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# Spaces with countable $sn$-networks

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*Abstract.* In this paper, we prove that a space $X$ is a sequentially-quotient $\pi$-image of a metric space if and only if $X$ has a point-star $sn$-network consisting of $cs^*$-covers. By this result, we prove that a space $X$ is a sequentially-quotient $\pi$-image of a separable metric space if and only if $X$ has a countable $sn$-network, if and only if $X$ is a sequentially-quotient compact image of a separable metric space; this answers a question raised by Shou Lin affirmatively. We also obtain some results on spaces with countable $sn$-networks.

*Keywords:* separable metric space, sequentially-quotient ($\pi$, compact) mapping, point-star $sn$-network, $cs*$-cover

*Classification:* Primary 54C05, 54C10; Secondary 54D65, 54E40

## 1. Introduction

In his book ([8]), Shou Lin proved that a space $X$ is a quotient compact image of a separable metric space if and only if $X$ is a quotient $\pi$-image of a separable metric space, if and only if $X$ has a countable weak base. Then, are there similar results on sequentially-quotient images of separable metric spaces? Related to this question, Shou Lin and Yan proved that a space $X$ is a sequentially-quotient compact image of a separable metric space if and only if $X$ has a countable $sn$-network ([10]). But they do not know whether sequentially-quotient $\pi$-images of separable metric spaces and sequentially-quotient compact images of separable metric spaces are equivalent. So Shou Lin raised the following question ([12]).

**Question 1.1.** *Has a sequentially-quotient $\pi$-image of a separable metric space a countable $sn$-network?*

In this paper, we give a characterization of sequentially-quotient $\pi$-images of metric spaces to prove that a space $X$ is a sequentially-quotient $\pi$-image of a separable metric space if and only if $X$ has a countable $sn$-network, which answers the above question affirmatively. We also obtain some results on spaces with countable $sn$-networks.

Throughout this paper, all spaces are assumed to be regular $T_1$, and all mappings are continuous and onto. $\mathbb{N}$ and $\omega$ denote the set of all natural numbers and

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the first infinite ordinal respectively. Let $x \in X$, $\mathcal{U}$ be a collection of subsets of $X$, $f$ be a mapping. Then $st(x,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : x \in U\}$, $f(\mathcal{U}) = \{f(U) : U \in \mathcal{U}\}$. The sequence $\{x_n : n \in \mathbb{N}\}$, the sequence $\{P_n : n \in \mathbb{N}\}$ of subsets and the sequence $\{\mathcal{P}_n : n \in \mathbb{N}\}$ of collections of subsets are abbreviated to $\{x_n\}$, $\{P_n\}$ and $\{\mathcal{P}_n\}$ respectively. We use the convention that every convergent sequence contains its limit point. For example, if we say that a sequence converging to $x$ is eventually in $A$, or frequently in $A$, it is to be understood that $x \in A$. $(M, d)$ denotes a metric space with metric $d$, $B(a, \varepsilon) = \{b \in M : d(a, b) < \varepsilon\}$. $(\alpha_n)$ denotes a point of a Tychonoff-product space, the $n$-th coordinate is $\alpha_n$. For terms which are not defined here we refer to [1].

**Definition 1.2** ([16])**.** Let $(M, d)$ be a metric space. $f : M \longrightarrow X$ is said to be a $\pi$-mapping, if $d(f^{-1}(x), X - f^{-1}(U)) > 0$ for every $x \in X$ and every open neighborhood $U$ of $x$.

**Definition 1.3** ([10])**.** Let $f : X \longrightarrow Y$ be a mapping. $f$ is a quotient mapping if whenever $f^{-1}(U)$ is open in $X$, then $U$ is open in $Y$; $f$ is a sequentially-quotient mapping, if for every convergent sequence $S$ in $Y$, there is a convergent sequence $L$ in $X$ such that $f(L)$ is a subsequence of $S$; $f$ is a compact mapping, if $f^{-1}(y)$ is compact in $X$ for every $y \in Y$; $f$ is a perfect mapping, if $f$ is a closed and compact mapping.

**Remark 1.4.** (1) compact mappings defined on metric spaces are $\pi$-mappings.

(2) If the domain is sequential, then quotient mapping $\Longrightarrow$ sequentially-quotient mapping ([8, Proposition 2.1.16]).

(3) If the image is sequential, then sequentially-quotient mapping $\Longrightarrow$ quotient mapping ([8, Proposition 2.1.16]).

**Definition 1.5** ([3])**.** Let $X$ be a space.

(1) Let $x \in X$. A subset $P$ of $X$ is a sequential neighborhood of $x$ (called a sequence barrier at $x$ in [9]) if every sequence $\{x_n\}$ converging to $x$ is eventually in $P$, i.e., $x \in P$ and there is $k \in \mathbb{N}$ such that $x_n \in P$ for all $n > k$.

(2) A subset $P$ of $X$ is sequentially open if $P$ is a sequential neighborhood of $x$ for every $x \in P$. $X$ is sequential if every sequentially open subset of $X$ is open.

**Remark 1.6.** $P$ is a sequential neighborhood of $x$ if and only if every sequence $\{x_n\}$ converging to $x$ is frequently in $P$, i.e., $x \in P$ and for every $k \in \mathbb{N}$, there is $n > k$ such that $x_n \in P$.

**Definition 1.7** ([10])**.** Let $\mathcal{P}$ be a cover of a space $X$.

(1) $\mathcal{P}$ is a $k$-network of $X$, if whenever $K$ is a compact subset of an open set $U$, there is a finite $\mathcal{F} \subset \mathcal{P}$ such that $K \subset \bigcup \mathcal{F} \subset U$.

(2) $\mathcal{P}$ is a $cs$-network of $X$, if every convergent sequence $S$ converging to a point $x \in U$ with $U$ open in $X$, is eventually in $P \subset U$ for some $P \in \mathcal{P}$.

(3) $\mathcal{P}$ is a $cs^*$-network of $X$, if every convergent sequence $S$ converging to a point $x \in U$ with $U$ open in $X$, is frequently in $P \subset U$ for some $P \in \mathcal{P}$.

**Definition 1.8** ([10])**.** Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a cover of a space $X$. Assume that $\mathcal{P}$ satisfies the following (a) and (b) for every $x \in X$.

(a) $\mathcal{P}$ is a network of $X$, that is, whenever $x \in U$ with $U$ open in $X$, then $x \in P \subset U$ for some $P \in \mathcal{P}_x$, where $\mathcal{P}_x$ is called a network at $x$.

(b) If $P_1, P_2 \in \mathcal{P}_x$, then there exists $P \in \mathcal{P}_x$ such that $P \subset P_1 \cap P_2$.

(1) $\mathcal{P}$ is called a weak base of $X$, if for $G \subset X$, $G$ is open in $X$ if and only if for every $x \in G$ there exists $P \in \mathcal{P}_x$ such that $P \subset G$, where $\mathcal{P}_x$ is called a weak neighborhood base at $x$.

(2) $\mathcal{P}$ is called an $sn$-network of $X$, if every element of $\mathcal{P}_x$ is a sequential neighborhood of $x$ for every $x \in X$, where $\mathcal{P}_x$ is called an $sn$-network at $x$.

**Definition 1.9** ([4], [5], [13], [17])**.** A space $X$ is a $g$-metric space (resp. an $sn$-metric space) if $X$ has a $\sigma$-locally finite weak base (resp. a $\sigma$-locally finite $sn$-network). A space $X$ is called $g$-first countable (resp. $sn$-first countable), if $X$ has a weak base (resp. an $sn$-network) $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ such that $\mathcal{P}_x$ is countable for every $x \in X$. A space $X$ is an $\aleph$-space if $X$ has a $\sigma$-locally finite $k$-network; $X$ is an $\aleph_0$-space if $X$ has a countable $k$-network.

**Remark 1.10.** (1) In [9], $sn$-networks are said to be universal $cs$-networks; $sn$-first countable is said to be universally $csf$-countable; $sn$-metric spaces are said to be spaces with $\sigma$-locally finite universal $cs$-networks.

(2) $\aleph_0$-spaces $\Longrightarrow$ $\aleph$-spaces, $\aleph$-spaces $\Longleftrightarrow$ spaces with a $\sigma$-locally finite $cs$-network [2, Theorem 4], $\aleph_0$-spaces $\Longleftrightarrow$ spaces with a countable $cs^*$-network (see [18, Proposition C]).

(3) For a space, weak base $\Longrightarrow$ $sn$-network $\Longrightarrow$ $cs$-network ([10]). So $g$-metric spaces $\Longrightarrow$ $sn$-metric spaces $\Longrightarrow$ $\aleph$-spaces, and $g$-first countable space $\Longrightarrow$ $sn$-first countable space. Spaces with countable weak base $\Longrightarrow$ spaces with countable $sn$-networks $\Longrightarrow$ $\aleph_0$-spaces.

(4) An $sn$-network for a sequential space is a weak base ([10]). Notice that $g$-first countable $\Longrightarrow$ sequential. $g$-first countable space $\Longleftrightarrow$ sequential, $sn$-first countable space. Spaces with countable weak base $\Longleftrightarrow$ sequential, spaces with countable $sn$-network.

(5) $g$-metric space $\Longleftrightarrow$ $k$, $sn$-metric space (see [9, Theorem 3.15 and Corollary 3.16]). So every $k$, $sn$-metric space is sequential.

**Definition 1.11** ([11])**.** Let $\{\mathcal{P}_n\}$ be a sequence of covers of a space $X$.

(1) $\{\mathcal{P}_n\}$ is a point-star network of $X$, if $\{st(x, \mathcal{P}_n)\}$ is a network at $x$ for every $x \in X$;

(2) $\{\mathcal{P}_n\}$ is a point-star $sn$-network, if $\{st(x, \mathcal{P}_n)\}$ is an $sn$-network at $x$ for every $x \in X$.

**Remark 1.12.** Spaces with a point-star $sn$-network are $sn$-first countable.

**Definition 1.13** ([11])**.** Let $\mathcal{P}$ be a cover of a space $X$. $\mathcal{P}$ is a $cs^*$-cover, if every convergent sequence in $X$ is frequently in $P$ for some $P \in \mathcal{P}$.

## 2. Main results

At first, we give a characterization of spaces with countable $sn$-networks. We have known that having countable weak bases, Lindelöf and separable are equivalent for $g$-metric spaces ([17]), and $\aleph_0$, (hereditarily) Lindelöf and hereditarily separable are equivalent for $\aleph$-spaces ([14], [15]). We have the following analogue for $sn$-metric spaces.

**Theorem 2.1.** *The following are equivalent for a space $X$:*
(1) *$X$ has a countable $sn$-network;*
(2) *$X$ is an $sn$-first countable, $\aleph_0$-space;*
(3) *$X$ is a (hereditarily) Lindelöf, $sn$-metric space;*
(4) *$X$ is a hereditarily separable, $sn$-metric space;*
(5) *$X$ is an $\omega_1$-compact, $sn$-metric space.*

Proof: (1) $\Longrightarrow$ (2) follows from Remark 1.10(3).

(2) $\Longrightarrow$ (3): $sn$-first countable, $\aleph$-spaces are $sn$-metric spaces ([9]), so $X$ is an $sn$-metric space. $\aleph_0$-spaces are hereditarily Lindelöf (see [7, Theorem 3.4]), so $X$ is hereditarily Lindelöf.

(3) $\Longrightarrow$ (4): $sn$-metric spaces are $\aleph$-spaces, and Lindelöf and hereditarily separable are equivalent for $\aleph$-spaces (see [7, Theorem 3.4]), so $X$ is hereditarily separable.

(4) $\Longrightarrow$ (5): It is clear that every Lindelöf space is $\omega_1$-compact.

(5) $\Longrightarrow$ (1): Let $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be an $sn$-network of $X$. We can assume $\mathcal{P}_n$ is a closed discrete collection of subsets of $X$ for every $n \in \mathbb{N}$ ([4]). We claim that $|\mathcal{P}_n| \leq \omega$ for every $n \in \mathbb{N}$. If not, there is $n \in \mathbb{N}$ such that $|\mathcal{P}_n| > \omega$. We pick $x_P \in P$ for every $P \in \mathcal{P}_n$. Then $\{x_P : P \in \mathcal{P}_n\}$ is a closed discrete subspace of $X$. This contradicts the $\omega_1$-compactness of $X$. So $X$ has a countable $sn$-network. □

Since perfect mappings inversely preserve $sn$-metric spaces if the domain spaces have $G_\delta$-diagonal ([4]), and inversely preserve Lindelöf spaces, we obtain the following result by the above theorem.

**Corollary 2.2.** *Let $f : X \longrightarrow Y$ be a perfect mapping. If $Y$ has a countable $sn$-network and $X$ has a $G_\delta$-diagonal, then $X$ has a countable $sn$-network.*

We give an example to show that "hereditarily separable" in Theorem 2.1 cannot relax to "separable".

**Example 2.3.** There is a separable, $sn$-metric space, which has not any countable $sn$-network.

Proof: Let $Y$ be a space in [6, Example 1]. Then $Y$ is a separable, $\aleph$-space, and is not an $\aleph_0$-space, hence $Y$ has not any countable $sn$-network. Notice that every convergent sequence in $Y$ is a finite subset of $Y$ and $Y$ has a $\sigma$-locally finite

$cs$-network, so $Y$ has a $\sigma$-locally finite $sn$-network. That is, $Y$ is an $sn$-metric space. □

**Lemma 2.4.** *Let $\{\mathcal{P}_n\}$ be a sequence of $cs^*$-covers of a space $X$, and $S$ be a sequence converging to a point $x \in X$. Then there is a subsequence $L$ of $S$ such that for every $n \in \mathbb{N}$, there is $P_n \in \mathcal{P}_n$ such that $L$ is eventually in $P_n$.*

PROOF: $\mathcal{P}_n$ is a $cs^*$-cover of $X$ for every $n \in \mathbb{N}$, so there is $P_n \in \mathcal{P}_n$ such that $S$ is frequently in $P_n$. Since $S$ is frequently in $P_1$, there is a subsequence $S_1$ of $S$ such that $S_1 \subset P_1$. Put $x_{n_1}$ is the first term of $S_1$. Similarly, $S_1$ is frequently in $P_2$, there is a subsequence $S_2$ of $S_1$ such that $S_2 \subset P_2$. Put $x_{n_2}$ is the second term of $S_2$. By the inductive method, for every $k \in \mathbb{N}$, since $S_{k-1}$ is frequently in $P_k$, there is a subsequence $S_k$ of $S_{k-1}$ such that $S_k \subset P_k$. Put $x_{n_k}$ is the $k$-th term of $S_k$. Let $L = \{x_{n_k} : k \in \mathbb{N}\} \cup \{x\}$. Then $L$, which is a subsequence of $S$, is eventually in $P_n$ for every $n \in \mathbb{N}$. In fact, for every $n \in \mathbb{N}$, if $k > n$, then $x_{n_k} \in S_k \subset S_n \subset P_n$. □

**Theorem 2.5.** *The following are equivalent for a space $X$:*
(1) *$X$ is a sequentially-quotient $\pi$-image of a metric space;*
(2) *$X$ has a point-star $sn$-network consisting of $cs^*$-covers;*
(3) *$X$ has a point-star network consisting of $cs^*$-covers.*

PROOF: (1) $\Longrightarrow$ (2). Let $f : M \longrightarrow X$ be a sequentially-quotient $\pi$-mapping, $(M, d)$ be a metric space. For every $n \in \mathbb{N}$, put $\mathcal{B}_n = \{B(a, 1/n) : a \in M\}$ and $\mathcal{P}_n = f(\mathcal{B}_n)$. Then $\{\mathcal{P}_n\}$ is a sequence of covers of $X$. Obviously, $\{st(x, \mathcal{P}_n)\}$ satisfies Definition 1.8(b) by the construction of $\{\mathcal{P}_n\}$.

(i) $\{\mathcal{P}_n\}$ is a point-star network of $X$: Let $x \in U$ with $U$ open in $X$. $f$ is a $\pi$-mapping, so there is $n \in \mathbb{N}$ such that $d(f^{-1}(x), M - f^{-1}(U)) > 2/n$, thus $st(x, \mathcal{P}_n) \subset U$. In fact, if $y \in st(x, \mathcal{P}_n)$, then there is $P = f(B(a, 1/n)) \in \mathcal{P}_n$ for some $a \in M$ such that $x, y \in P$. Let $b, c \in B(a, 1/n)$ such that $f(b) = x$ and $f(c) = y$. Then $d(c, f^{-1}(x)) \leq d(c, b) < 2/n$, so $c \notin M - f^{-1}(U)$, thus $y = f(c) \in U$.

(ii) $\mathcal{P}_n$ is a $cs^*$-cover of $X$ for every $n \in \mathbb{N}$: Let $S$ be a sequence in $X$ converging to the point $x \in X$. $f$ is sequentially-quotient, so there is a sequence $L$ in $M$ converging to a point $a \in f^{-1}(x)$ such that $f(L) = S_1$ is a subsequence of $S$. $L$ is eventually in $B(a, 1/n)$, so $S_1 = f(L)$ is eventually in $P = f(B(a, 1/n)) \in \mathcal{P}_n$. Thus $S$ is frequently in $P$.

(iii) $st(x, \mathcal{P}_n)$ is a sequential neighborhood of $x$ for every $x \in X$ and $n \in \mathbb{N}$: Let $S$ be a sequence converging to the point $x \in X$. By the proof in the above (ii), $S$ is frequently in some $P \in \mathcal{P}_n$ and $x \in P$, so $S$ is frequently in $st(x, \mathcal{P}_n)$. By Remark 1.6, $st(x, \mathcal{P}_n)$ is a sequential neighborhood of $x$.

By the above (i)–(iii), $X$ has a point-star $sn$-network consisting of $cs^*$-covers.

(2) $\Longrightarrow$ (3) is obvious.

(3) $\Longrightarrow$ (1). Let $\{\mathcal{P}_n\}$ be a point-star network consisting of $cs^*$-covers of $X$. We can assume $\mathcal{P}_n$ is a collection of closed subsets of $X$ for every $n \in \mathbb{N}$.

Put $\mathcal{P}_n = \{P_\alpha : \alpha \in A_n\}$ for every $n \in \mathbb{N}$, the topology on $A_n$ is the discrete topology. Put $M = \{a = (\alpha_n) \in \Pi_{n\in\mathbb{N}} A_n : \{P_{\alpha_n}\}$ is a network at some $x_a \in X\}$. Then $M$, which is a subspace of the product space $\Pi_{n\in\mathbb{N}} A_n$, is a metric space with metric $d$ defined as follows:

Let $a = (\alpha_n), b = (\beta_n) \in M$. Then $d(a,b) = 0$ if $a = b$, and $d(a,b) = 1/\min\{n \in \mathbb{N} : \alpha_n \neq \beta_n\}$ if $a \neq b$.

Define $f : M \longrightarrow X$ by $f(a) = x_a$ for every $a = (\alpha_n) \in M$, where $\{P_{\alpha_n}\}$ is a network at $x_a$. It is easy to see that $x_a$ is unique for every $a \in M$ by $T_1$-property of $X$, so $f$ is a function.

(i) $f$ is onto: Let $x \in X$. For every $n \in \mathbb{N}$, there is $\alpha \in A_n$ such that $x \in P_{\alpha_n}$. For $\{\mathcal{P}_n\}$ is a point-star network of $X$, $\{P_{\alpha_n}\}$ is a network for $x$. Put $a = (\alpha_n)$, then $f(a) = x$.

(ii) $f$ is continuous: Let $a = (\alpha_n) \in M$, $U$ be a neighborhood of $x = f(a)$. Then there is $k \in \mathbb{N}$ such that $P_{\alpha_k} \subset U$. Put $V = \{b = (\beta_n) \in M : \beta_k = \alpha_k\}$. Then $V$ is open in $M$ containing $a$ and $f(V) \subset P_{\alpha_k} \subset U$, thus $f$ is continuous.

(iii) $f$ is a $\pi$-mapping: Let $x \in U$ with $U$ open in $X$. For $\mathcal{P}_n$ is a point-star network of $X$, there is $n \in \mathbb{N}$ such that $st(x, \mathcal{P}_n) \subset U$. Then $d(f^{-1}(x), M - f^{-1}(U)) \geq 1/2n > 0$. In fact, let $a = (\alpha_n) \in M$ such that $d(f^{-1}(x), a) < 1/2n$. Then there is $b = (\beta_n) \in f^{-1}(x)$ such that $d(a,b) < 1/n$, so $\alpha_k = \beta_k$ if $k \leq n$. Notice that $x \in P_{\beta_n} \in \mathcal{P}_n$, $P_{\alpha_n} = P_{\beta_n}$, so $f(a) \in P_{\alpha_n} = P_{\beta_n} \subset st(x, \mathcal{P}_n) \subset U$, hence $a \in f^{-1}(U)$. Thus $d(f^{-1}(x), a) \geq 1/2n$ if $a \in M - f^{-1}(U)$, so $d(f^{-1}(x), M - f^{-1}(U)) \geq 1/2n > 0$.

(iv) $f$ is sequentially-quotient: Let $S$ be a sequence converging to a point $x \in X$. Notice that $\{\mathcal{P}_n\}$ is a sequence of $cs^*$-covers of $X$. By Lemma 2.4, there is a subsequence $L = \{x_k : k \in \mathbb{N}\} \cup \{x\}$ of $S$ such that for every $n \in \mathbb{N}$, there is $\alpha_n \in A_n$ such that $L$ is eventually in $P_{\alpha_n}$. Put $a = (\alpha_n)$. Since $\{\mathcal{P}_n\}$ is a point-star network, $a \in M$ and $f(a) = x$. We pick $b_k \in f^{-1}(x_k)$ for every $x_k \in L$ as follows. For every $n \in \mathbb{N}$, if $x_k \in P_{\alpha_n}$, put $\beta_{k_n} = \alpha_n$; if $x_k \notin P_{\alpha_n}$, pick $\alpha_{k_n} \in A_n$ such that $x_k \in P_{\alpha_{k_n}}$, and put $\beta_{k_n} = \alpha_{k_n}$. Put $b_k = (\beta_{k_n}) \in \Pi_{n\in\mathbb{N}} A_n$. Obviously, $b_k \in M$ and $f(b_k) = x_k$. It is easy to prove that $L' = \{b_k : k \in \mathbb{N}\} \cup \{a\}$ is a sequence in $M$ converging to the point $a$. In fact, let $U$ be open in $M$ containing $a$. By the definition of Tychonoff-product spaces, we can assume there is $m \in \mathbb{N}$ such that $U = ((\Pi\{\{\alpha_n\} : n \leq m\}) \times (\Pi\{A_n : n > m\})) \cap M$. For every $n \leq m$, $L$ is eventually in $P_{\alpha_n}$, so there is $k(n) \in \mathbb{N}$ such that $x_k \in P_{\alpha_n}$ if $k > k(n)$, thus $\beta_{k_n} = \alpha_n$. Put $k_0 = \max\{k(1), k(2), \ldots, k(m), m\}$. It is easy to see that $b_k \in U$ if $k > k_0$, so $L'$ converges to $a$. Thus there is a converging sequence $L'$ in $M$ such that $f(L') = L$ is a subsequence of $S$, so $f$ is sequentially-quotient. □

The following lemma belongs to Shou Lin ([8, Proposition 3.7.14(2)]).

**Lemma 2.6.** *Let $f : X \longrightarrow Y$ be a sequentially-quotient mapping, and $X$ be an $\aleph_0$-space. Then $Y$ is an $\aleph_0$-space.*

PROOF: Since $X$ is an $\aleph_0$-space, it is easy to prove that $X$ has a countable $cs$-network $\mathcal{P}$(also see [8, Proposition 1.6.7]). Put $\mathcal{P}' = f(\mathcal{P})$. By Remark 1.10(2), we need only prove that $\mathcal{P}'$ is a $cs^*$-network of $Y$.

Let $S$ be a sequence in $Y$ converging to a point $y \in U$ with $U$ open in $Y$. $f$ is sequentially-quotient, so there is a sequence $L$ in $X$ converging to a point $x \in f^{-1}(y) \subset f^{-1}(U)$ such that $f(L)$ is a subsequence of $S$. Since $\mathcal{P}$ is a $cs$-network of $X$, there exists $P \in \mathcal{P}$ such that $L$ is eventually in $P$ and $P \subset f^{-1}(U)$. Thus $f(L)$ is eventually in $f(P) \subset U$, and so $S$ is frequently in $f(P) \subset U$. Notice that $f(P) \in \mathcal{P}'$. So $\mathcal{P}'$ is a $cs^*$-network of $Y$. □

**Theorem 2.7.** *The following are equivalent for a space $X$:*
(1) *$X$ has a countable $sn$-network;*
(2) *$X$ is a sequentially-quotient compact image of a separable metric space;*
(3) *$X$ is a sequentially-quotient $\pi$-image of a separable metric space.*

PROOF: (1) $\Longleftrightarrow$ (2) from [10], and (2) $\Longrightarrow$ (3) from Remark 1.4. We only need to prove (3) $\Longrightarrow$ (1).

Let $f : M \longrightarrow X$ be a sequentially-quotient $\pi$-mapping, $M$ be a separable metric space. Then $X$ is $sn$-first countable from Theorem 2.5 and Remark 1.12. Sequentially-quotient mappings preserve $\aleph_0$-spaces by Lemma 2.6, so $X$ is an $\aleph_0$-space. Thus $X$ has a countable $sn$-network by Theorem 2.1.

□

Every $k$-space with a countable $sn$-network is sequential by Remark 1.10(5), so it has a countable weak base. Thus we have the following corollary.

**Corollary 2.8.** *The following are equivalent for a $k$-space $X$:*
(1) *$X$ has a countable weak base;*
(2) *$X$ has a countable $sn$-network;*
(3) *$X$ is a quotient compact image of a separable metric space;*
(4) *$X$ is a sequentially-quotient compact image of a separable metric space;*
(5) *$X$ is a quotient $\pi$-image of a separable metric space;*
(6) *$X$ is a sequentially-quotient $\pi$-image of a separable metric space.*

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