_0| = \left|\widehat{C}(X_1)\right| \cdot |D_1|$.

Therefore, $A_{\widehat{0}}(X) \cong \Box\mathbb{R}^{X_0} \cong C_\Box(X)$. $\square$

Next, the main result of this section.

2.4 Theorem. *Let X be a space such that X_0 is an F_σ-subset of X and $\emptyset \neq X_1 \subset \mathrm{cl}_X X_0$. Then,*

$$A_{\widehat{0}}(X) \cong C_\Box(X) \cong \Box\mathbb{R}^{X_0}.$$

Proof: In order to prove this theorem we only have to construct two partitions $\mathcal{F} = \{F_n : n < \omega\}$ and $\mathcal{C} = \{C_\alpha : \alpha < |X_0|\}$ of X_0 satisfying (2) and (3) in Proposition 2.3.

Since X_0 is an F_σ-subset of X, there is a disjoint family $\mathcal{E} = \{E_i : i < \omega\}$ of closed subsets of X covering X_0.

First case: For each $i < \omega$, $|E_i| < \aleph_0$.

In this case, we can modify the partition $\mathcal{E}$ in such a way that we obtain a new partition $\mathcal{F} = \{F_i : i < \omega\}$ of X_0 constituted by closed subsets of X with $|F_i| = 1$. We name x_n the only element belonging to F_n for each $n < \omega$. Now, we take a partition $A_0, A_1, \dots, A_n, \dots$ of ω where each A_i is infinite. We define $C_k = \{x_n : n \in A_k\}$ for each $k < \omega$. The collections $\mathcal{F}$ and $\{C_k : k < \omega\}$ satisfy (2) and (3) in Proposition 2.3.

<u>Second case:</u> There is an infinite subset of natural numbers J such that, for each $n \in J$, $|E_n| \geq \aleph_0$.

In this case, we modify partition $\mathcal{E}$ in such a way that we obtain a partition $\{H_i : i < \omega\}$ of X_0 of closed subsets of X with $|H_i| \geq \aleph_0$ for every $i < \omega$. Now, let κ_i be the cardinality of H_i for each $i < \omega$. Let us list H_i as $\{x_i^\lambda : \lambda < \kappa_i\}$ with $x_i^\lambda \neq x_i^\xi$ if $\lambda \neq \xi$. Of course $|X_0| = \kappa = sup\{\kappa_i : i < \omega\}$. For each $\delta < \kappa$, we take $G_\delta = \{x_i^\delta : \kappa_i > \delta\}$ and $T = \{\delta < \kappa : |G_\delta| = \aleph_0\}$.

If $\kappa \setminus T = \emptyset$, then $\{H_n : n < \omega\}$ and $\{G_\delta : \delta < \kappa\}$ satisfy the requirements.

Now, assume that $\kappa \setminus T \neq \emptyset$. For each $\delta \in \kappa \setminus T$, G_δ is a finite set $\{x_{s_1}^\delta, \dots, x_{s_t}^\delta\}$. We denote by M_δ the set of natural numbers $\{s_1, \dots, s_t\}$ related with G_δ. Let δ_0 be the first element of $\kappa \setminus T$.

Claim: For each $\delta \in \kappa \setminus T$, $M_\delta \subset M_{\delta_0}$.

Indeed, let δ be an element of $\alpha \setminus T$ and $G_\delta = \{x_{s_1}^\delta, \dots, x_{s_t}^\delta\}$. By definition of G_δ, we have that $\delta < \kappa_{s_l}$ for every $l \in \{1, \dots, t\}$. Since $\delta_0 \leq \delta$, $\delta_0 < \kappa_{s_l}$ for each $l \in \{1, \dots, t\}$. Then, for each $l \in \{1, \dots, t\}$, $x_{s_l}^{\delta_0} \in G_{\delta_0}$; that is, $s_l \in M_{\delta_0}$ for every $l \in \{1, \dots, t\}$. We conclude that $M_\delta \subset M_{\delta_0}$.

Let n_0 be equal to the greatest element in M_{δ_0}, and let us call H the set $\bigcup_{i \leq n_0} H_i$. We partition H in a family of infinite countable subsets: $H = \bigcup_{\lambda < |H|} D_\lambda$, $|D_\lambda| = \aleph_0$ for all $\lambda < |H|$, and $D_\lambda \cap D_\xi = \emptyset$ for every $\lambda, \xi < |H|$ with $\lambda \neq \xi$. For each $\lambda < |H|$ we enumerate the elements of D_λ as $\{z_1^\lambda, z_2^\lambda, \dots, z_n^\lambda, \dots\}$ in such a way that $z_j^\lambda \neq z_i^\lambda$ if $j \neq i$.

We take $\mathcal{C}$ as the family $\{\widetilde{G}_\delta : \delta \in T\} \cup \{D_\lambda : \lambda < |H|\}$, where, for each $\delta \in T$, $\widetilde{G}_\delta = \{x_i^\delta \in G_\delta : i > n_0\}$.

Now, we are going to define the family $\mathcal{F}$. Let $F_1 = H_{n_0+1} \cup \{z_1^\lambda : \lambda < |H|\}$, $\dots$, $F_k = E_{n_0+k} \cup \{z_k^\lambda : \lambda < |H|\}$, $\dots$

It is not difficult now to prove that families $\mathcal{F}$ and $\mathcal{C}$ satisfy properties (2) and (3) in Proposition 2.3.

<u>Third case:</u> There is $n_0 < \omega$ such that $\{n < \omega : |E_n| \geq \aleph_0\} \subset \{0, \dots, n_0\}$.

In this case, we can modify partition $\mathcal{E}$ in such a way that we obtain a new partition $\{H_i : i < \omega\}$ of X_0 with each H_i closed in X, $|H_0| \geq \aleph_0$ and $|H_n| = 1$ for every $n > 0$; say, $H_n = \{x_n\}$ for each $n > 0$.

We partition H_0 in a family of infinite countable subsets: $H_0 = \bigcup_{\lambda < |H_0|} D_\lambda$,

$|D_\lambda| = \aleph_0$ for all $\lambda < |H_0|$, and $D_\lambda \cap D_\xi = \emptyset$ for every $\lambda, \xi < |H_0|$ with $\lambda \neq \xi$. For each $\lambda < |H_0|$ we enumerate the elements of D_λ as $\{z_1^\lambda, z_2^\lambda, \dots, z_n^\lambda, \dots\}$ in such a way that $z_j^\lambda \neq z_i^\lambda$ if $j \neq i$.

We take $\mathcal{C}$ as the family $\{C\} \cup \{D_\lambda : \lambda < |H_0|\}$, where $C = \{x_n : n > 0\}$.

Now, we are going to define the family $\mathcal{F}$. Let $F_1 = H_1 \cup \{z_1^\lambda : \lambda < |H_0|\}, \dots, F_k = H_k \cup \{z_k^\lambda : \lambda < |H_0|\}, \dots$

The families $\mathcal{F}$ and $\mathcal{C}$ satisfy properties (2) and (3) in Proposition 2.3.

We have already finished the proof. □

2.5 Examples.

(1) If the set X_0 of isolated points of a topological space X is countable (in particular, if X has countable cellularity) and $\emptyset \neq X \setminus X_0 = X_1 \subset \mathrm{cl}_X X_0$, then $C_\square(X) \cong \square\mathbb{R}^{\aleph_0}$. We then have $C_\square(\beta\omega) \cong \square\mathbb{R}^\omega$. More generally, if κ is an infinite cardinal number of countable cofinality, and Y is a subset of uniform ultrafilters on κ, then $Y \subset \mathrm{cl}_{\kappa\cup Y}\,\kappa$ and $(\kappa \cup Y)_1 = Y$ is a G_δ subset of $\kappa \cup Y$; so, $C_\square(\kappa \cup Y) \cong \square\mathbb{R}^\kappa$. Also, for every almost disjoint family $\mathcal{A}$ of ω, $C_\square(\Psi(\mathcal{A})) \cong \square\mathbb{R}^\omega$.

(2) Of course, if X is perfect, X_0 is F_σ in X; so, spaces of a wide class fulfill the conditions in Theorem 2.4. In particular, if X is metrizable (or even semi-stratifiable or developable) and $\emptyset \neq X_1 \subset \mathrm{cl}_X X_0$, then $C_\square(X) \cong \square\mathbb{R}^{X_0}$. In particular, for every countable ordinal number α, $C_\square([0,\alpha)) \cong \square\mathbb{R}^\omega$. On the other hand, $[0,\omega_1)$ is a locally compact first countable orderable space such that its subset of isolated points is not an F_σ-subset, and the Michael line M is an hereditarily paracompact quasi-developable space such that $M_0 = \{\text{irrational numbers}\}$ is not an F_σ-subset of M.

(3) A way to obtain examples of spaces X which satisfy the conditions in Theorem 2.4, now that we have mentioned the Michael line, is the following: Take a dense-in-itself non-countable separable space $(X, \mathcal{T})$. Let Q be a countable dense subset of X. We define a new space $(X_Q, \mathcal{T}_Q)$ as follows: let $X_Q = X$ and $B \in \mathcal{T}_Q$ if and only if $B = C \cup D$ with $C \in \mathcal{T}$ and $D \subset Q$ (see [E, p. 306]). The space X_Q satisfies conditions in Theorem 2.4, so $C_\square(X_Q) \cong \square\mathbb{R}^\omega$. Observe that $(X_Q)_0 = Q$ and $(X_Q)_1 = X \setminus Q$.

(4) Let Y be a space with $iw(Y) = \aleph_0$. So $C_p(Y)$ is a dense-in-itself non-countable separable space ([Ar]). Let Q be a countable dense subset of $C_p(Y)$. Because of the previous example we obtain: $C_\square(C_p(Y)_Q) \cong \square\mathbb{R}^\omega$. When $Y = \mathbb{R}$, the set Q of polynomials with rational coefficients is a countable dense subset of $C_p(\mathbb{R})$. Then, $C_\square(C_p(\mathbb{R})_Q) \cong \square\mathbb{R}^\omega$.

For each $x \in X$, we have denoted by $\mathcal{N}(x)$ the system of neighborhoods of x in X. We will use the symbol $\mathcal{N}_0(x)$ to designate the set $\{V \cap X_0 : V \in \mathcal{N}(x)\}$. Observe that if $x \in \mathrm{cl}_X X_0$, $\mathcal{N}_0(x)$ is a filter on X_0. We are going to obtain a converse of Theorem 2.4 when $\mathcal{N}_0(x)$ is an ultrafilter for every $x \in X_1$. First we prove the following:

2.6 Proposition. *If X_0 is an F_σ-subset of X, then X_0 is a-ω-rwrtX^b.*

Proof: Let $\mathcal{F} = \{F_n : n < \omega\}$ be a partition of X_0 constituted by closed subsets of X. We are going to prove that $\mathcal{F}$ is a resolution of X_0 with respect to X^b. Let $p \in X^b$ and $V \in \mathcal{N}(p)$. Since $p \in \mathrm{cl}_X(X_0)$, then $V \cap X_0 \neq \emptyset$. Let $A = \{n < \omega : F_n \cap V \neq \emptyset\}$, and assume that there exists $n_0 < \omega$ such that for all $n > n_0$, $V \cap F_n = \emptyset$. We know that $V \cap (X \setminus \bigcup_{n \leq n_0} F_n) \in \mathcal{N}(p)$, but $V \cap (X \setminus \bigcup_{n \leq n_0} F_n) \cap X_0 = \emptyset$, which is not possible. So, A must be infinite. □

2.7 Theorem. *Let $\emptyset \neq X_1 \subset \mathrm{cl}_X X_0$ and assume that $\mathcal{N}_0(x)$ is an ultrafilter on X_0 for each $x \in X_1$. Then the following assertions are equivalent.*

(1) *X_0 is F_σ in X.*
(2) *X_0 is a-ω-rwrtX_1.*
(3) *$C_\square(X) \cong \square\mathbb{R}^{X_0}$ and X_0 is a-ω-rwrtX_1.*

Proof: (1) $\Rightarrow$ (2): This is a consequence of Proposition 2.6.

(2) $\Rightarrow$ (1): Let $\{F_n : n < \omega\}$ be a resolution of X_0 with respect to X_1. Let us fix $n < \omega$, and suppose that there exists $x \in \mathrm{cl}\, F_n \cap X_1$. That is, for each $V \in \mathcal{N}(x)$, we have $V \cap F_n \neq \emptyset$. Thus, for each $V \cap X_0 \in \mathcal{N}_0(x)$, $V \cap F_n = (V \cap X_0) \cap F_n \neq \emptyset$. Since $\mathcal{N}_0(x)$ is an ultrafilter, F_n must belong to $\mathcal{N}_0(x)$. Let V' be an element of $\mathcal{N}(x)$ such that $F_n = V' \cap X_0$. Then we obtain $F_m \cap V' = \emptyset$ when $n \neq m$. But this last assertion contradicts our hypotheses. Therefore, F_n must be closed in X.

We obtain (1) $\Rightarrow$ (3) by Theorem 2.4 and Proposition 2.6, and (3) $\Rightarrow$ (2) is trivial. □

The statement "X_0 is a-ω-rwrtX_1" in (2) and (3) in the previous theorem cannot be replaced by the weaker proposition "X is a-ω-rwrtX_1". In fact, let α be an infinite cardinal number with uncountable cofinality. Since $X = \beta\alpha$ is an $\aleph_0$-resolvable space, X is a-ω-rwrtX_1 and $A_{\hat{0}}(\beta\alpha)$ is open in $C_\square(\beta\alpha)$. Nevertheless, X_0 is not an F_σ-subset of X.

2.8 Problem. *Assume that we have the same assumptions given in Theorem 2.7. Suppose also that $C_\square(X) \cong \square\mathbb{R}^{X_0}$. Is X_0 then a-ω-rwrtX_1?*

3. Spaces $C_\square(X)$ when X_0 is an F_σ-subset of X

In this section we are going to consider spaces $C_\square(X)$ when the set of isolated points of X, X_0, is an F_σ-subset of X and $\emptyset \neq X \setminus \mathrm{cl}_X X_0 = Z$. For example, for the subspace $Y = \{r \in \mathbb{R} : r \leq 0\} \cup \{1/n : n \in \mathbb{N}\}$ of $\mathbb{R}$, $Y_0 = \{1/n : n \in \mathbb{N}\}$ is an F_σ-subset of Y, $Y^b = \{0\}$ and $Z = \{r \in \mathbb{R} : r < 0\}$. As usual, CN is the class of cardinal numbers, and for a cardinal number κ, κ^+ is the minimum cardinal number strictly greater than κ.

Observe that the product space $E(\kappa) = [0, \omega] \times D(\kappa)$ where $[0, \omega]$ is the space of ordinals $\leq \omega$ with its order topology, and $D(\kappa)$ is the discrete space of cardinality κ, satisfies the conditions in Theorem 2.4. Thus, by Corollary 1.7 and

Theorem 2.4, we have

$$\Box\mathbb{R}^\kappa \cong C_\Box(E(\kappa)) \cong \oplus_{\widehat{x}\in C((E(\kappa))_1)}(\Box\mathbb{R}^\kappa)_{\widehat{x}}.$$

Since $(E(\kappa))_1$ coincides with $D(\kappa)$, we have $C((E(\kappa))_1) = \mathbb{R}^\kappa$. Furthermore, $|C((E(\kappa))_1)| = 2^\kappa$. So we have proved:

3.1 Lemma. *For each infinite cardinal κ, $\Box\mathbb{R}^\kappa$ accepts a partition of 2^κ clopen subsets, each of them homeomorphic to $\Box\mathbb{R}^\kappa$.*

3.2 Definitions. A partition $\mathcal{C}$ of a topological space X is a *homeomorphic clopen partition of* X if each element of $\mathcal{C}$ is clopen and homeomorphic to X. *The homeomorphic clopen partition number of* X, $\mathrm{hop}(X)$, is the cardinal number $\min\{\kappa \in CN :$ there is no homeomorphic clopen partition of X of cardinality $\kappa\}$.

It is easy to see that $\mathrm{hop}(X) \leq |X|^+$ for every space X.

3.3 Proposition. *For each infinite cardinal κ,* $\mathrm{hop}(\Box\mathbb{R}^\kappa) = (2^\kappa)^+$.

Proof: Because of Lemma 3.1, $(2^\kappa)^+ \leq \mathrm{hop}(\mathbb{R}^\kappa)$. Moreover, $\mathrm{hop}(\mathbb{R}^\kappa) \leq |\mathbb{R}^\kappa|^+ = (2^\kappa)^+$; so, $\mathrm{hop}(\Box\mathbb{R}^\kappa) = (2^\kappa)^+$. □

3.4 Corollary. *Let τ, γ and κ be infinite cardinals. Then*

$$\bigoplus_{\alpha<\kappa} (\Box\mathbb{R}^\tau)_\alpha \cong \Box\mathbb{R}^\gamma \Leftrightarrow \kappa \leq 2^\tau \ \ \textit{and} \ \ \gamma = \tau.$$

Proof: Suppose that $\psi : \oplus_{\alpha<\kappa} (\Box\mathbb{R}^\tau)_\alpha \longrightarrow \Box\mathbb{R}^\gamma$ is a homeomorphism. Let $\alpha < \kappa$ and $x \in (\Box\mathbb{R}^\tau)_\alpha$. We have that the connected component of x in $\oplus_{\alpha<\kappa} (\Box\mathbb{R}^\tau)_\alpha$, $\sigma_x^\Box\mathbb{R}^\tau$, must be homeomorphic to $\sigma_{\psi(x)}^\Box\mathbb{R}^\gamma$. Then, because of Proposition 1.12, $\tau = \gamma$. Therefore, since $|\oplus_{\alpha<\kappa} (\Box\mathbb{R}^\tau)_\alpha| = |\Box\mathbb{R}^\gamma|$, $\kappa \cdot 2^\tau = 2^\gamma = 2^\tau$; thus, $\kappa \leq 2^\tau$.

Now, assume that $\kappa \leq 2^\tau$ and $\gamma = \tau$. By Proposition 3.3, we have $\oplus_{\alpha<\kappa} (\Box\mathbb{R}^\tau)_\alpha \cong \Box\mathbb{R}^\tau$. Hence, $\oplus_{\alpha<\kappa} (\Box\mathbb{R}^\tau)_\alpha \cong \Box\mathbb{R}^\gamma$. □

As has been the custom in this article, for a space X, we denote by X_0 the set of isolated points of X, $X_1 = X \setminus X_0$, $X^b = (\mathrm{cl}_X X_0) \cap X_1 \neq \emptyset$ and $Z = X \setminus \mathrm{cl}_X X_0$. Our next result generalizes Proposition 1.6.

3.5 Proposition. *Let X be a space such that X_0 is F_σ and $X^b \neq \emptyset \neq Z$. Then, the following assertions are equivalent.*

(1) *Z is almost-ω-resolvable.*
(2) *X is a-ω-rwrtX_1.*
(3) *$A_{\widehat{0}}(X) = \{f \in C(X) : f \restriction X_1 \equiv 0\}$ is a clopen subgroup of $C_\Box(X)$.*

Proof: (1) $\Rightarrow$ (2): Let $\mathcal{H} = \{H_n : n < \omega\}$ be a resolution of Z, and let $\mathcal{F} = \{F_n : n < \omega\}$ be a partition of X_0 constituted by closed subsets of X. We define $C_0 = H_0 \cup X^b \cup F_0$ and $C_i = H_i \cup F_i$ for $i \geq 1$. It happens that $\mathcal{C} = \{C_n : n < \omega\}$ is a resolution of X with respect to X_1.

(2) $\Rightarrow$ (1): Let $\mathcal{C} = \{C_n : n < \omega\}$ be a resolution of X with respect to X_1. Then, $\mathcal{H} = \{C_n \cap Z : n < \omega\}$ is a resolution of Z.

The equivalence (2) $\Leftrightarrow$ (3) is Proposition 1.6. $\square$

3.6 Corollary. *If X_0 is an F_σ-subset of X, $X^b \neq \emptyset$ and Z is almost-ω-resolvable, then*

$$C_\Box(X) = \oplus_{\widehat{x} \in \widehat{C}(X_1)} A_{\widehat{x}}(X).$$

3.7 Theorem. *Let X be a topological space with X_0 being an F_σ-subset of X and $X^b \neq \emptyset$. If Z is almost-ω-resolvable, then $A_{\widehat{0}}(X) \cong \Box\mathbb{R}^{X_0}$ and $C_\Box(X)$ is the free topological sum of $\leq |\widehat{C}(X_1)|$ copies of $\Box\mathbb{R}^{X_0}$.*

Proof: We take the subspace $Y = X_0 \cup X^b$ of X. The set of isolated points of Y, Y_0, is an F_σ-subset of Y and Y_1 is not empty; in fact, $Y_0 = X_0$ and $Y_1 = X^b$. Hence, $A_{\widehat{0}}(Y)$ is equal to $\{f \in C(Y) : f \upharpoonright X^b \equiv 0\}$. We claim that the natural projection $\pi_Y : A_{\widehat{0}}(X) \to A_{\widehat{0}}(Y)$ defined by $\pi_Y(f) = f \upharpoonright Y$ is a homeomorphism. By Lemma 1.2, $\pi_Y \upharpoonright A_{\widehat{0}}(X)$ is an embedding.

Let $h \in A_{\widehat{0}}(Y)$. We define $h' \in \mathbb{R}^X$ as $h'(z) = 0$ for every $z \in Z$ and $h' \upharpoonright Y = h$. Let $z \in Z = X_1 \setminus X^b$ and $\epsilon > 0$. Since Z is open in X and $h'[Z] = \{0\} \subset (-\epsilon, \epsilon)$, h' is continuous in z. Now let $z \in X^b$. By the continuity of h, there is an open neighborhood W of z in X such that $h[W \cap Y] \subset (-\epsilon, \epsilon)$. Since $W = (W \cap Y) \cup (W \cap Z)$, $h'[W] \subset (-\epsilon, \epsilon)$. It is clear that $h' \in A_{\widehat{0}}(X)$ and $\pi_Y(h') = h$. Thus $\pi_Y[A_{\widehat{0}}(X)] = A_{\widehat{0}}(Y)$.

Because of Corollary 3.6, $C_\Box(X) = \oplus_{\widehat{x} \in \widehat{C}(X_1)} (A_{\widehat{0}}(X))_{\widehat{x}}$ where $(A_{\widehat{0}}(X))_{\widehat{x}}$ is a copy of $A_{\widehat{0}}(X)$. Thus, $C_\Box(X) \cong \oplus_{\widehat{x} \in \widehat{C}(X_1)} (A_{\widehat{0}}(Y))_{\widehat{x}}$. On the other hand, Y_0 is an F_σ-subset of Y, $\emptyset \neq Y_1 \subset \mathrm{cl}_Y Y_0$ and $Y_0 = X_0$; so $A_{\widehat{0}}(X) \cong A_{\widehat{0}}(Y) \cong \Box\mathbb{R}^{X_0}$ (Theorem 2.4). $\square$

As a consequence of the previous theorem and Corollary 3.4 we obtain:

3.8 Corollary. *Assume that X_0 is an F_σ-subset of X, $X^b \neq \emptyset$ and Z is almost-ω-resolvable. Then $C_\Box(X) \cong \Box\mathbb{R}^{X_0}$ if and only if $|\widehat{C}(X_1)| \leq 2^{|X_0|}$.*

The following result is a consequence of Theorems 0.1 and 3.7.

3.9 Corollary. *It is consistent with* ZFC *that for every space X for which X_0 is an F_σ-subset and $X^b \neq \emptyset$, $C_\Box(X) \cong \oplus_{f \in \widehat{C}(X_1)} (\Box\mathbb{R}^{X_0})_f$, and $C_\Box(X) \cong \Box\mathbb{R}^{X_0}$ if, in addition, $|\widehat{C}(X_1)| \leq 2^{|X_0|}$.*

It is known (see [CH]) that $|C(X)| \leq (wX)^{wcX} \leq 2^{d(X)}$ for every infinite space X, where wX is the least cardinal of an open basis, and wcX is the least κ for which each open cover of X has a subfamily with κ or fewer elements whose union is dense; so:

3.10 Corollary. *Let X be a topological space satisfying: X_0 is an F_σ-subset of X, $X^b \neq \emptyset$ and Z is almost-ω-resolvable. If $(wX_1)^{wcX_1} \leq 2^{|X_0|}$ (in particular, if $w(X) \leq |X_0|$ or $d(X_1) \leq |X_0|$), then $C_\square(X) \cong \square\mathbb{R}^{X_0}$.*

3.11 Examples.

(1) Since every first countable space is almost-ω-resolvable, $C_\square(X) \cong \square\mathbb{R}^{X_0}$ if X is semi-stratifiable or developable (metrizable), $X^b \neq \emptyset$ and $w(X) \leq |X_0|$.

(2) We now use the ideas of Example 2.5.(3). Let X be an almost-ω-resolvable separable space with a non-countable open subset A such that $X \setminus \mathrm{cl}_X A \neq \emptyset$. Let Q be a countable dense subset of A. The space X_Q satisfies conditions in Corollary 3.10, so $C_\square(X_Q) \cong \square\mathbb{R}^\omega$. Observe that, in this case, $(X_Q)_0 = Q$, $(X_Q)^b = \mathrm{cl}_X A \setminus Q$, and $Z(X_Q) = X \setminus \mathrm{cl}_X A \neq \emptyset$ which is an almost-ω-resolvable space.

(3) Because of the previous example, $C_\square(\mathbb{R}^2_Q) \cong \square\mathbb{R}^\omega$, where $Q = \{(x,y) \in \mathbb{Q}^2 : y > 0\}$.

Now we are going to prove a generalization of Theorem 2.7.

3.12 Theorem. *Let X be a space with $X^b \neq \emptyset$ and Z being an almost-ω-resolvable space. Assume that for each $p \in X^b$, $\mathcal{N}_0(p)$ is an ultrafilter on X_0. Then, the following assertions are equivalent.*

(1) *X_0 is F_σ in X.*

(2) *X_0 is a-ω-rwrtX^b.*

(3) *$C_\square(X)$ is a free topological sum of $\leq |\widehat{C}(X_1)|$ copies of $\square\mathbb{R}^{X_0}$ and X_0 is a-ω-rwrtX^b.*

(4) *$A_{\widehat{0}}(X)$ is open in $C_\square(X)$ and X_0 is a-ω-rwrtX^b.*

(5) *$A_{\widehat{0}}(X) \cong \square\mathbb{R}^{X_0}$ and X_0 is a-ω-rwrtX^b.*

Proof: The implication (1) $\Rightarrow$ (2) is Proposition 2.6, (2) $\Rightarrow$ (1) can be proved in a similar way to (2) $\Rightarrow$ (1) in Theorem 2.7, and (1) $\Rightarrow$ (5) is a consequence of Theorem 3.7.

(1) $\Rightarrow$ (3): If X_0 is F_σ, then, using Theorem 3.7, we obtain that $C_\square(X)$ is a free topological sum of $\leq |\widehat{C}(X_1)|$ copies of $\square\mathbb{R}^{X_0}$. The remainder is obtained by Proposition 2.6.

(3) $\Rightarrow$ (2), (4) $\Rightarrow$ (2) and (5) $\Rightarrow$ (2) are obvious.

(2) $\Rightarrow$ (4): This is (2) $\Rightarrow$ (3) in Theorem 2.7 plus Theorem 2.4 if $Z = \emptyset$. Assume now that $Z \neq \emptyset$. Since X_0 is c-ω-rcraX^b, X is a-ω-rwrtX_1, and we have only to apply Proposition 3.5. $\square$

We finish this section with the following result which summarizes everything we have obtained until now in this section, plus Rudin's and Lawrence's results [Ru2], [La].

3.13 Corollary. *Let X be a topological space such that X_0 is an F_σ-subset of X, $\emptyset \neq X^b$ and Z is almost-ω-resolvable. Then:*

(1) *$C_\Box(X)$ is not normal if $|X_0| > \aleph_0$;*
(2) *The Continuum Hypothesis implies that $C_\Box(X)$ is paracompact if $|X_0| = \aleph_0$.*

4. Σ-products and spaces $C_\Box(X)$

We are now going to consider spaces X with $|X_1| = 1$ and we will calculate $C_\Box(X)$ for this kind of spaces. Recall that a filter $\mathcal{F}$ on a set X_0 is *ω^+-complete* if for every $\{F_n : n < \omega\} \subset \mathcal{F}$, the set $\bigcap_{n<\omega} F_n$ belongs to $\mathcal{F}$. For a filter $\mathcal{F}$ on a set X_0, we define $\Sigma_{\mathcal{F}}\mathbb{R}^{X_0}$, $\Sigma_{*,\mathcal{F}}\mathbb{R}^{X_0}$, $\widehat{\Sigma}_{\mathcal{F}}\mathbb{R}^{X_0}$ in the following way:

$$\begin{aligned}
\Sigma_{\mathcal{F}}\mathbb{R}^{X_0} &= \{f \in \mathbb{R}^{X_0} : \{x \in X_0 : f(x) = 0\} \in \mathcal{F}\},\\
\Sigma_{*,\mathcal{F}}\mathbb{R}^{X_0} &= \{f \in \mathbb{R}^{X_0} : \text{ for all } \ \epsilon > 0,\ \{x \in X_0 : |f(x)| < \epsilon\} \in \mathcal{F}\},\\
\widehat{\Sigma}_{\mathcal{F}}\mathbb{R}^{X_0} &= \{f \in \mathbb{R}^{X_0} : \text{ for all } \ \epsilon > 0,\ \{x \in X_0 : |f(x)| \geq \epsilon\} \notin \mathcal{F}\}.
\end{aligned}$$

For a filter $\mathcal{F}$ on X_0, $\Sigma_{\mathcal{F}}\mathbb{R}^{X_0} \subset \Sigma_{*,\mathcal{F}}\mathbb{R}^{X_0} \subset \widehat{\Sigma}_{\mathcal{F}}\mathbb{R}^{X_0}$. When $\mathcal{F}$ is an ultrafilter, then $\Sigma_{*,\mathcal{F}}\mathbb{R}^{X_0} = \widehat{\Sigma}_{\mathcal{F}}\mathbb{R}^{X_0}$.

The symbols $\Sigma^{\Box}_{\mathcal{F}}\mathbb{R}^{X_0}$, $\Sigma^{\Box}_{*,\mathcal{F}}\mathbb{R}^{X_0}$, $\widehat{\Sigma}^{\Box}_{\mathcal{F}}\mathbb{R}^{X_0}$ mean that we are considering the sets $\Sigma_{\mathcal{F}}\mathbb{R}^{X_0}$, $\Sigma_{*,\mathcal{F}}\mathbb{R}^{X_0}$, $\widehat{\Sigma}_{\mathcal{F}}\mathbb{R}^{X_0}$ with their box product topology.

Recall that for a space X, X_0 is the set of isolated points of X, $X_1 = X \setminus X_0$ and always $\emptyset \neq X_1 \subset \mathrm{cl}_X X_0$.

We begin our analysis by obtaining some results about resolvability of X when $|X_1| = 1$.

4.1 Proposition. *Let X be a space such that $X_1 = \{p\}$. Then, X is a-ω-rwrtX_1 iff X_0 is a-ω-rwrtX_1.*

PROOF: Assume that X is a-ω-rwrtX_1 and let $\{F_n : n < \omega\}$ be a resolution of X with respect to X_1. Let n_0 be a natural number such that $p \in F_{n_0}$. So, $\{F_n : n \in \omega \setminus \{n_0\}\} \cup \{F_{n_0} \cap X_0\}$ is a resolution of X_0 with respect to X_1. □

4.2 Proposition. *Let $X_1 = \{p\}$, and assume that $\mathcal{N}_0(p) = \{X_0 \cap N : N$ is a neighborhood of $p\}$ is a non-ω^+-complete filter. Then, X_0 is a-ω-rwrtX_1.*

PROOF: Let $\{V_n : n < \omega\} \subset \mathcal{N}_0(p)$ such that $\bigcap_{n<\omega} V_n \notin \mathcal{N}_0(p)$ with $V_0 = X_0$. Let $W_n = \bigcap_{i\leq n} V_i$, $F_0 = \bigcap_{n<\omega} W_n$ and $F_{n+1} = W_n \setminus W_{n+1}$. We are going to prove that $\{F_n : n < \omega\}$ is a resolution of X_0 with respect to X_1.

Suppose the contrary; that is, assume that there are $V \in \mathcal{N}(p)$ and $n_0 < \omega$ such that $V \cap F_n = \emptyset$ for all $n \geq n_0$. We take $M_0 = W_{n_0+1} \cap V$. The set M_0 belongs to $\mathcal{N}(p)$ and $M_0 \cap F_n = \emptyset$ for all $n \geq n_0$. We claim that M_0 is contained in $\bigcap_{n>n_0} W_n$. In fact, if $x \in M_0$, then $x \in W_{n_0+1}$. If $x \notin W_{n_0+2}$, x must belong to F_{n_0+2}, which is not possible. The same reasoning used in an inductive process gives us that x must belong to $\bigcap_{n>n_0} W_n$. But $\mathcal{N}(p)$ is a filter, $M_0 \in \mathcal{N}(p)$ and $M_0 \subset \bigcap_{n>n_0} W_n = \bigcap_{n<\omega} W_n$; so, we obtain a contradiction. Thus $\{F_n : n < \omega\}$ is a resolution of X_0 with respect to X_1. $\square$

As a consequence of Theorem 2.7 and Proposition 4.2 we can prove:

4.3 Theorem. *If $\mathcal{N}_0(p)$ is an ultrafilter which is not an ω^+-complete filter, then X_0 is F_σ in X and $C_\square(X) \cong \square\mathbb{R}^{X_0}$.*

So, for an infinite cardinal number κ, if $p \in \beta\kappa \setminus \kappa$ is not ω^+-complete, and $\{p\} \cup \kappa$ is considered with its topology inherited from $\beta\kappa$, then

$$C_\square(\{p\} \cup \kappa) \cong \square\mathbb{R}^\kappa.$$

Now, we are going to see that the "Σ-products" defined at the beginning of this section are related to our study of $C_\square(X)$.

4.4 Proposition. *If $X_1 = \{p\}$, then $A_{\widehat{0}}(X) \cong \Sigma^\square_{*,\mathcal{N}_0(p)}\mathbb{R}^{X_0}$.*

Proof: The range of the isomorphic embedding $\phi : A_{\widehat{0}}(X) \to \mathbb{R}^{X_0}$ defined as $\phi(f) = f \restriction X_0$ (see Lemma 1.2) is equal to $\Sigma^\square_{*,\mathcal{N}_0(p)}\mathbb{R}^{X_0}$. Indeed, if $\epsilon > 0$ and $f \in A_{\widehat{0}}(X)$, there is $V \in \mathcal{N}(p)$ such that $f[V] \subset (-\epsilon, \epsilon)$. Thus, $V \cap X_0 \in \mathcal{N}_0(p)$ and $V \cap X_0 \subset \{x \in X : |f(x)| < \epsilon\}$. Therefore, $\{x \in X_0 : |f(x)| < \epsilon\} \in \mathcal{N}_0(p)$. This means that $f \restriction X_0 \in \Sigma^\square_{*,\mathcal{N}_0(p)}\mathbb{R}^{X_0}$. Moreover, if $g \in \Sigma^\square_{*,\mathcal{N}_0(p)}\mathbb{R}^{X_0}$ and $\widehat{g} \in \mathbb{R}^X$ is such that $\widehat{g} \restriction X_0 = g$ and $\widehat{g}(p) = 0$, then $\widehat{g} \in A_{\widehat{0}}(X)$ and $\phi(\widehat{g}) = g$. $\square$

4.5 Proposition. *If $X_1 = \{p\}$ and $\mathcal{N}_0(p)$ is an ω^+-complete filter, then*

$$\Sigma^\square_{\mathcal{N}_0(p)}\mathbb{R}^{X_0} = \Sigma^\square_{*,\mathcal{N}_0(p)}\mathbb{R}^{X_0}.$$

Proof: We only have to prove that $\Sigma^\square_{*,\mathcal{N}_0(p)}\mathbb{R}^{X_0} \subset \Sigma^\square_{\mathcal{N}_0(p)}\mathbb{R}^{X_0}$. Let $F \in \Sigma^\square_{*,\mathcal{N}_0(p)}\mathbb{R}^{X_0}$, and for each $n < \omega$ we take $D_n = \{x \in X_0 : |f(x)| < \frac{1}{2^n}\}$. It is clear that $D_n \in \mathcal{N}_0(p)$ for all $n < \omega$. So, $\bigcap_{n<\omega} D_n \in \mathcal{N}_0(p)$. But $\bigcap_{n<\omega} D_n = \{x \in X_0 : f(x) = 0\}$. Therefore, $f \in \Sigma^\square_{\mathcal{N}_0(p)}\mathbb{R}^{X_0}$. $\square$

Because of the last two propositions we obtain the following corollary.

4.6 Corollary. *If $X_1 = \{p\}$ and $\mathcal{N}_0(p)$ is an ω^+-complete filter, then*

$$A_{\widehat{0}}(X) \cong \Sigma^{\square}_{\mathcal{N}_0(p)}\mathbb{R}^{X_0}.$$

So, if $X_1 = \{p\}$, then the following assertions hold.

(1) If $\mathcal{N}_0(p)$ is an ω^+-complete filter, and X is a-ω-rwrtX_1, then

$$C_{\square}(X) \cong \bigoplus_{x\in\mathbb{R}}[\Sigma^{\square}_{\mathcal{N}_0(p)}\mathbb{R}^{X_0}]_x.$$

(2) If $\mathcal{N}_0(p)$ is a filter which is not ω^+-complete, then

$$C_{\square}(X) \cong \bigoplus_{x\in\mathbb{R}}[\Sigma^{\square}_{*,\mathcal{N}_0(p)}\mathbb{R}^{X_0}]_x.$$

Now we are going to consider the one-point compactification $K(X_0)$ of the discrete space X_0. Let p be the point of $K(X_0)$ which compactifies X_0.

4.7 Proposition. *Let $X_1 = \{p\}$. Then, $X = K(X_0)$ if and only if, for every $A \in [X_0]^{\aleph_0}$ and $V \in \mathcal{N}(p)$, we have $A \cap V \neq \emptyset$.*

4.8 Theorem. *If $X = K(X_0)$, then $\mathcal{N}(p)$ is not ω^+-complete. So, in this case we have*

$$C_{\square}(X) \cong \bigoplus_{x\in\mathbb{R}}[\Sigma^{\square}_{*,\mathcal{N}_0(p)}\mathbb{R}^{X_0}]_x.$$

Proof: Let $A \in [X_0]^{\aleph_0}$ with $A = \{a_n : n < \omega\}$ and $a_n \neq a_m$ if $n \neq m$. The set $V_n = X_0 \setminus \{a_0, \dots, a_n\}$ belongs to $\mathcal{N}_0(p)$ for every $n < \omega$, and $\bigcap_{n<\omega} V_n \notin \mathcal{N}_0(p)$. $\square$

In order to reduce (1) and (2) formulated after Corollary 4.6, we need to calculate the *hop* number of our "Σ-products".

4.9 Theorem. *Let $X_1 = \{p\}$ and let $\kappa \geq \aleph_0$. If there is $A \in [X_0]^{\kappa}$ such that $X \setminus A \in \mathcal{N}(p)$, then*

$$\mathrm{hop}(\Sigma^{\square}_{*,\mathcal{N}_0(p)}\mathbb{R}^{X_0}) \geq (2^{\kappa})^+$$

and

$$\mathrm{hop}(\Sigma^{\square}_{\mathcal{N}_0(p)}\mathbb{R}^{X_0}) \geq (2^{\kappa})^+.$$

Proof: We will sketch the proof of the first inequality. The proof of the second inequality is similar. Let $A \in [X_0]^{\kappa}$ such that $X \setminus A \in \mathcal{N}(p)$. Because of Proposition 3.3, there exists a clopen partition of $\square\mathbb{R}^A$ of cardinality 2^{κ} such that each of its elements is homeomorphic to $\square\mathbb{R}^A$. Let $\{A_\lambda : \lambda < 2^{\kappa}\}$ be such a partition. For each $\lambda < 2^\kappa$ we take a homeomorphism $\psi_\lambda : A_\lambda \to \Box\mathbb{R}^A$, and we take $B_\lambda \subset \mathbb{R}^{X_0}$ such that $\pi_x(B_\lambda) = \mathbb{R}$ if $x \notin A$ and $\pi_A[B_\lambda] = A_\lambda$. Now we define for $\lambda < 2^\kappa$, $C_\lambda = B_\lambda \cap \Sigma^\Box_{*,\mathcal{N}_0(p)}\mathbb{R}^{X_0}$. The set C_λ is a non-empty subset of $\Sigma^\Box_{*,\mathcal{N}_0(p)}\mathbb{R}^{X_0}$ for each $\lambda < 2^\kappa$ because $X \setminus A \in \mathcal{N}(p)$. The function $\Psi_\lambda : C_\lambda \to \Sigma^\Box_{*,\mathcal{N}_0(p)}\mathbb{R}^{X_0}$ defined by $\pi_A \circ \Psi_\lambda(f) = \psi_\lambda(f \restriction A)$ and for $x \notin A$, $\pi_x \circ \Psi_\lambda(f) = f(x)$, is a homeomorphism. (The hypothesis $X \setminus A \in \mathcal{N}(p)$ also gives us the surjectivity of Ψ_λ.) Furthermore, it is possible to prove that $\{C_\lambda : \lambda < 2^\kappa\}$ is a clopen partition of $\Sigma^\Box_{*,\mathcal{N}_0(p)}\mathbb{R}^{X_0}$. □

As a consequence of Proposition 4.7 and Theorem 4.9 we have:

4.10 Corollary. *Let $X_1 = \{p\}$ and $X \neq K(X_0)$. Then,*

$$\mathrm{hop}(\Sigma^\Box_{*,\mathcal{N}_0(p)}\mathbb{R}^{X_0}) \geq (2^{\aleph_0})^+$$

and

$$\mathrm{hop}(\Sigma^\Box_{\mathcal{N}_0(p)}\mathbb{R}^{X_0}) \geq (2^{\aleph_0})^+.$$

For a set S, the "Σ-product" $\Sigma = \{f \in \mathbb{R}^S : \forall\, \epsilon > 0,\ |\{x \in S : |f(x)| \geq \epsilon\}| < \aleph_0\}$ of $\mathbb{R}^S$ coincides with the Σ-product $\Sigma^\Box_{*,\mathcal{F}_0}\mathbb{R}^S$, where $\mathcal{F}_0$ is the Fréchet filter on S ($F \in \mathcal{F}_0$ iff $S \setminus F$ is finite). That is, $\Sigma = \Sigma^\Box_{*,\mathcal{N}(p)}\mathbb{R}^S$ where p is the point which compactifies the discrete space S.

4.11 Proposition. *Let $X = K(X_0)$. Then,* $\mathrm{hop}(\Sigma^\Box_{*,\mathcal{N}_0(p)}\mathbb{R}^{X_0}) \geq (2^{\aleph_0})^+$.

Proof: We have already mentioned that $\Sigma^\Box_{*,\mathcal{N}_0(p)}\mathbb{R}^{X_0}$ is equal to the subspace $\{f \in \mathbb{R}^{X_0} : \forall\, \epsilon > 0,\ |\{x \in X_0 : |f(x)| \geq \epsilon\}| < \aleph_0\}$ of $\Box\mathbb{R}^{X_0}$. Let us fix an infinite countable subset of X_0: $B = \{x_n : n < \omega\}$. The collection $\mathcal{F} = \{\{x_n\} : n < \omega\}$ is a partition of B. So, we can consider the clopen subgroup $E(\mathcal{F})$ of $\Box\mathbb{R}^{X_0}$ defined by $\mathcal{F}$ (see the definition before Proposition 1.3). It is now easy to verify that $C = \Sigma^\Box_{*,\mathcal{N}_0(p)}\mathbb{R}^{X_0} \cap E(\mathcal{F})$ is a non-empty clopen subgroup of $\Sigma^\Box_{*,\mathcal{N}_0(p)}\mathbb{R}^{X_0}$. Thus, $\Sigma^\Box_{*,\mathcal{N}_0(p)}\mathbb{R}^{X_0}$ can be expressed as the free topological sum of the cosets of the quotient group $G = \Sigma^\Box_{*,\mathcal{N}_0(p)}\mathbb{R}^{X_0}/C$:

$$\Sigma^\Box_{*,\mathcal{N}_0(p)}\mathbb{R}^{X_0} \cong \bigoplus_{D \in G} D \cong \bigoplus_{\beta < |G|} (C)_\beta.$$

Now, we are going to prove that $|G|$ is greater or equal to 2^ω. In fact, take an almost disjoint family $\mathcal{B}$ of B of cardinality 2^ω, and for each $T \in \mathcal{B}$ we define $f_T : X_0 \to \mathbb{R}$ as

$$f_T(x) = \begin{cases} \frac{1}{2^{n-1}} & \text{if } x = x_n \text{ and } x_n \in T \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $f_T \in \Sigma^{\square}_{*,\mathcal{N}_0(p)}\mathbb{R}^{X_0}$. Furthermore, if $T_1, T_2 \in \mathcal{B}$ are different, $f_{T_1} - f_{T_2}$ does not belong to C. In fact, since $T_1 \setminus T_2$ is an infinite set, there is a sequence $\{n_k : k < \omega\}$ of natural numbers, strictly increasing, such that $\{x_{n_k} : k < \omega\} \subset T_1 \setminus T_2$. For each $m < \omega$, there is k_m such that $n_{k_m} > m$; so, $(f_{T_1} - f_{T_2})(x_{n_{k_m}}) = \frac{1}{2^{n_{k_m}-1}} \notin [-\frac{1}{2^{n_{k_m}}}, \frac{1}{2^{n_{k_m}}}]$. Then, $f_{T_1} - f_{T_2} \notin E_{0,m}(\mathcal{F})$. Therefore, $f_{T_1} - f_{T_2} \notin E_0(\mathcal{F})$. We conclude that $|G| \geq 2^\omega$.

Now, because of Theorem 4.8 and the facts we have already shown in this proof,

$$C_\square(X) \cong \bigoplus_{\alpha<2^{\aleph_0}} \Sigma^{\square}_{*,\mathcal{N}_0(p)}\mathbb{R}^{X_0} \cong \bigoplus_{\alpha<2^{\aleph_0}\cdot|G|} (C)_\alpha \cong \bigoplus_{\alpha<|G|} (C)_\alpha \cong \Sigma^{\square}_{*,\mathcal{N}_0(p)}\mathbb{R}^{X_0}.$$

This means that $\mathrm{hop}(\Sigma^{\square}_{*,\mathcal{N}_0(p)}\mathbb{R}^{X_0}) \geq (2^{\aleph_0})^+$. □

Next, we summarize the results given in Theorem 4.8, Corollary 4.10, Proposition 4.11 and in formulas (1), (2) which appear after Corollary 4.6.

4.12 Theorem. *Let X be a space with $X_1 = \{p\}$. Then, the following propositions hold.*

(1) *If $\mathcal{N}_0(p)$ is ω^+-complete and X is a-ω-rwrtX_1, then*

$$C_\square(X) \cong \Sigma^{\square}_{\mathcal{N}_0(p)}\mathbb{R}^{X_0}.$$

(2) *If $\mathcal{N}_0(p)$ is not ω^+-complete, then*

$$C_\square(X) \cong \Sigma^{\square}_{*,\mathcal{N}_0(p)}\mathbb{R}^{X_0}.$$

4.13 Examples.

(1) Let κ be an infinite cardinal and let $p \in \beta\kappa \setminus \kappa$. Consider the subspace $X = \kappa \cup \{p\}$ of $\beta\kappa$. If p is not ω^+-complete, then

$$\Sigma^{\square}_{*,p}\mathbb{R}^\kappa \cong \square\mathbb{R}^\kappa$$

(see Theorem 4.12.(2) and the remark after Theorem 4.3).

(2) Let $L_\kappa(X_0) = \{p\} \cup X_0$ be the $<\kappa^+$-Lindelöfication of the discrete space X_0 where $p \notin X_0$ (that is, every point in X_0 is isolated and a system of neighborhoods of p is $\{V \subset L_\kappa(X_0) : p \in V$ and $|L_\kappa(X_0) \setminus V| < \kappa^+\}$). The filter $\mathcal{N}_0(p)$ is ω^+-complete, and if $\mathrm{cof}(|X_0|) > \kappa$, then $L_\kappa(X_0)$ is a-ω-rwrt$\{p\}$. Since $\{p\} = (L_\kappa(X_0))_1$, $C_\square(L_\kappa(X_0)) \cong \Sigma^{\square}_{\mathcal{N}_0(p)}\mathbb{R}^{X_0}$.

(3) In particular, if $|X_0| = \aleph_1$, then the space of the real-valued continuous functions defined in the Lindelöfication of X_0 with the box product topology, $C_\square(L(X_0))$, is homeomorphic to the Σ-product $\Sigma^{\square}_{\widehat{0}}\mathbb{R}^{\aleph_1}$ of $\square\mathbb{R}^{\aleph_1}$ based on $\widehat{0}$ with the box product topology.

This last example raises the following problem:

4.14 Problem. *Is $\Sigma_{\widehat{0}}^{\square}\mathbb{R}^{\aleph_1}$ homeomorphic to $\square\mathbb{R}^{\aleph_1}$ if $2^{\omega} = 2^{\omega_1}$?*

The last results of this section take advantage, once more, of Theorem 2.7 in order to obtain relations between product spaces $\square\mathbb{R}^{\kappa}$ and their Σ-products, and they give positive answers to some variations of Problem 4.14.

For infinite cardinal numbers γ and κ with $\gamma \leq \kappa$, we define

$$\Sigma_{\widehat{0},\gamma}^{\square}\mathbb{R}^{\kappa} = \{f \in \square\mathbb{R}^{\kappa} : |\{\lambda < \kappa : f(\lambda) \neq 0\}| < \gamma\}.$$

4.15 Proposition. *For uncountable cardinals γ and κ with $\gamma \leq \kappa$, $\mathrm{cof}(\gamma) > \aleph_0$ and $\mathrm{cof}(\kappa) = \aleph_0$, $\square\mathbb{R}^{\kappa}$ is homeomorphic to $\bigoplus_{\lambda<2^{\kappa}}(\Sigma_{\widehat{0},\gamma}^{\square}\mathbb{R}^{\kappa})_{\lambda}$.*

Proof: Let $Y = \{q \in \kappa^* : |F| \geq \gamma \;\forall\, F \in q\}$. Take $X = \kappa \cup Y$ with its topology inherited from $\beta\kappa$. Since $\mathrm{cof}(\kappa) = \aleph_0$, κ $(= X_0)$ is a-ω-rwrtX_1. Then, Theorem 2.7 guarantees that $C_{\square}(X)$ is homeomorphic to $\square\mathbb{R}^{\kappa}$.

On the other hand, the isomorphic embedding ϕ from $A_{\widehat{0}}(X)$ to $\square\mathbb{R}^{\kappa}$, defined in Lemma 1.2, satisfies $\phi[A_{\widehat{0}}(X)] = \Sigma_{\widehat{0},\gamma}^{\square}\mathbb{R}^{\kappa}$ (here we use the hypothesis $\mathrm{cof}(\gamma) > \omega$). Moreover, X is normal and Y is closed in X, so $\widehat{C}(Y) = C(Y)$. Since Y is a compact space, $|C(Y)| = 2^{\kappa}$ (Proposition 1.14). Hence, by Corollary 1.7, $C_{\square}(X) \cong \bigoplus_{\lambda<2^{\kappa}}(\Sigma_{\widehat{0},\gamma}^{\square}\mathbb{R}^{\kappa})_{\lambda}$.

$\square$

As a consequence of the previous result we obtain:

4.16 Corollary. *Let κ and γ be two cardinal numbers which satisfy the same properties given in the hypotheses of Proposition 4.15. Then, $\square\mathbb{R}^{\kappa} \cong \Sigma_{\widehat{0},\gamma}^{\square}\mathbb{R}^{\kappa}$ iff $\mathrm{hop}(\Sigma_{\widehat{0},\gamma}^{\square}\mathbb{R}^{\kappa}) = (2^{\kappa})^{+}$.*

4.17 Corollary. *Let $\gamma < \kappa$, $\mathrm{cof}(\kappa) = \omega$, $\mathrm{cof}(\gamma) > \omega$. Assume that there is a cardinal number $\tau < \gamma$ such that $2^{\tau} = 2^{\kappa}$. Then, $\mathrm{hop}(\Sigma_{\widehat{0},\gamma}^{\square}\mathbb{R}^{\kappa}) = (2^{\kappa})^{+}$ and $\square\mathbb{R}^{\kappa} \cong \Sigma_{\widehat{0},\gamma}^{\square}\mathbb{R}^{\kappa}$.*

Proof: We have that $\Sigma_{\widehat{0},\gamma}^{\square}\mathbb{R}^{\kappa}$ is equal to $\Sigma_{\widehat{0},\mathcal{N}(p)}^{\square}\mathbb{R}^{X_0}$ where X is the $<\gamma$-Lindelöfication of the discrete space of cardinality κ. Corollary 4.9 implies that $\mathrm{hop}(\Sigma_{\widehat{0},\gamma}^{\square}\mathbb{R}^{\kappa}) \geq (2^{\tau})^{+}$. This inequality plus the hypothesis $2^{\tau} = 2^{\kappa}$ plus Proposition 4.15 imply $\square\mathbb{R}^{\kappa} \cong \Sigma_{\widehat{0},\gamma}^{\square}\mathbb{R}^{\kappa}$. $\square$

5. Spaces $C_{\square}(X)$ when X is countably compact

We have already pointed out that for a set S the "Σ-product" $\Sigma = \{f \in \mathbb{R}^{S} : \forall\, \epsilon > 0,\ |\{x \in S : |f(x)| \geq \epsilon\}| < \aleph_0\}$ of $\mathbb{R}^{S}$, coincides with the Σ-product

$\Sigma^{\square}_{*,\mathcal{F}_0}\mathbb{R}^S$ where $\mathcal{F}_0$ is the Fréchet filter on S ($F \in \mathcal{F}_0$ iff $S \setminus F$ is finite). That is, Σ is equal to $\Sigma^{\square}_{*,\mathcal{N}(p)}\mathbb{R}^S$ where p is the point which compactifies the discrete space S. Recall that for a topological space X, X_0 is its subset of isolated points, $X_1 = X \setminus X_0$, $X^b = X_1 \cap \mathrm{cl}_X X_0$ and $Z = X_1 \setminus X^b$. Moreover, every space X in this article satisfies $X_0 \neq \emptyset \neq X^b$.

5.1 Proposition. *Let X be such that every infinite subspace of X_0 has a cluster point in X. Then $A_{\widehat{0}}(X)$ is homeomorphic to $\Sigma^{\square}_{*,\mathcal{F}_0}\mathbb{R}^{X_0}$.*

PROOF: It is easy to verify that the range of the isomorphic embedding $\phi : A_{\widehat{0}}(X) \to \square\mathbb{R}^{X_0}$ (see Lemma 1.2) coincides with $\Sigma^{\square}_{*,\mathcal{F}_0}\mathbb{R}^{X_0}$. $\square$

5.2 Theorem. *Let X be such that every infinite subset of X_0 has a cluster point in X, and such that X is a-ω-rwrtX_1. Then $C_\square(X)$ is homeomorphic to $\bigoplus_{\widehat{x}\in\widehat{C}(X_1)}(\Sigma^{\square}_{*,\mathcal{F}_0}\mathbb{R}^{X_0})_{\widehat{x}}$. In particular, $C_\square(X) \cong \Sigma^{\square}_{*,\mathcal{F}_0}\mathbb{R}^{X_0}$ if $|\widehat{C}(X_1)| \leq 2^\omega$.*

PROOF: Since X is a-ω-rwrtX_1, $A_{\widehat{0}}(X)$ is clopen in $C_\square(X)$. So,

$$C_\square(X) \cong \bigoplus_{\widehat{x}\in\widehat{C}(X_1)} A_{\widehat{x}}(X) \cong \bigoplus_{\widehat{x}\in\widehat{C}(X_1)} (\Sigma^{\square}_{*,\mathcal{F}_0}\mathbb{R}^{X_0})_{\widehat{x}}$$

(Proposition 5.1). The last assertion follows from Proposition 4.11. $\square$

Since every countably compact space is almost-ω-resolvable, the following result is a consequence of the previous theorem.

5.3 Corollary. *If X is a countably compact space, then*

$$C_\square(X) \cong \bigoplus_{\widehat{x}\in\widehat{C}(X_1)} (\Sigma^{\square}_{*,\mathcal{F}_0}\mathbb{R}^{X_0})_{\widehat{x}}.$$

If in addition $|\widehat{C}(X_1)| \leq 2^\omega$, we obtain $C_\square(X) \cong \Sigma^{\square}_{*,\mathcal{F}_0}\mathbb{R}^{X_0}$.

Observe that if X is a pseudocompact space and N is an infinite countable subset of X_0, hence N must have a cluster point in X. So, it is natural to ask:

5.4 Problem. *Is X a-ω-rwrtX_1 if X is a pseudocompact space?*

5.5 Corollary. (1) *For every infinite cardinal number κ, $C_\square(\beta\kappa)$ is homeomorphic to $\bigoplus_{\lambda<2^\kappa}(\Sigma^{\square}_{*,\mathcal{F}_0}\mathbb{R}^\kappa)_\lambda$.*

(2) *For every $q \in \omega^*$, $\square\mathbb{R}^\omega \cong \Sigma^{\square}_{*,\mathcal{F}_0}\mathbb{R}^\omega \cong \Sigma^{\square}_{*,q}\mathbb{R}^\omega$.*

PROOF: Assertion (1) follows from Corollary 5.3 and from the equalities: $(\beta\kappa)_1 = \kappa^*$, $\widehat{C}(\kappa^*) = C(\kappa^*)$, $w(\kappa^*) = 2^\kappa$ and (by Proposition 1.14) $|C(\kappa^*)| = w(\kappa^*)^\omega = 2^\kappa$.

Assertion (2) is implied by Theorem 2.7 and Corollary 5.3. $\square$

Again, using Proposition 1.14, we will prove the following lemma.

5.6 Lemma. *For every infinite ordinal number α, $|C([0,\alpha))| = |\alpha|^\omega$.*

Proof: For every ordinal number α, the space $[0,\alpha]$ is compact; so, $|C([0,\alpha])| = |\alpha|^\omega$ by Proposition 1.14. If α is a successor ordinal, $[0,\alpha)$ is compact and, again, $|C([0,\alpha))| = |\alpha|^\omega$. Moreover, if α is a limit ordinal with uncountable cofinality, then $|C([0,\alpha))| = |C([0,\alpha])| = |\alpha|^\omega$.

Now, we only have to prove the equality $|C([0,\alpha))| = |\alpha|^\omega$ when α is a limit ordinal with countable cofinality. In this case, if $\alpha_0 < \alpha_1 < \dots < \alpha_n < \dots$ and $sup\{\alpha_n : n < \omega\} = \alpha$, thus $C([0,\alpha)) \cong \prod_{n<\omega} C((\alpha_n, \alpha_{n+1}])$. Then, $|C([0,\alpha))| \leq \prod_{n<\omega} |C([0,\alpha_n])| = \prod_{n<\omega} |\alpha_n|^\omega \leq |\alpha|^\omega$.

On the other hand, $|\alpha|^\omega = |C([0,\alpha])| \leq |C([0,\alpha))|$ because the relation $f \to f \restriction [0,\alpha)$ is a one-to-one function from $C([0,\alpha])$ to $C([0,\alpha))$. $\square$

So, the following results are a consequence of Corollary 5.3 and Lemma 5.6.

5.7 Corollary. *Let α be an infinite ordinal. Then,*

(1) $C_\square([0,\alpha]) \cong \bigoplus_{\lambda<|\alpha|^\omega} (\Sigma^\square_{*,\mathcal{F}_0}\mathbb{R}^{|\alpha|})_\lambda$;

(2) $C_\square([0,\alpha)) \cong \bigoplus_{\lambda<|\alpha|^\omega} (\Sigma^\square_{*,\mathcal{F}_0}\mathbb{R}^{|\alpha|})_\lambda$ *if* $\operatorname{cof}(\alpha) > \aleph_0$;

(3) $\square\mathbb{R}^\omega \cong C_\square([0,\omega])$ *and* $C_\square([0,\omega_1]) \cong C_\square([0,\omega_1)) \cong \Sigma^\square_{*,\mathcal{F}_0}\mathbb{R}^{\omega_1}$.

Observe that the connected component of a point x in $\Sigma^\square_{*,\mathcal{F}_0}\mathbb{R}^{|\alpha|}$ is homeomorphic to $\sigma^\square_x\mathbb{R}^{|\alpha|}$. By Proposition 1.12 and the previous results we obtain:

5.8 Corollary. *Let α and β be two infinite ordinals, both with cofinality different from $\aleph_0$. Then, $C_\square([0,\alpha)) \cong C_\square([0,\beta))$ if and only if $|\alpha| = |\beta|$.*

Consider the set

$$\widetilde{\Sigma}_{*,\mathcal{F}_0}\mathbb{R}^{[0,\alpha)} = \big\{f \in \mathbb{R}^{[0,\alpha)} : \text{ for all } \epsilon > 0 \text{ and } \beta < \alpha, \\ |\{\lambda < \beta : |f(\lambda)| \geq \epsilon\}| < \aleph_0\big\}.$$

We denote this set with its box topology by $\widetilde{\Sigma}^\square_{*,\mathcal{F}_0}\mathbb{R}^{[0,\alpha)}$. We leave the proof of the following result to the reader.

5.9 Proposition. *Let α be an ordinal number such that $|\alpha| > \aleph_0$ and $\operatorname{cof}(\alpha) = \aleph_0$. Then,*

$$C_\square([0,\alpha)) \cong \bigoplus_{\lambda<|\alpha|^\omega} (\widetilde{\Sigma}^\square_{*,\mathcal{F}_0}\mathbb{R}^{[0,\alpha)})_\lambda.$$

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