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A d.c. C^1 function need not be difference of convex C^1 functions

David Pavlica

Abstract. In [2] a delta convex function on $\mathbb{R}^2$ is constructed which is strictly differentiable at 0 but it is not representable as a difference of two convex function of this property. We improve this result by constructing a delta convex function of class $C^1(\mathbb{R}^2)$ which cannot be represented as a difference of two convex functions differentiable at 0. Further we give an example of a delta convex function differentiable everywhere which is not strictly differentiable at 0.

Keywords: differentiability, delta-convex functions

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Let X be a normed vector space. We say that a function $f\colon X \to \mathbb{R}$ is *delta convex* (d.c.) if there exist continuous convex functions f_1, f_2 on X such that $f = f_1 - f_2$.

We denote $B(a,r) = \{x \in X : \|x-a\| \le r\}$. Let g be a function defined on an open set $A \subset X$. We say that $L \in X^*$ is the *strict derivative at a point* $a \in A$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for each $x, y \in B(a,\delta)$ we have

$$|g(x) - g(y) - L(x-y)| \le \varepsilon\|x-y\|.$$

Note that if a convex function on X is Fréchet differentiable at a point a then it is strictly differentiable at a ([6, Proposition 3.8]).

If X is a finite dimensional space then every function $f \in C^2(X)$ can be represented as $f = f_1 - f_2$, where f_1, f_2 are convex and $f_1 \in C^2(X)$, $f_2 \in C^\infty(X)$ (see [3], where other related results are obtained).

In [2], a d.c. function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is constructed which is strictly differentiable at 0 and is not representable as a difference of two convex functions with this property. But this function is not differentiable everywhere. We shall improve the construction of [2] to obtain a d.c. function of class $C^1(\mathbb{R}^2)$ not representable as a difference of convex functions differentiable at 0.

We shall denote λ_n the Lebesgue measure on $\mathbb{R}^n$. We say that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is *Lipschitz* with the constant L if for each $x, y \in \mathbb{R}^2$ is $|f(x) - f(y)| \le L\|x-y\|$.

In the following we shall use the notion of the dual convex function.

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Definition. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a convex function. The *dual function* f^* of the function f is defined on $(\mathbb{R}^n)^*$ by

$$f^*(x^*) = \sup_{x\in\mathbb{R}^n} \left(\langle x, x^*\rangle - f(x)\right), \qquad x^* \in (\mathbb{R}^n)^*.$$

It follows immediately from the definition that if $f, g\colon \mathbb{R}^n \to \mathbb{R}$ are convex functions, $f \le g$ and f^* is finite everywhere then g^* is finite everywhere. Therefore if $f \ge \|\cdot\|^2 - 1$ then f^* is finite everywhere.

As usual, we identify the dual space $(\mathbb{R}^n)^*$ with $\mathbb{R}^n$ and $\langle\cdot,\cdot\rangle$ denotes both the duality and the scalar product.

Facts. *If $f\colon \mathbb{R}^n \to \mathbb{R}$ is a convex function and f^* is finite everywhere then*

$$(f^*)^* = f, \tag{1}$$

$$x^* \in \partial f(x) \Leftrightarrow x \in \partial f^*(x^*). \tag{2}$$

The statement (1) can be found in [4, Theorem 12.2] and (2) in [4, Theorem 23.5].

In [2] a function $\bar{G}\colon \mathbb{R}^2 \to \mathbb{R}$ is constructed in the following way.

Fix a sequence of positive integers $\{k_i\}$ such that $\cos(\frac{2\pi}{k_i}) \ge 1 - 2^{-i-3}$ for $i \in \mathbb{N}$. Let us denote

$$M := \left\{ \left(2^{-i}\cos\left(\frac{2\pi k}{k_i}\right), 2^{-i}\sin\left(\frac{2\pi k}{k_i}\right)\right) : i \in \mathbb{N}, k \in \{1, \dots, k_i\} \right\}.$$

Set

$$F(x) = \|x\| + 4\|x\|^2 \qquad \text{for } \ x \in \mathbb{R}^2.$$

For each $z \in M$ define

$$G_z(x) = F(z) + \langle F'(z), x - z\rangle = (8\|z\| + 1)\frac{\langle x, z\rangle}{\|z\|} - 4\|z\|^2.$$

Since F is convex we have $G_z \le F$ on $\mathbb{R}^2$. Let us define for $x \in \mathbb{R}^2$

$$\bar{G}(x) = \sup\left\{G_z(x) : z \in M\right\}, \qquad G(x) = \max\{\bar{G}(x), \|x\|^2 - 1\}.$$

Obviously $\bar{G}$ and G are convex functions,

The following 3 lemmas are proved in [2] (Lemmas 3,4,5).

Lemma 1. *The function $\bar{G}$ satisfies*

$$\|x\| + \|x\|^2 \le \bar{G}(x) \le \|x\| + 4\|x\|^2 = F(x)$$

for $\|x\| < 1$.

Corollary 1. *Therefore $G \equiv \bar{G}$ on $B(0,1)$ and $\partial G(0) = \partial \bar{G}(0) = B(0,1)$. (Indeed, $\partial(\|\cdot\| + a\|\cdot\|^2)(0) = B(0,1)$ for each $a \geq 0$.)*

Lemma 2. *If $x \in \mathbb{R}^2$, $\|x\| < 1$, $z \in M$, $\|z\| \leq \frac{\|x\|}{9}$ then*

$$G_z(x) \leq \bar{G}(x) - \frac{\|x\|^2}{9}.$$

Lemma 3. *If $x \in \mathbb{R}^2$, $0 < \|x\| < \frac{1}{16}$ and*

$$M_x := \{z \in M : \|z\| \leq 2\|x\|, \langle x, z\rangle \geq \|z\| \cdot \|x\|(1 - 8\|z\|)\}$$

then

$$\bar{G}(x) = \sup\{G_z(x) : z \in M_x\}.$$

Corollary 2. *Let $x \in \mathbb{R}^2$, $0 < \|x\| < \frac{1}{16}$. Then there exists a neighbourhood W of x such that, for $w \in W$,*

$$G(w) = \sup\{G_z(w) : z \in M_x\}$$

holds.

Proof: The set $N_z := \{u \in \mathbb{R}^2 : 0 < \|u\| < \frac{1}{16}, z \notin M_u\}$ is obviously open for all $z \in M$. Hence

$$U := \bigcap_{z \in M \setminus (M_x \cup B(0, \frac{\|x\|}{18}))} N_z$$

is a neighbourhood of x. Since $M_x \cup B(0, \frac{\|x\|}{18}) \supset M_w$ for every $w \in U$, we conclude, using Lemma 2 and Lemma 3 for w, that we can put $W = U \cap B(x, \|x\|/2)$. $\square$

Lemma 4. *Let $\hat{G}_\alpha : \mathbb{R}^2 \to \mathbb{R}$, $\alpha \in A$, be a family of affine functions with the Lipschitz constant L and $\hat{G}(w) = \sup\{\hat{G}_\alpha(w) : \alpha \in A\}$ for $w \in \mathbb{R}^2$, $\hat{G} : \mathbb{R}^2 \to \mathbb{R}$. Let $x \in \mathbb{R}^2$ and $u^* \in \partial \hat{G}(x)$. Then $\|u^*\| \leq L$.*

Proof: The function $\hat{G}(w)$ is obviously Lipschitz with the constant L. Therefore $\|u^*\| \leq L$. $\square$

Lemma 5. *If $x \in \mathbb{R}^2$, $0 < \|x\| < \frac{1}{16}$ and $x^* \in \partial G(x)$, then*

$$\left\| x^* - \frac{x}{\|x\|} \right\| \leq 24\|x\|^{1/2}.$$

Proof: Let $z \in M_x$ and

$$y^* = \frac{z}{\|z\|} + 8z \in \partial G_z(x).$$

Clearly

$$\left\| \frac{z}{\|z\|} - \frac{x}{\|x\|} \right\|^2 = 2 - \frac{2\langle z, x\rangle}{\|z\| \cdot \|x\|}$$

and by the definition of M_x we have $1 - \frac{\langle z,x\rangle}{\|z\| \|x\|} \leq 8\|z\|$ and $\|z\| \leq 2\|x\|$. Therefore

$$\begin{aligned} \left\| y^* - \frac{x}{\|x\|} \right\| &\leq 8\|z\| + \left\| \frac{z}{\|z\|} - \frac{x}{\|x\|} \right\| \\ &= 8\|z\| + \left(2 - \frac{2\langle z, x\rangle}{\|z\| \cdot \|x\|} \right)^{1/2} \leq 16\|x\| + (2 \cdot 8\|z\|)^{1/2} \\ &\leq 16\|x\|^{1/2} + (32\|x\|)^{1/2} \leq 24\|x\|^{1/2}. \end{aligned}$$

Therefore $G_z - \langle \frac{x}{\|x\|}, \cdot \rangle$ is Lipschitz with the constant $24\|x\|^{1/2}$ for $z \in M_x$. Using Corollary 2 and Lemma 4 applied for $G_z - \langle \frac{x}{\|x\|}, \cdot \rangle$, $z \in M_x$, we obtain $\|u^*\| \leq 24\|x\|^{1/2}$ for $u^* \in \partial(G - \langle \frac{x}{\|x\|}, \cdot \rangle)(x)$. Since

$$x^* - \frac{x}{\|x\|} \in \partial \left(G - \left\langle \frac{x}{\|x\|}, \cdot \right\rangle \right)(x),$$

whenever $x^* \in \partial G(x)$, this completes the proof of Lemma 5. □

By Corollary 1, $G^* \equiv 0$ on $B(0,1)$ since $G(0) = 0$.

Define a function $\alpha \colon [0, +\infty) \to \mathbb{R}$,

$$\begin{aligned} \alpha(t) &= 0, & t &\in [0,1), \\ &= (t-1)^4, & t &\in [1, +\infty), \end{aligned}$$

and $\psi(x^*) := \alpha(\|x^*\|)$, for $x^* \in \mathbb{R}^2$. Then ψ is a convex function on $\mathbb{R}^2$, since $\|\cdot\|$ is convex and α is convex and increasing. Notice that

$$\psi'(x^*) = 4(\|x^*\| - 1)^3 \frac{x^*}{\|x^*\|}$$

for $\|x^*\| \geq 1$.

Set $K := G^* + \psi$ and $\tilde{G} := K^*$.

The function $\tilde{G}$ is differentiable on $\mathbb{R}^2 \setminus \{(0,0)\}$. Otherwise there exist $x \in \mathbb{R}^2 \setminus \{(0,0)\}$ and $x^*, y^* \in \partial\tilde{G}(x)$, $x^* \neq y^*$. Then $x \in \partial K(x^*) \cap \partial K(y^*)$.

It is easy to see that then K is affine on $\operatorname{conv}\{x^*, y^*\}$ and $x \in \partial K(z^*)$, for each $z^* \in \operatorname{conv}\{x^*, y^*\}$. Since $K \equiv 0$ on $B(0,1)$ and $x \neq 0$, the interior of $B(0,1)$ is disjoint with $\operatorname{conv}\{x^*, y^*\}$. Further there is no line segment in $\partial B(0,1)$, consequently the function K is affine on some line segment in $\mathbb{R}^2 \setminus B(0,1)$. Also ψ is affine on this line segment (since ψ and G^* are convex). But it is impossible since ψ' is one-to-one on $\mathbb{R}^2 \setminus B(0,1)$.

Lemma 6. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in \mathbb{R}^2$, $0 < \|x\| < \delta$, then*

$$\left\| (\tilde{G})'(x) - \frac{x}{\|x\|} \right\| \leq \varepsilon.$$

Proof: Set

$$\delta := \min \left\{ \left(\frac{\varepsilon}{9 \cdot 24^3} \right)^2, \frac{1}{16} \right\}.$$

Let $0 < \|x\| < \delta$. Denote $x^* := (\tilde{G})'(x)$ and $x' := x - \psi'(x^*)$. Then, by Fact (2), $x \in \partial K(x^*)$ and therefore, since $K \equiv 0$ on $B(0,1)$, we have $\|x^*\| \geq 1$.

Clearly $x' \in \partial(K - \psi)(x^*) = \partial G^*(x^*)$ and, using again Fact (2), $x^* \in \partial G(x')$. Further $x' \neq 0$, since if $\|x^*\| = 1$ then clearly $x' = x$ and if $\|x^*\| > 1$ we use $x^* \in \partial G(x')$ and Corollary 1.

Since ∂G is monotone, $0 \in \partial G(0)$ and $x^* \in \partial G(x')$, we have $\langle x', x^* \rangle \geq 0$. Hence

$$\langle x', \psi'(x^*) \rangle = \langle x', x^* \rangle \frac{4(\|x^*\| - 1)^3}{\|x^*\|} \geq 0.$$

Consequently $\|x'\|^2 = \langle x', x - \psi'(x^*) \rangle \leq \langle x', x \rangle \leq \|x'\| \cdot \|x\|$ which implies $\|x'\| \leq \|x\| < \delta$. Now we compute, using Lemma 5 for x',

$$\begin{aligned}
\left\| (\tilde{G})'(x) - \frac{x}{\|x\|} \right\| &\leq \left\| x^* - \frac{x'}{\|x'\|} \right\| + \left\| \frac{x'}{\|x'\|} - \frac{x}{\|x\|} \right\| \\
&\leq 24\|x'\|^{1/2} + \left\| \frac{(\|x\|x' - \|x'\|x') + (\|x'\|x' - \|x'\|x)}{\|x\| \cdot \|x'\|} \right\| \\
&\leq 24\|x'\|^{1/2} + \frac{2\|x - x'\|}{\|x\|} = 24\|x'\|^{1/2} + \frac{2\|\psi'(x^*)\|}{\|x\|} \\
&\leq 24\delta^{1/2} + \frac{8(\|x^*\| - 1)^3}{\|x\|} \leq 24\delta^{1/2} + 8\frac{24^3\|x'\|^{3/2}}{\|x\|} \\
&\leq \delta^{1/2}(24 + 8 \cdot 24^3) \leq \varepsilon,
\end{aligned}$$

since by Lemma 5 we also have $\|x^*\| - 1 \leq 24\|x'\|^{1/2}$. □

Theorem. *The function $H := \tilde{G} - \|\cdot\|$ is a C^1 delta-convex function on $\mathbb{R}^2$ and there does not exists a convex function h differentiable at the origin such that $H + h$ is convex.*

Proof: As was already proved, $\tilde{G}$ is differentiable on $\mathbb{R}^2 \setminus \{(0,0)\}$ and therefore, since it is convex, $\tilde{G}$ is also C^1 on $\mathbb{R}^2 \setminus \{(0,0)\}$. Obviously $\|\cdot\|$ is C^1 on $\mathbb{R}^2 \setminus \{(0,0)\}$. Hence $H \in C^1(\mathbb{R}^2 \setminus \{(0,0)\})$. The Fréchet derivative of H at the origin is 0 since, by Lemma 6, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|H(u) - H(0)| = \left| \int_0^1 \langle u, H'(tu) \rangle \, dt \right| \leq \int_0^1 \|H'(tu)\| \, dt \|u\| \leq \varepsilon \|u\|,$$

for each $u \in \mathbb{R}^2$, $0 < \|u\| < \delta$. It also follows immediately from Lemma 6 that H' is continuous at the origin.

Now we shall prove that H has no control function differentiable at 0. For a contradiction let us suppose that $h, h+H$ are convex functions on $\mathbb{R}^2$ and h is differentiable at 0. We may assume $h'(0) = 0$. Then 0 is the strict derivative of h at 0 ([3, Proposition 3.8]). Find $0 < R < 1/(8^2 \cdot 24^6)$ such that

$$|h(x) - h(y)| < \frac{1}{48}\|x - y\| \quad \text{if } \ x, y \in B(0, 2R).$$

Denote for $z \in M$

$$\begin{aligned} S_z &:= \{x \in [-R/2, R/2]^2 : G(x) = G_z(x)\}, \\ \hat{S}_z &:= S_z + \psi'(F'(z)), \quad \hat{S} := \bigcup_{z \in M} \hat{S}_z. \end{aligned}$$

Claim 1. *The function $\tilde{G}$ is affine on $\hat{S}_z$ for each $z \in M$. Further, for $z_1, z_2 \in M, z_1 \neq z_2$, we have $\operatorname{int} \hat{S}_{z_1} \cap \operatorname{int} \hat{S}_{z_2} = \emptyset$.*

Proof of Claim 1:

If $z \in M$ and $u \in S_z$ then clearly $F'(z) \in \partial G(u)$. By Fact (2) we have $u \in \partial G^*(F'(z))$. Hence $u + \psi'(F'(z)) \in \partial K(F'(z))$. Now, again by Fact (2), $F'(z) \in \partial \tilde{G}(u + \psi'(F'(z)))$. Therefore $\tilde{G}$ is affine on $\hat{S}_z$.

Finally $\operatorname{int} \hat{S}_{z_1} \cap \operatorname{int} \hat{S}_{z_2} = \emptyset$ since $F'(z_1) \neq F'(z_2)$, for $z_1 \neq z_2$. □

Claim 2. $\hat{S}_z \subset [-R, R]^2$ *for $z \in M$.*

Proof of Claim 2:

Let $z \in M$, $u \in S_z$. By Lemma 5, since $F'(z) \in \partial G(u)$, we have $\|F'(z)\| - 1 \leq 24\|u\|^{1/2} \leq 24 \cdot (R)^{1/2}$.

We easily compute

$$\|F'(z)\| = \left\| \frac{z}{\|z\|} + 8z \right\| = 1 + 8\|z\| > 1.$$

Hence

$$\begin{aligned} \|\psi'(F'(z))\| &= \left\| 4(\|F'(z)\| - 1)^3 \cdot \frac{F'(z)}{\|F'(z)\|} \right\| \leq 4 \cdot 24^3 \cdot (R)^{3/2} \\ &< 4 \cdot 24^3 \left(\frac{1}{8^2 \cdot 24^6} \right)^{1/2} R = \frac{R}{2}\,. \end{aligned}$$

This proves Claim 2. □

According to Lemma 2, for each $0 < \delta < 1$, $G = \sup\{G_z : z \in M \setminus B(0, \delta/9)\}$ on $B(0,1) \setminus B(0,\delta)$.

Hence, for each $\delta > 0$, the function G is defined on $B(0,1) \setminus B(0,\delta)$ as a supremum of finitely many G_z. Therefore $\bigcup_{z\in M} S_z = [-R/2, R/2] \setminus \{(0,0)\}$. Since S_z are convex we get by Claim 1

$$\lambda_2(\hat{S}) = \sum_{z\in M} \lambda_2(S_z) = R^2.$$

Without loss of generality we may assume

$$\lambda_2(\hat{S} \cap \{(t_1,t_2) \in \mathbb{R}^2 : 0 \le t_1 \le R, -t_1 \le t_2 \le t_1\}) \ge \frac{R^2}{4}\,.$$

By Fubini's Theorem

$$\int_0^R \lambda_1(\{t_2 \in [-t_1, t_1] : (t_1,t_2) \in \hat{S}\})\, dt_1 \ge \frac{R^2}{4}\,.$$

Thus there exists $0 < r < R$ such that

$$\lambda_1(\{t_2 \in [-r,r] : (r,t_2) \in \hat{S}\}) \ge \frac{R}{4} > \frac{r}{4}\,.$$

Let us denote for $t \in [-r,r]$

$$\begin{aligned}\phi(t) &:= \|(r,t)\|,\\ \gamma(t) &:= \tilde{G}((r,t)),\\ \kappa(t) &:= h((r,t)).\end{aligned}$$

By Claim 1 the function γ is affine on the interval $\bar{S}_z := \{t \in [-r,r] : (r,t) \in \hat{S}_z\}$ for $z \in M$ and $\lambda_1(\bigcup_{z\in M} \bar{S}_z) \ge r/4$. Therefore there exist $-r \le s_1 < t_1 \le s_2 < t_2 \le \cdots \le s_k < t_k \le r$, $k \in \mathbb{N}$, such that γ is affine on $[s_i, t_i]$, for every $1 \le i \le k$, and $\sum_{i=1}^k (t_i - s_i) \ge r/5$.

Since $\kappa + \gamma - \phi$ is convex on $[-r,r]$, for each $i = 1, \dots, k$

$$\kappa'_-(t_i) - \kappa'_+(s_i) + \gamma'_-(t_i) - \gamma'_+(s_i) - \phi'(t_i) + \phi'(s_i) \ge 0$$

holds. Obviously $\gamma'_-(t_i) = \gamma'_+(s_i)$, $i = 1, \dots, k$.

Hence, by convexity of κ, we have $\kappa'_-(r) - \kappa'_+(-r) \ge \sum_{i=1}^k (\kappa'_-(t_i) - \kappa'_+(s_i)) \ge \sum_{i=1}^k (\phi'(t_i) - \phi'(s_i))$. Since κ is Lipschitz with the constant $1/48$ on $[-r,r]$, we have

$$|\kappa'_-(r)| \le \frac{1}{48}, \quad |\kappa'_+(-r)| \le \frac{1}{48}\,.$$

By the Mean Value Theorem there exist $\xi_i \in]s_i, t_i[$ such that $\phi'(t_i) - \phi'(s_i) = \phi''(\xi_i)(t_i - s_i)$, $i = 1, \dots, k$.

$$\phi''(\xi_i) = \frac{(r^2+\xi_i^2)^{1/2} - \frac{\xi_i^2}{(r^2+\xi_i^2)^{1/2}}}{r^2+\xi_i^2} = \frac{r^2}{(r^2+\xi^2)^{3/2}} \geq \frac{r^2}{(2r^2)^{3/2}} \geq \frac{1}{4r}\,.$$

Finally we obtain

$$\begin{aligned}\frac{1}{24} &\geq \kappa'_-(r) - \kappa'_+(-r) \geq \sum_{i=1}^{k}(\phi'(t_i) - \phi'(s_i)) \\ &\geq \frac{1}{4r}\sum_{i=1}^{k}(t_i - s_i) \geq \frac{1}{20}\,,\end{aligned}$$

a contradiction.

If a convex function on a Hilbert space is Fréchet differentiable at some point then it is strictly differentiable at this point. For d.c. functions this need not be true. First example (on $\mathbb{R}^2$) of this phenomenon is probably due to A. Shapiro (see [5], [1] or [6]). But none of these functions is differentiable everywhere.

We shall give an example of a d.c. function on $\mathbb{R}^2$ differentiable at 0 which is of class C^1 on $\mathbb{R}^2 \setminus \{(0,0)\}$, but is not strictly differentiable at 0.

Set for $(x, y) \in \mathbb{R}^2$

$$\begin{aligned} f_1(x,y) &= y && \text{for } \ y \geq x^2, \\ &= x^2 + \frac{y^2}{x^2} - y && \text{for } \ x^2 > y > 0, \\ &= x^2 - y && \text{for } \ 0 \geq y. \end{aligned}$$

It is easy to check that f_1 is a continuous function with a continuous derivative on $\mathbb{R}^2 \setminus \{(0,0)\}$. The Hess's matrix of y, $x^2 + \frac{y^2}{x^2} - y$ and $x^2 - y$ is nonnegative definite for $y > x^2$, for $x^2 > y > 0$ and for $0 > y$, respectively. Since the function f_1 has a supporting affine functional at 0 and f_1 is differentiable at the points of the sets $\{y = x^2, x \neq 0\}$ and $\{y = 0, x \neq 0\}$, the function f_1 is convex on every line, therefore it is convex.

Analogously we prove that

$$\begin{aligned} f_2(x,y) &= x^2 + y && \text{for } \ y \geq 0, \\ &= x^2 + \frac{y^2}{x^2} + y && \text{for } \ 0 > y > -x^2, \\ &= -y && \text{for } \ -x^2 \geq y \end{aligned}$$

is a convex function with continuous derivative on $\mathbb{R}^2 \setminus \{(0,0)\}$.

It is easy to prove that for $(x,y) \in \mathbb{R}^2$

$$|f_1(x,y) - f_2(x,y)| \leq 3x^2,$$

therefore $f := f_1 - f_2$ is a d.c. function which is differentiable also at 0. Since

$$\frac{\partial f}{\partial y}(x,0) = -2 \quad \text{for} \quad x \neq 0$$

and

$$\frac{\partial f}{\partial y}(0,0) = 0,$$

the function f is not strictly differentiable at $(0,0)$.

References

[1] Hiriart-Urruty J.-B., *Generalized differentiability, duality and optimization for problem dealing with difference of convex functions*, Lecture Notes in Econom. and Math. Systems 256, J. Ponstein, Ed., Springer, Berlin, 1985, pp. 37–70.

[2] Kopecká E., Malý J., *Remarks on delta-convex functions*, Comment. Math. Univ. Carolinae **31.3** (1990), 501–510.

[3] Penot J.-P., Bougeard M.L., *Approximation and decomposition properties of some classes of locally d.c. functions*, Math. Programming **41** (1988), 195–227.

[4] Rockafellar R.T., *Convex Analysis*, Princeton University Press, Princeton, 1970.

[5] Shapiro A., *On functions representable as a difference of two convex functions in inequality constrained optimization*, Research report University of South Africa, 1983.

[6] Veselý L., Zajíček L., *Delta-convex mappings between Banach spaces and applications*, Dissertationes Math. **289** (1989), 1–52.

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