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Commentationes Mathematicae Universitatis Carolinae, Vol. 45 (2004), No. 1, 119--137

Persistent URL: `http://dml.cz/dmlcz/119441`

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Abstract. We calculate the cardinal characteristics of the σ-ideal $\mathcal{HN}(G)$ of Haar null subsets of a Polish non-locally compact group G with invariant metric and show that $\mathrm{cov}(\mathcal{HN}(G)) \leq \mathfrak{b} \leq \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\} \leq \mathrm{non}(\mathcal{HN}(G)) \leq \mathrm{cof}(\mathcal{HN}(G)) > \min\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$. If $G = \prod_{n\geq 0} G_n$ is the product of abelian locally compact groups G_n, then $\mathrm{add}(\mathcal{HN}(G)) = \mathrm{add}(\mathcal{N})$, $\mathrm{cov}(\mathcal{HN}(G)) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{N})\}$, $\mathrm{non}(\mathcal{HN}(G)) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$ and $\mathrm{cof}(\mathcal{HN}(G)) \geq \mathrm{cof}(\mathcal{N})$, where $\mathcal{N}$ is the ideal of Lebesgue null subsets on the real line. Martin Axiom implies that $\mathrm{cof}(\mathcal{HN}(G)) > 2^{\aleph_0}$ and hence G contains a Haar null subset that cannot be enlarged to a Borel or projective Haar null subset of G. This gives a negative (consistent) answer to a question of S. Solecki. To obtain these estimates we show that for a Polish non-locally compact group G with invariant metric the ideal $\mathcal{HN}(G)$ contains all o-bounded subsets (equivalently, subsets with the small ball property) of G.

Keywords: Polish group, Haar null set, Martin Axion, cardinal characteristics of an ideal, o-bounded set, the small ball property

Classification: 03E04, 03E15, 03E17, 03E35, 03E50, 03E75, 22A10, 28C10, 54A25, 54H11

A subset N of a topological group G is called *Haar null* if it is contained in a universally measurable set $B \subset G$ for which there exists a σ-additive Borel probability measure μ on G such that $\mu(gBh) = 0$ for all $g, h \in G$ (a subset B of a topological space X is *universally measurable* if it is measurable with respect to any Borel σ-additive probability measure on X). The family $\mathcal{HN}(G)$ of Haar null subsets of a Polish group G is closed under translations, taking subsets and countable unions, see [THJ, 2.4.5]. The notion of Haar null sets is a natural extension of the notion of sets of Haar measure zero: if G happens to be locally compact, then Haar null sets are precisely the sets of Haar measure zero. Since the publication of Christensen's paper [C] who introduced this new notion, Haar null sets have found many applications, see [BL], [PZ].

In this paper we estimate the principal cardinal characteristics of the σ-ideal $\mathcal{HN}(G)$ of Haar null subsets of a Polish group G. There is nothing surprising about $\mathcal{HN}(G)$ if the group G is locally compact and non-discrete. In this case the ideal $\mathcal{HN}(G)$ is isomorphic to the σ-ideal $\mathcal{N}$ of Lebesgue null subsets of the real line $\mathbb{R}$ in the sense that there is a Borel isomorphism $h : G \to \mathbb{R}$ such that a subset $A \subset G$ belongs to $\mathcal{HN}(G)$ if and only if $h(A) \in \mathcal{N}$ (this follows from the classical

Partly supported by the grant SLO–UKR 02/03–04.

theorem on isomorphism of Borel measure spaces, see [Ke, 17.41]). Consequently, for a non-discrete locally compact Polish group G the σ-ideals $\mathcal{HN}(G)$ and $\mathcal{N}$ have the same cardinal characteristics. Let us remind their definitions, see [V].

Given a σ-ideal $\mathcal{I}$ of subsets of a set X let

$$\begin{aligned}
&\operatorname{add}(\mathcal{I}) = \min\big\{|\mathcal{J}| : \mathcal{J} \subset \mathcal{I} \text{ and } \textstyle\bigcup\mathcal{J} \notin \mathcal{I}\big\};\\
&\operatorname{cov}(\mathcal{I}) = \min\big\{|\mathcal{J}| : \mathcal{J} \subset \mathcal{I} \text{ and } \textstyle\bigcup\mathcal{J} = X\big\};\\
&\operatorname{non}(\mathcal{I}) = \min\big\{|A| : A \subset X \text{ and } A \notin \mathcal{I}\big\};\\
&\operatorname{cof}(\mathcal{I}) = \min\big\{|\mathcal{J}| : \mathcal{J} \subset \mathcal{I} \text{ and } \mathcal{I} = \{A \subset X : \exists\, E \in \mathcal{J} \text{ with } A \subset E\}\big\}.
\end{aligned}$$

It is easy to see that these cardinals are related as follows:

$$\aleph_1 \le \operatorname{add}(\mathcal{I}) \le \min\{\operatorname{non}(\mathcal{I}), \operatorname{cov}(\mathcal{I})\} \le \max\{\operatorname{non}(\mathcal{I}), \operatorname{cov}(\mathcal{I})\} \le \operatorname{cof}(\mathcal{I}).$$

It follows from the famous Cichoń diagram (see [V], [BS]) that $\aleph_1 \le \operatorname{add}(\mathcal{N}) \le \mathfrak{b} \le \mathfrak{d} \le \operatorname{cof}(\mathcal{N})$, where $\mathfrak{b}$ and $\mathfrak{d}$ are two well-known small cardinals introduced by E. van Douwen in his seminal paper [vD]. Since for any non-discrete locally compact Polish group G the cardinal characteristics of the ideals $\mathcal{HN}(G)$ and $\mathcal{N}$ coincide, we get

$$\aleph_1 \le \operatorname{add}(\mathcal{HN}(G)) \le \mathfrak{b} \le \mathfrak{d} \le \operatorname{cof}(\mathcal{HN}(G)).$$

In [S_2, 3.4] S. Solecki proved that the same estimates hold also for any non-locally compact Polish group G with invariant metric. There is however one crucial difference between locally compact and non-locally compact cases: for a Polish non-locally compact group the cardinal $\operatorname{cof}(\mathcal{HN}(G))$ always exceeds $\aleph_1$. Moreover, under Martin Axiom, it exceeds the size of continuum. Thus for a non-locally compact Polish group G the σ-ideal $\mathcal{HN}(G)$ differs substantially from other classical ideals whose cardinal characteristics lie between $\aleph_1$ and $\mathfrak{c}$ (and thus fall into the category of so-called small cardinals). Unlike to the cofinality $\operatorname{cof}(\mathcal{HN}(G))$, the other cardinal characteristics of the σ-ideal $\mathcal{HN}(G)$ behave not so wildly and for some special groups (like $\mathbb{R}^\omega$ or $\mathbb{Z}^\omega$) they can be expressed via known small cardinals $\mathfrak{b}$, $\mathfrak{d}$, $\operatorname{add}(\mathcal{N})$, $\operatorname{cov}(\mathcal{N})$, and $\operatorname{non}(\mathcal{N})$.

To calculate the cardinal characteristics of the ideal $\mathcal{HN}(G)$ for a non-locally compact Polish group G with invariant metric we shall prove that for such a group G the ideal $\mathcal{HN}(G)$ contains the σ-ideal $o\mathcal{B}(G)$ of o-bounded subsets of G. Following O. Okunev and M. Tkachenko [Tk, 3.9] we define a subset B of a topological group G to be *o-bounded* if for any sequence $(U_n)_{n\ge0}$ of neighborhoods of the neutral element of G there is a sequence $(F_n)_{n\ge0}$ of finite subsets of G such that $B \subset \bigcup_{n\ge0} F_nU_n$ (this is equivalent to saying that there is a sequence $(F_n)_{n\ge0}$ of finite subsets of G with $B \subset \bigcap_{k\ge0}\bigcup_{n\ge k} F_nU_n$, see [HRT, 2.7]). Recently o-bounded sets attracted a lot of attention, see [Tk2], [HRT], [Her], [Ba1], [Ba2],

[BNS], [Ts]. It should be mentioned that in Banach space theory they are known as sets with the small ball property, i.e., sets which can be covered by a sequence of small balls whose radii tend to zero, see [BK]. It is easy to see that the family $o\mathcal{B}(G)$ of all o-bounded subsets of a topological group G forms a σ-ideal containing all compact subsets of G.

Our main instrument in estimation of cardinal characteristics of the ideal $\mathcal{HN}(G)$ is

Theorem 1. *Let G be a non-locally compact Polish group.*

1. *If G admits an invariant metric, then $o\mathcal{B}(G) \subset \mathcal{HN}(G)$;*
2. *If $G = \prod_{n\geq 0} G_n$ is the countable product of locally compact groups, then $o\mathcal{B}(G) \subset \mathcal{HN}(G)$;*
3. *For a continuous homomorphism $h : G \to H$ onto a non-discrete (locally compact) Polish group H, a subset $A \subset H$ is Haar null (if and) only if its preimage $h^{-1}(H)$ is Haar null in G, which implies that $\mathrm{cov}(\mathcal{HN}(G)) \leq \mathrm{cov}(\mathcal{HN}(H))$ and $\mathrm{non}(\mathcal{HN}(G)) \geq \mathrm{non}(\mathcal{HN}(H))$.*

The first statement of Theorem 1 generalizes the result of Dougherty [D] who proved that for a Polish non-locally compact group with invariant metric the ideal $\mathcal{HN}(G)$ contains all compact subsets of G (for abelian G this fact was proven by Christensen [C]). Theorem 1 will help us to make the following estimations of the cardinal characteristics of the ideal $\mathcal{HN}(G)$.

Theorem 2. *Suppose G is a non-discrete Polish group.*

1. *If G is locally compact, then $\mathrm{add}(\mathcal{HN}(G)) = \mathrm{add}(\mathcal{N})$, $\mathrm{cov}(\mathcal{HN}(G)) = \mathrm{cov}(\mathcal{N})$, $\mathrm{non}(\mathcal{HN}(G)) = \mathrm{non}(\mathcal{N})$, and $\mathrm{cof}(\mathcal{HN}(G)) = \mathrm{cof}(\mathcal{N})$.*
2. *If G is not locally compact and has an invariant metric, then $\mathrm{cov}(\mathcal{HN}(G)) \leq \mathfrak{b}$, $\mathrm{non}(\mathcal{HN}(G)) \geq \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$ and $\mathrm{cof}(\mathcal{HN}(G)) > \min\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$.*
3. *If G contains a closed normal subgroup H such that either H or G/H is locally compact and not discrete, then $\mathrm{add}(\mathcal{HN}(G)) \leq \mathrm{add}(\mathcal{N})$, $\mathrm{cov}(\mathcal{HN}(G)) \leq \mathrm{cov}(\mathcal{N})$, $\mathrm{non}(\mathcal{HN}(G)) \geq \mathrm{non}(\mathcal{N})$ and $\mathrm{cof}(\mathcal{HN}(G)) \geq \mathrm{cof}(\mathcal{N})$.*
4. *If the center $Z = \{g \in G : \forall x \in G\ gx = xg\}$ of G is not locally compact, then $\mathrm{cov}(\mathcal{HN}(G)) \leq \mathrm{cov}(\mathcal{HN}(Z)) \leq \mathfrak{b}$, and $\mathrm{non}(\mathcal{HN}(G)) \geq \mathrm{non}(\mathcal{HN}(Z)) \geq \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$.*
5. *If G admits a surjective continuous homomorphism onto a non-locally compact group H with invariant metric, then $\mathrm{cov}(\mathcal{HN}(G)) \leq \mathrm{cov}(\mathcal{HN}(H)) \leq \mathfrak{b}$, and $\mathrm{non}(\mathcal{HN}(G)) \geq \mathrm{non}(\mathcal{HN}(H)) \geq \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$.*

For linear complete metric spaces, Theorem 2(2),(3) implies

Corollary 1. *If X is an infinite-dimensional linear complete metric space, then*

1. $\mathrm{add}(\mathcal{HN}(X)) \le \mathrm{add}(\mathcal{N})$;
2. $\mathrm{cov}(\mathcal{HN}(X)) \le \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{N})\}$;
3. $\mathrm{non}(\mathcal{HN}(X)) \ge \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$;
4. $\mathrm{cof}(\mathcal{N}) \le \mathrm{cof}(\mathcal{HN}(X)) > \min\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$.

For groups G which are countable products of locally compact amenable groups the first three inequalities of Corollary 1 can be reversed. Haar null subsets in such groups were characterized by S. Solecki [S]. We remind that a locally compact group G is *amenable* if it admits a left invariant mean on the space $L^\infty(G)$ of all essentially bounded complex functions measurable with respect to the Haar measure. It is well-known [Pa, 4.10] that a locally compact group G endowed with a left-invariant Haar measure μ is amenable if and only if it satisfies the Følner condition: for any $\varepsilon > 0$ and any compact subset $C \subset G$ there is a compact subset $K \subset G$ such that $\mu(xK \triangle K) < \varepsilon\mu(K)$ for all $x \in C$. The class of amenable locally compact groups contains all abelian (and even exponentially bounded) locally compact groups, see [Pa, Chapter 6].

Another class containing all abelian groups is the class of groups admitting a finitely supported kaleidoscopical measure. A probability measure λ on a topological group G is called *kaleidoscopical* if there is a partition $G = A_1 \cup \dots \cup A_n$ of G into $n > 1$ λ-measurable pieces such that $\mu(xA_iy) = \frac{1}{n}$ for every $i \le n$ and all $x, y \in G$. Groups admitting a kaleidoscopical finitely supported measure will be called *kaleidoscopical*, cf. [BP, §8]. We shall say that a group G is *almost kaleidoscopical* if for any $\varepsilon > 0$ there is a finitely supported probability measure μ on G and a partition $G = A_1 \cup \dots \cup A_n$, $n > 1$, such that $\left|\mu(xA_iy) - \frac{1}{n}\right| < \frac{\varepsilon}{n}$ for all $x, y \in G$ and $i \le n$.

The following result proved in [BP, §8] shows that the class of (almost) kaleidoscopical groups is quite large. We recall that a topological group G is a *SIN-group* (abbreviated from "Small Invariant Neighborhoods") if it has a neighborhood base $\mathcal{B}$ at the unit such that $gUg^{-1} = U$ for any $U \in \mathcal{B}$ and $g \in G$. It is well-known that each first countable SIN-group admits an invariant metric and that a topological group is a SIN-group if it is *totally bounded* in the sense that for any neighborhood $U \subset G$ of the unit there is a finite subset $F \subset G$ with $G = UF = FU$.

Proposition 1 ([BP, §8]). 1. *A group admitting a homomorphism onto an* (*almost*) *kaleidoscopical group is* (*almost*) *kaleidoscopical.*

2. *A group G is kaleidoscopical provided G admits a homomorphism onto a group containing a finite non-trivial normal subgroup.*
3. *A group G is almost kaleidoscopical provided G admits a homomorphism onto a topological SIN-group containing a totally bounded non-trivial normal subgroup.*

Question 1. *Is every (amenable) group almost kaleidoscopical?*

Now we can give some estimates of cardinal characteristics of the ideal $\mathcal{HN}(G)$ for Polish groups which are products of locally compact groups.

Theorem 3. *Suppose that a Polish non-locally compact group $G = \prod_{n\geq 0} G_n$ is the countable product of locally compact groups G_n. Then*

1. $\mathrm{cov}(\mathcal{HN}(G)) \leq \mathfrak{b}$, $\mathrm{non}(\mathcal{HN}(G)) \geq \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$, *and* $\mathrm{cof}(\mathcal{HN}(G)) > \min\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$;
2. *if all but finitely many groups G_n are amenable, then* $\mathrm{add}(\mathcal{HN}(G)) \geq \mathrm{add}(\mathcal{N})$, $\mathrm{cov}(\mathcal{HN}(G)) \geq \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{N})\}$ *and* $\mathrm{non}(\mathcal{HN}(G)) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$;
3. *if some group G_n is non-discrete or infinitely many of the groups G_n are almost kaleidoscopical, then* $\mathrm{add}(\mathcal{HN}(G)) \leq \mathrm{add}(\mathcal{N})$, $\mathrm{cov}(\mathcal{HN}(G)) \leq \mathrm{cov}(\mathcal{N})$, *and* $\mathrm{cof}(\mathcal{HN}(G)) \geq \mathrm{cof}(\mathcal{N})$;
4. *if G is abelian, then* $\mathrm{add}(\mathcal{HN}(G)) = \mathrm{add}(\mathcal{N})$, $\mathrm{cov}(\mathcal{HN}(G)) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{N})\}$, $\mathrm{non}(\mathcal{HN}(G)) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$, *and* $\mathrm{cof}(\mathcal{N}) \leq \mathrm{cof}(\mathcal{HN}(G)) > \min\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$.

The strict inequality $\mathrm{cof}(\mathcal{HN}(G)) > \min\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$ together with Martin Axiom has very strange consequences displaying a striking difference between properties of the σ-ideal $\mathcal{HN}(G)$ in the locally compact and non-locally compact cases.

It is well-known that any subset of zero Haar measure in a locally compact Polish group can be enlarged to a G_δ-set of zero Haar measure. In [S, p. 208] S. Solecki asked if the same is true for Haar null subsets in non-locally compact groups. We shall show that the answer to this question is negative under Martin Axiom. More precisely, in each non-locally compact Polish group we shall find a universally null subset that cannot be enlarged to a σ-projective Haar null (more generally, 2-Zorn) set.

Following [Ke, 39.15] we call a subset A of a Polish space X *σ-projective* if it belongs to the smallest σ-algebra $\sigma\mathbf{P}(X)$ containing X and such that the image $f(A)$ of any set $A \in \sigma\mathbf{P}(X)$ under a continuous map $f : A \to X$ belongs to $\sigma\mathbf{P}(X)$. The σ-algebra $\sigma\mathbf{P}(X)$ contains all analytic and consequently all Borel subsets of X.

Generalizing the notion of a Zorn set [PZ] let us call a subset Z of a group G a *κ-Zorn set*, where κ is a cardinal, if $G \neq F \cdot Z$ for any subset $F \subset G$ of size $|F| \leq \kappa$. It is clear that each Haar null subset of a Polish group G is κ-Zorn for any $\kappa < \mathrm{cov}(\mathcal{HN}(G))$. The family of all κ-Zorn subsets of a topological G group will be denoted by $\mathcal{Z}_\kappa(G)$. By $\mathcal{UN}(G)$ we denote the ideal of all universally null subsets of G (a subset $N \subset G$ is universally null if it has zero measure with respect to any Borel non-atomic measure on G). Denote by $\mathrm{cof}(\mathcal{UN}(G), \mathcal{Z}_2(G))$ the smallest size $|\mathcal{Z}|$ of a family $\mathcal{Z} \subset \mathcal{Z}_2(G)$ of 2-Zorn subsets of G such that each universally null subset of G lies in some set $Z \in \mathcal{Z}$. Since $\mathcal{UN}(G) \subset \mathcal{HN}(G) \subset \mathcal{Z}_2(G)$ we get $\mathrm{cof}(\mathcal{UN}(G), \mathcal{Z}_2(G)) \le \mathrm{cof}(\mathcal{HN}(G))$.

It is well-known that Martin Axiom implies $\mathfrak{b} = \mathfrak{d} = \mathrm{add}(\mathcal{N}) = \mathfrak{c}$, where $\mathfrak{c}$ is the size of continuum.

Theorem 4. *Let G be a Polish non-locally compact group.*

1. *If* $\mathrm{non}(\mathcal{N}) \ge \mathfrak{d}$, *then* $\mathrm{cof}(\mathcal{HN}(G)) \ge \mathrm{cof}(\mathcal{UN}(G), \mathcal{Z}_2(G)) > \mathfrak{d}$.
2. *If* $\mathrm{non}(\mathcal{N}) = \mathfrak{d} = \mathfrak{c}$ (*which holds under Martin Axiom*), *then the group G contains a universally null* (*and thus Haar null*) *subset that cannot be enlarged to a σ-projective Haar null* (*more generally,* 2*-Zorn*) *subset of G.*

It should be mentioned that the strict inequality $\mathfrak{c} < \mathrm{cof}(\mathcal{UN}(G), \mathcal{Z}_2(G))$ from Theorem 4 cannot be proven in ZFC. According to [La] there is a model of ZFC in which $2^{\aleph_1} = \mathfrak{c}$ and each universally null set has size $\le \aleph_1$. In this model $\mathrm{cof}(\mathcal{UN}(G), \mathcal{Z}_2(G)) \le \mathrm{cof}(\mathcal{UN}(G)) \le \mathfrak{c}^{\aleph_1} = \mathfrak{c}$.

Problem 1. *Is the inequality* $\mathrm{cof}(\mathcal{HN}(G)) \le \mathfrak{c}$ *consistent with ZFC for some Polish non-locally compact group G?*

Assuming Martin Axiom we get $\mathrm{add}(\mathcal{N}) = \mathfrak{b} = \mathrm{non}(\mathcal{N}) = \mathfrak{c}$ and thus $\mathrm{non}(\mathcal{HN}(G)) = \mathfrak{c} < \mathrm{cof}(\mathcal{HN}(G))$ for any Polish non-locally compact group G.

Problem 2. *Let G be a nondiscrete Polish group* (*with invariant metric*). *Is* $\mathrm{add}(\mathcal{HN}(G)) = \mathrm{cov}(\mathcal{HN}(G)) = \mathfrak{c}$ *under MA or PFA?*

Two topological groups G, H are called *Haar null isomorphic* if there is a bijection $h : G \to H$ such that a subset $N \subset G$ is Haar null in G if and only if $h(N)$ is Haar null in H. It follows from Isomorphism Theorem for non-atomic measure spaces [Ke, 17.41] that any two non-discrete locally compact Polish groups are Haar null isomorphic. On the other hand, the failure of the countable chain condition for the ideal $\mathcal{HN}(G)$ in the non-locally compact case [S_1] implies that a Polish locally compact group cannot be Haar null isomorphic to a Polish non-locally compact group with invariant metric.

Problem 3. *Are there two Polish non-locally compact groups* (*with invariant metric*) *that fail to be Haar null isomorphic? In particular, is the Hilbert space ℓ^2 Haar null isomorphic to $\mathbb{R}^\omega$ or $\mathbb{Z}^\omega$? Have the ideals $\mathcal{HN}(\ell^2)$ and $\mathcal{HN}(\mathbb{R}^\omega)$ the same cardinal characteristics?*

Cardinal characteristics of the ideal $o\mathcal{B}(G)$

In this section we shall estimate the cardinal characteristics of the ideal $o\mathcal{B}(G)$ of o-bounded sets in a Polish non-locally compact group G with invariant metric. First we remind the definition of the small cardinals $\mathfrak{b}$ and $\mathfrak{d}$. For two functions $f, g \in \mathbb{N}^\omega$ we write $f \le^* g$ if $f(n) \le g(n)$ for all sufficiently large n. A subset $B \subset \mathbb{N}^\omega$ is called

- *bounded* in $\mathbb{N}^\omega$ if there is $f \in \mathbb{N}^\omega$ such that $g \le^* f$ for all $g \in B$;

- *dominating* if for any $f \in \mathbb{N}^\omega$ there is $g \in B$ with $f \le^* g$.

By definition, $\mathfrak{b}$ is the smallest size of an unbounded subset of $\mathbb{N}^\omega$ while $\mathfrak{d}$ is the smallest size of a dominating subset of $\mathbb{N}^\omega$, see [vD] or [V].

It is well-known (and easily seen) that the family $\mathcal{B}$ (resp. $\mathcal{ND}$) of bounded (resp. non-dominating) subsets of $\mathbb{N}^\omega$ forms a σ-ideal. As we shall see, the ideal $\mathcal{ND}$ is closely related to the ideal $o\mathcal{B}(G)$ while $\mathcal{B}$ is related to the σ-ideal $\mathcal{B}(G)$ generated by compact subsets of G.

Lemma 1. *If G is a Polish non-locally compact group, then* $\mathrm{cov}(o\mathcal{B}(G)) \le \mathrm{cov}(\mathcal{ND}) = \mathfrak{b} \le \mathfrak{d} = \mathrm{non}(\mathcal{ND}) \le \mathrm{non}(o\mathcal{B}(G))$.

Proof: To prove the lemma we shall construct a function $\psi : G \to \mathbb{N}^\omega$ such that for any non-dominating subset $D \subset \mathbb{N}^\omega$ the set $\psi^{-1}(D)$ is o-bounded in G. Fix a decreasing neighborhood base $(U_n)_{n\ge 0}$ at the unit of the group G and a countable dense subset $\{a_k\}_{k\in\omega}$ of G. Define a function $\psi : G \to \mathbb{N}^\omega$ assigning to each $x \in G$ the function $y \in \mathbb{N}^\omega$ such that $y(n)$ is the smallest number with $x \in a_{y(n)} U_n$. We claim that the map $\psi : G \to \mathbb{N}^\omega$ satisfies our requirements.

Fix any non-dominating subset $D \subset \mathbb{N}^\omega$ and consider the preimage $\psi^{-1}(D) \subset G$. To show that $\psi^{-1}(D)$ is o-bounded in G, fix any sequence $(W_n)_{n\ge 0}$ of neighborhoods of the origin of G. By induction construct an increasing function $f : \omega \to \omega$ such that $U_{f(n)} \subset W_n$ for all $n \in \omega$. Since D is not dominating, there is an increasing function $y \in \mathbb{N}^\omega$ such that $y \not\le^* z$ for all $z \in D$. Take any function $g \in \mathbb{N}^\omega$ such that $\min\{g(i) : f(k) \le i < f(k+1)\} \ge y(f(k+1))$ for every $k \ge 0$.

For every $n \ge 0$ let $F_n = \{a_k : k \le g(n)\}$. We claim that $\psi^{-1}(D) \subset \bigcup_{n\ge 0} F_n U_n$. Assuming the converse find a point $x \in \psi^{-1}(D) \setminus \bigcup_{n\ge 0} F_n U_n$. Consider the function $z = \psi(x) \in D$. It follows from the definition of ψ that $z(i) > g(i)$ for all $i \ge 0$. Let us show that $z(i) \ge y(i)$ for all $i \ge f(0)$. Indeed, given such an i, find $k \ge 0$ with $f(k) \le i < f(k+1)$ and observe that $z(i) \ge g(i) \ge y(f(k+1)) \ge y(i)$. Thus $y \le^* z \in D$ which contradicts the choice of y.

It follows from the property of the function ψ that $\mathrm{cov}(o\mathcal{B}(G)) \le \mathrm{cov}(\mathcal{ND})$ and $\mathrm{non}(o\mathcal{B}(G)) \ge \mathrm{non}(\mathcal{ND})$. To complete the proof it rests to note that $\mathrm{cov}(\mathcal{ND}) = \mathfrak{b}$ and $\mathrm{non}(\mathcal{ND}) = \mathfrak{d}$. To establish these equalities observe that a subset $D \subset \mathbb{N}^\omega$ is not dominating if and only if $D \subset \{x \in \mathbb{N}^\omega : f \not\le^* x\}$ for some $f \in \mathbb{N}^\omega$. $\square$

In the sequel we shall also need some information concerning cardinal characteristics of the σ-ideal $\mathcal{B}(G)$ generated by compact subsets of a topological group G.

Lemma 2. *Suppose G is a Polish non-locally compact group. Then* $\mathrm{add}(\mathcal{B}(G)) = \mathrm{add}(\mathcal{B}) = \mathfrak{b} = \mathrm{non}(\mathcal{B}) = \mathrm{non}(\mathcal{B}(G))$ *and* $\mathrm{cov}(\mathcal{B}(G)) = \mathrm{cov}(\mathcal{B}) = \mathfrak{d} = \mathrm{cof}(\mathcal{B}) = \mathrm{cof}(\mathcal{B}(G))$.

Proof: Let $\bar{G}$ be any metrizable compactification of G and $f : K \to \bar{G}$ be a continuous surjective map from a zero-dimensional compact space. Consider the preimage $f^{-1}(G)$ and by Zorn Lemma find a minimal closed subset $Z \subset f^{-1}(G)$ with $f(Z) = G$. Then Z, being Polish, zero-dimensional and nowhere locally compact, is homeomorphic to $\mathbb{N}^\omega$ according to the Aleksandrov-Urysohn Theorem [Ke, 7.7]. Since the map $f|Z$ is proper (that is the preimages of compact subsets are compact) we get that the space G is the image of the space $\mathbb{N}^\omega$ under a continuous proper map $\pi : \mathbb{N}^\omega \to G$.

Call a subset of G *σ-bounded* if it lies in a σ-compact subset of G. Observe that a subset $B \subset \mathbb{N}^\omega$ lies in a σ-compact subset of $\mathbb{N}^\omega$ if and only if it is bounded in the sense of the pre-order $\le^*$. Consequently, for any bounded subset A of $(\mathbb{N}^\omega, \le^*)$ the image $\pi(A)$ is σ-bounded in G and for any σ-bounded subset $B \subset G$ the preimage $\pi^{-1}(B)$ is bounded in $(\mathbb{N}^\omega, \le^*)$. This observation together with known equalities $\mathrm{add}(\mathcal{B}) = \mathfrak{b} = \mathrm{non}(\mathcal{B})$ and $\mathrm{cov}(\mathcal{B}) = \mathfrak{d} = \mathrm{cof}(\mathcal{B})$ allow us to conclude that $\mathrm{add}(\mathcal{B}(G)) = \mathrm{add}(\mathcal{B}) = \mathfrak{b} = \mathrm{non}(\mathcal{B}) = \mathrm{non}(\mathcal{B}(G))$ and $\mathrm{cov}(\mathcal{B}(G)) = \mathrm{cov}(\mathcal{B}) = \mathfrak{d} = \mathrm{cof}(\mathcal{B}) = \mathrm{cof}(\mathcal{B}(G))$. □

Finally let us prove another useful lemma which probably belongs to the mathematical folklore.

Lemma 3. *Let $\mathcal{F}$ be a family of universally measurable subsets of a Polish space X. If $|\mathcal{F}| < \mathrm{add}(\mathcal{N})$, then the union $\bigcup \mathcal{F}$ is universally measurable in X.*

Proof: Fix any finite Borel measure μ. We have to show that the union $\bigcup \mathcal{F}$ is μ-measurable. Let $C = \{x \in X : \mu(\{x\}) > 0\}$. It is clear that the set C is at most countable and thus Borel. Consider the discrete measure $\nu = \sum_{x \in C} \mu(\{x\})\delta_x$ where δ_x is the Dirac measure concentrated at x. Then $\eta = \mu - \nu$ is a non-atomic measure. Since each subset of X is ν-measurable, it suffices to show that the set $\bigcup \mathcal{F}$ is η-measurable. That is so if $\eta = 0$. So we consider the case of non-trivial measure η. Multiplying η by a suitable constant we may assume that η is a probability measure. Then by Isomorphism Theorem for non-atomic probability measures [Ke, 17.41] the measure η is equivalent to the Lebesgue measure λ on $[0, 1]$. Hence we may assume that $X = [0, 1]$ and $\eta = \lambda$. Let $\lambda_*(\bigcup \mathcal{F}) = \sup\{\lambda(S) : S \subset \cup \mathcal{F} \text{ is } \sigma\text{-compact}\}$ and find a σ-compact subset $S \subset \bigcup \mathcal{F}$ with $\lambda(S) = \lambda_*(\bigcup \mathcal{F})$. Then $\lambda(B) = 0$ for any measurable subset $B \subset \bigcup \mathcal{F} \setminus S$. It follows that $\lambda(F \setminus S) = 0$ for each $F \in \mathcal{F}$. Since $|\mathcal{F}| < \mathrm{add}(\mathcal{N})$ we conclude $\lambda(\bigcup \mathcal{F} \setminus S) = 0$ which implies that $\bigcup \mathcal{F} = S \cup (\bigcup \mathcal{F} \setminus S)$ is λ-measurable. □

Proof of Theorem 1

We divide the proof of Theorem 1 into three lemmas.

Lemma 4. *If G is a Polish non-locally compact group with invariant metric, then $o\mathcal{B}(G) \subset \mathcal{HN}(G)$.*

Proof: Fix any complete invariant metric d on the group G. Since G is not locally compact, no non-empty open subset of G is totally bounded. Using this observation we can inductively construct a sequence $(\varepsilon_n)_{n\geq 0} \subset (0,1]$ of positive reals such that for every $n \geq 0$ the ε_n-ball $B(\varepsilon_n) = \{x \in G : d(x,0) < \varepsilon_n\}$ around the origin of G fails to have a finite $6\varepsilon_{n+1}$-net. Then $\varepsilon_{n+1} \leq \frac{1}{6}\varepsilon_n \leq \frac{1}{6^n}$ for all n. By the invariance of the metric d we get $F \cdot B(\varepsilon_n) = B(\varepsilon_n) \cdot F$ for any finite subset $F \subset G$.

To show that $o\mathcal{B}(G) \subset \mathcal{HN}(G)$, fix any o-bounded subset $B \subset G$ and find a sequence $(F_n)_{n\geq 0}$ of finite subsets of G such that $B \subset \bigcap_{k\geq 0}\bigcup_{n\geq k} F_n B(\varepsilon_n)$. Observe that the set $M = \bigcap_{k\geq 0}\bigcup_{n\geq k} F_n B(\varepsilon_n)$ is Borel in G.

Using the fact that the ε_n-ball $B(\varepsilon_n)$ admits no finite $6\varepsilon_{n+1}$-net, for every $n \geq 0$ fix a finite subset $D_n \subset B(\varepsilon_n)$ of size $|D_n| = 2^{n+1}|F_{n+2}|$ which is $6\varepsilon_{n+1}$-separated in the sense that $d(x,y) \geq 6\varepsilon_{n+1}$ for any distinct points $x, y \in D_n$.

Let $D = \bigcup_{n\geq 0}\prod_{k\leq n} D_k$ and let $\bar{D} = \prod_{k\geq 0} D_k$ be the infinite product endowed with the Tychonov product topology. Consider the map $\psi : D \to G$ assigning to each finite sequence $(x_0, \dots, x_n) \in D$ the product $x_0 \cdots x_n$ in G. Also let $\varphi : \bar{D} \to G$ be the continuous map assigning to each infinite sequence $(x_n)_{n\geq 0}$ the limit $\lim_{n\to\infty} x_0 \cdots x_n$ of the sequence $(x_0 \cdots x_n)_{n\geq 0}$. It can be shown that for any distinct sequences $x = (x_n)_{n\geq 0}$ and $y = (y_n)_{n\geq 0}$ in D with $k = \min\{n \in \omega : x_n \neq y_n\}$ we get $d(\varphi(x), \varphi(y)) \geq 6\varepsilon_{k+1} - 2\sum_{i=k+1}^{\infty} \varepsilon_i > 2\varepsilon_{k+1}$. This implies that for any $g, h \in G$ and any $k \geq 1$ the preimage $\varphi^{-1}(gB(\varepsilon_{k+1})h)$ is small in the sense that there is a finite sequence $(x_0, \dots, x_{k-1}) \in D$ such that for any $y \in \varphi^{-1}(gB(\varepsilon_{k+1})h)$ we get $y_i = x_i$ for all $i < k$.

Let $\lambda = \bigotimes_{n\geq 0} \lambda_n$ be the tensor product of probability counting measures λ_n on D_n (i.e., $\lambda_n(A) = |A|/|D_n|$ for $A \subset D_n$) and μ be the image of the measure λ under the map φ (i.e., $\mu(A) = \lambda(\varphi^{-1}(A))$ for a Borel subset $A \subset G$).

We claim that $\mu(gMh) = 0$ for each $g, h \in G$. For this we note that for any $g, h \in G$ and $k \geq 1$ we get $\mu(gB(\varepsilon_{k+1})h) = \lambda\big(\varphi^{-1}(gB(\varepsilon_{k+1})h)\big) \leq \big(\prod_{i<k}|D_i|\big)^{-1}$. Consequently, $\mu(gF_{k+1}B(\varepsilon_{k+1})h) \leq |F_{k+1}|\big(\prod_{i<k}|D_i|\big)^{-1} \leq \frac{|F_{k+1}|}{|D_{k-1}|} = \frac{1}{2^k}$ and

$$\mu(gMh) \leq \mu\Big(\bigcup_{i>k} gF_iB(\varepsilon_i)h\Big) \leq \sum_{i>k} \frac{1}{2^{i-1}} = \frac{1}{2^{k-1}}.$$

Sending k to ∞ we get $\mu(gMh) = 0$, which means that B lies in the Haar null G_δ-subset M of G. □

Lemma 5. *If $\pi : G \to H$ is a continuous surjective homomorphism from a Polish group G onto a non-discrete (locally compact) Polish group H, then a subset $A \subset H$ is Haar null (if and) only if its preimage $\pi^{-1}(A)$ is Haar null in G, which implies* $\mathrm{cov}(\mathcal{HN}(G)) \leq \mathrm{cov}(\mathcal{HN}(H))$ *and* $\mathrm{non}(\mathcal{HN}(G)) \geq \mathrm{non}(\mathcal{HN}(H))$.

Proof: To prove the "only if" part assume that a subset A is Haar null in H. Without loss of generality, A is universally measurable in H. Then its preimage

$\pi^{-1}(A)$ is universally measurable in G. Fix any probability measure μ on H with $\mu(xAy) = 0$ for all $x, y \in H$ and find any probability measure η on G that maps onto μ by the homomorphism π (the existence of such a measure η follows from the Jankov, von Neumann Uniformization Theorem [Ke, 18.1]). Then for any $x, y \in G$ we get $\eta(x\pi^{-1}(A)y) = \eta\big(\pi^{-1}(\pi(x)A\pi(y))\big) = \mu(\pi(x)A\pi(y)) = 0$, which means that $\pi^{-1}(A)$ is Haar null.

To prove that $\mathrm{cov}(\mathcal{HN}(G)) \le \mathrm{cov}(\mathcal{HN}(H))$ take any cover $\mathcal{C}$ of H by Haar null sets with $|\mathcal{C}| = \mathrm{cov}(\mathcal{HN}(H))$ and observe that $\pi^{-1}(\mathcal{C}) = \{\pi^{-1}(C) : C \in \mathcal{C}\}$ is a cover of G by Haar null sets, which yields $\mathrm{cov}(\mathcal{HN}(G)) \le |\pi^{-1}(\mathcal{C})| \le \mathrm{cov}(\mathcal{HN}(H))$.

To prove that $\mathrm{non}(\mathcal{HN}(G)) \ge \mathrm{non}(\mathcal{HN}(H))$ take any subset $A \subset G$ of size $|A| < \mathrm{non}(\mathcal{HN}(H))$. Then $|\pi(A)| < \mathrm{non}(\mathcal{HN}(H))$ and hence $\pi(A)$ is Haar null in H while $\pi^{-1}(\pi(A)) \supset A$ is Haar null in G. Thus $\mathrm{non}(\mathcal{HN}(G)) \ge \mathrm{non}(\mathcal{HN}(H))$.

To prove the "if" part, suppose that the group H is locally compact and $A \subset H$ is such that $\pi^{-1}(A)$ is Haar null in G. Let λ denote a left invariant Haar measure on H. We should show that $\lambda(A) = 0$. The set $\pi^{-1}(A)$, being Haar null, is contained in a universally measurable subset $M \subset G$ for which there exists a probability measure μ with compact support on G such that $\mu(xMy) = 0$ for all $x, y \in G$. Find a locally finite Borel measure η on G that maps onto the Haar measure λ by the homomorphism π. Consider the convolution $\eta * \mu$ of the measures η and μ, i.e., a measure assigning to a continuous function $f : G \to \mathbb{R}$ the integral $\int_\eta \int_\mu f(xy)\,dx\,dy$. It follows from the Fubini Theorem that $\eta * \mu(M) = 0$, see [THJ, 2.4.4].

Let us show that the homomorphism π maps the measure $\eta * \mu$ onto the Haar measure λ. Indeed, given a Borel subset $B \subset H$ denote by $\chi_B : H \to \{0, 1\}$ the characteristic function of the set B and applying the Fubini Theorem conclude that

$$\begin{aligned}
\eta * \mu(\pi^{-1}(B)) &= \int_\eta \int_\mu \chi_B \circ \pi(xy)\,dx\,dy \\
&= \int_\mu \int_\eta \chi_B \circ \pi(xy)\,dy\,dx = \int_\mu \eta\big(\pi^{-1}(\pi(x^{-1})B)\big)\,dx \\
&= \int_\mu \lambda\big(\pi(x^{-1})B\big)\,dx = \int_\mu \lambda(B)\,dx = \lambda(B).
\end{aligned}$$

Since $\eta*\mu(M) = 0$ there is a σ-compact set $S \subset G\setminus M$ such that $\eta*\mu(G\setminus S) = 0$. Then $\lambda(H \setminus \pi(S)) = 0$ and hence $\lambda(A) = 0$ since $A \cap \pi(S) = \emptyset$. $\square$

Lemma 6. *If a non-locally compact Polish group $G = \prod_{n\ge 0} G_n$ is the product of locally compact groups, then $o\mathcal{B}(G) \subset \mathcal{HN}(G)$.*

Proof: We remind that the *modular function* on a locally compact group H endowed with a left invariant Haar measure λ is a unique homomorphism $\triangle$:

$H \to \mathbb{R}_+$ into the multiplicative group of positive real numbers such that $\lambda(Bx) = \triangle(x)\lambda(B)$ for any $x \in H$ and a Borel subset $B \subset H$, see [He, §1.2] or [Za, §4]. A locally compact group H is *unimodular* if its modular function is constant (this is equivalent to saying that any left invariant Haar measure on H is right invariant).

To prove that $o\mathcal{B}(G) \subset \mathcal{HN}(G)$ fix any o-bounded subset $B \subset G = \prod_{n\geq 0} G_n$. Without loss of generality, we can assume that all the groups G_n are not compact.

If infinitely many groups G_n fail to be unimodular, then the Polish abelian group $H = \prod_{n\geq 0} G_n/\operatorname{Ker}(\triangle_n)$ is not locally compact and consequently, the group G admits a continuous homomorphism $\pi : G \to H$ onto the Polish non-locally compact abelian group H. By [Tk, 3.10] the set $\pi(B)$ is o-bounded in H. Since the abelian group H has invariant metric we may apply Lemma 4 to conclude that $o\mathcal{B}(H) \subset \mathcal{HN}(H)$ and thus the set $\pi(B)$ is Haar null in H. Applying Lemma 5 we get that the preimage $\pi^{-1}(\pi(B)) \supset B$ is Haar null in G.

Now consider the case when almost all the groups G_n are unimodular. Without loss of generality, we can assume that the groups G_n are unimodular for all $n \geq 1$. For every $n \geq 0$ fix a left invariant Haar measure λ_n on the locally compact group G_n and a neighborhood $W_n \subset G_n$ of the unit, having compact closure in G_n. Let $U_n = \{(x_i)_{i\geq 0} \in G : x_i \in W_i \text{ for } i \leq n\}$, $n \geq 1$. Using the o-boundedness of the set B find a sequence $(F_n)_{n\geq 1}$ of finite subsets of the group G such that $B \subset \bigcap_{k\geq 1}\bigcup_{n\geq k} F_n U_n$. Note that the set $M = \bigcap_{k\geq 1}\bigcup_{n\geq k} F_n U_n$ is Borel and hence universally measurable. We claim that it is Haar null in G.

To find a suitable measure μ on G, for every $n \geq 0$ fix a compact subset $K_n \subset G_n$ with $\lambda_n(K_n) \geq 2^n |F_n| \lambda_n(W_n)$ (such a set K_n exists since G_n is not compact and the measure λ_n is unbounded). Next, consider the probability measure μ_n on G_n defined by $\mu_n(B) = \frac{\lambda_n(B\cap K_n)}{\lambda_n(K_n)}$ for a Borel subset $B \subset G_n$. Finally consider the tensor product $\mu = \bigotimes_{n\geq 0} \mu_n$ of the measures μ_n.

We claim that $\mu(xMy) = 0$ for any $x, y \in G$. For this notice that by the invariance of the measures λ_n, for every $n \geq 1$ we get

$$\mu(xF_nU_ny) \leq |F_n| \frac{\lambda_n(W_n)}{\lambda_n(K_n)} \leq \frac{1}{2^n}\,.$$

Consequently, for every $k \geq 0$

$$\mu(xMy) \leq \mu\Big(\bigcup_{n\geq k} xF_nU_ny\Big) \leq \sum_{n\geq k} \frac{1}{2^n} = \frac{1}{2^{k-1}}\,.$$

Sending k to ∞ we get $\mu(xMy) = 0$ which means that $M \supset B$ is Haar null in G. □

Proof of Theorem 2

Suppose that G is a non-discrete Polish group.

1. If G is locally compact, then a subset $A \subset G$ is Haar null if and only if A has zero measure with respect to a left-invariant Haar measure λ on G, see [THJ, p. 374]. Replace the Haar measure λ by a Borel probability measure μ equivalent to λ in the sense that $\mu(B) = 0$ for a Borel subset $B \subset G$ if and only if $\lambda(B) = 0$. By Theorem [Ke, 17.41] on the isomorphism of measure spaces, there is a Borel isomorphism $f : G \to [0, 1]$ such that for any Borel subset $B \subset G$ $\mu(B) = \tau(f(B))$ where τ is the Lebesgue measure on $[0, 1]$. This shows that the ideal $\mathcal{HN}(G)$ is isomorphic to the ideal $\mathcal{N}$ of Lebesgue null subsets of $[0, 1]$ and consequently these ideals have the same cardinal characteristics, i.e., $\mathrm{add}(\mathcal{HN}(G)) = \mathrm{add}(\mathcal{N})$, $\mathrm{cov}(\mathcal{HN}(G)) = \mathrm{cov}(\mathcal{N})$, $\mathrm{non}(\mathcal{HN}(G)) = \mathrm{non}(\mathcal{N})$, and $\mathrm{cof}(\mathcal{HN}(G)) = \mathrm{cof}(\mathcal{N})$.

2. Suppose that G is not locally compact and admits an invariant metric. The inclusion $o\mathcal{B}(G) \subset \mathcal{HN}(G)$ and estimates $\mathrm{cov}(o\mathcal{B}(G)) \le \mathfrak{b}$, $\mathrm{non}(o\mathcal{B}(G)) \ge \mathfrak{d}$ proved in Lemmas 4 and 1 imply $\mathrm{cov}(\mathcal{HN}(G)) \le \mathrm{cov}(o\mathcal{B}(G)) \le \mathfrak{b}$ and $\mathrm{non}(\mathcal{HN}(G)) \ge \mathrm{non}(o\mathcal{B}(G)) \ge \mathfrak{d}$. The estimate $\mathrm{non}(\mathcal{N}) \le \mathrm{non}(\mathcal{HN}(G))$ follows from the inclusion $\mathcal{UN}(G) \subset \mathcal{HN}(G)$ and the well-known equality $\mathrm{non}(\mathcal{UN}) = \mathrm{non}(\mathcal{N})$ (holding because of the Isomorphism Theorem for non-atomic measure spaces [Ke, 17.41]). Therefore $\mathrm{cov}(\mathcal{HN}(G)) \le \mathfrak{b} \le \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\} \le \mathrm{non}(\mathcal{HN}(G)) \le \mathrm{cof}(\mathcal{HN}(G))$.

To show that $\mathrm{cof}(\mathcal{HN}(G)) > \min\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$ we first prove that $\mathrm{non}(\mathcal{N}) \ge \mathfrak{d}$ implies $\mathrm{cof}(\mathcal{HN}(G)) \ge \mathrm{cof}(\mathcal{UN}(G), \mathcal{Z}_2(G)) > \mathfrak{d}$ (this will be used for the proof of Theorem 4).

Assuming that $\mathrm{non}(\mathcal{N}) \ge \mathfrak{d}$ and $\mathrm{cof}(\mathcal{UN}(G), \mathcal{Z}_2(G)) \le \mathfrak{d}$, fix a family $\{Z_\alpha\}_{\alpha<\mathfrak{d}}$ of 2-Zorn subsets of G such that each universally null subsets of G lies in Z_α for some ordinal $\alpha < \mathfrak{d}$ (as usual, we identify cardinals with initial ordinals). Since $\mathrm{cof}(\mathcal{B}(G)) = \mathrm{cof}(\mathcal{B}) = \mathfrak{d}$, we can also fix a family $\{C_\alpha\}_{\alpha<\mathfrak{d}}$ of σ-compact subsets of G such that each σ-compact sets C lies in some C_α.

Let us show that for any ordinal $\alpha < \mathfrak{d}$ we get $G \ne Z_\alpha \cup \big(\bigcup_{\beta\le\alpha} C_\alpha\big)$. Let $S_\alpha = \bigcup_{\beta\le\alpha} C_\beta$ and consider the set $S_\alpha \cdot S_\alpha^{-1} = \bigcup_{\beta,\gamma\le\alpha} C_\beta \cdot C_\gamma^{-1}$ which is the union of $< \mathfrak{d}$ compact subsets of G. Since $\mathrm{cov}(\mathcal{B}(G)) = \mathrm{cov}(\mathcal{B}) = \mathfrak{d}$, there is an element $g \in G \setminus (S_\alpha \cdot S_\alpha^{-1})$. For this element g we get $S_\alpha \cap gS_\alpha = \emptyset$. Assuming that $G = Z_\alpha \cup S_\alpha$ we would get $gS_\alpha \subset Z_\alpha$ and $S_\alpha \subset g^{-1}Z_\alpha$. Then $G = Z_\alpha \cup g^{-1}Z_\alpha$ which is not possible as Z_α is 2-Zorn. Consequently, $G \ne Z_\alpha \cup S_\alpha$ and we can pick a point $x_\alpha \in G \setminus (Z_\alpha \cup S_\alpha)$.

We claim that the subset $X = \{x_\alpha : \alpha < \mathfrak{d}\}$ is universally null. Fix any probability non-atomic measure μ on G and find a σ-compact subset $C \subset G$ with $\mu(G \setminus C) = 0$, see [Ke, 17.11]. By the choice of the family $\{C_\alpha\}$, there is an ordinal $\alpha < \mathfrak{d}$ with $C \subset C_\alpha$. It follows from the construction of X that $X \cap C_\alpha \subset \{x_\beta : \beta \le \alpha\}$ and $|X \cap C_\alpha| < \mathfrak{d}$. Since $\mathfrak{d} \le \mathrm{non}(\mathcal{N}) = \mathrm{non}(\mathcal{UN}(G))$, the set $X \cap C_\alpha$ is universally null. Consequently, $\mu(X) \le \mu(X \cap C_\alpha) + \mu(G \setminus C_\alpha) = 0$, i.e., X is universally null and hence Haar null. By the choice of the family $\{Z_\alpha\}$, there is an ordinal $\alpha < \mathfrak{d}$ with $X \subset Z_\alpha$. On the other hand $X \setminus Z_\alpha \ni x_\alpha$, which is a contradiction.

Thus $\mathrm{cof}(\mathcal{HN}(G)) \geq \mathrm{cof}(\mathcal{UN}(G), \mathcal{Z}_2(G)) > \mathfrak{d} \geq \min\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$ under $\mathrm{non}(\mathcal{N}) \geq \mathfrak{d}$. If $\mathrm{non}(\mathcal{N}) < \mathfrak{d}$, then again $\mathrm{cof}(\mathcal{HN}(G)) \geq \mathrm{non}(\mathcal{HN}(G)) \geq \mathfrak{d} > \min\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$.

3. Assume that H is a non-discrete locally compact group and either H is a closed normal subgroup of G or else H is a quotient of G. In both cases we shall construct a map $p : G \to H$ such that a subset $N \subset H$ is Haar null in H if and only if $p^{-1}(N)$ is Haar null in G. If H is a quotient group of G, then let $p : G \to H$ be the quotient homomorphism and apply Lemma 5.

So now consider the case when H is a closed normal subgroup of G. According to [Ke, 12.17] the quotient homomorphism $\pi : G \to G/H$ admits a Borel section $s : G/H \to G$. The set $T = s(G/H)$, being the image of the Polish space G/H under an injective Borel map, is Borel in G, see [Ke, 15.1].

Consider the map $p : G \to H$ assigning to a point $x \in G$ the point $p(x) = (s \circ \pi(x))^{-1}x$. We claim that $p^{-1}(B) = TB$ for any subset $B \subset H$. Indeed, for any $t \in T$ and $b \in B$

$$p(tb) = (s \circ \pi(tb))^{-1}tb = (s \circ \pi(t))^{-1}tb = t^{-1}tb = b \in B.$$

On the other hand, if $p(x) = b \in B$, then $b = p(x) = (s \circ \pi(x))^{-1}x$ and thus $x = (s \circ \pi(x))b \in TB$.

We claim that a subset $N \subset H$ is Naar null in H if and only if TN is Haar null in G. Fix a left-invariant Haar measure λ on H.

Suppose that N is Haar null in H. Then $\lambda(N) = 0$ and we can assume that N is Borel in H. The product $TN = p^{-1}(N)$, being the image of the Borel space $T \times N$ under an injective continuous map, is Borel and thus universally measurable in G, see [Ke, 15.1]. We claim that $\lambda(xTNy) = 0$ for all $x, y \in G$. Given points $x, y \in G$ let $t = s(x^{-1}y^{-1})$ and observe that $xTNy \cap H = xtNy$ and hence $\lambda(xTNy) = \lambda(xtNy) = \lambda(Ny) = \triangle(y)\lambda(N) = 0$ which means that TN is Haar null in G.

Now assume conversely, that TN is Haar null in G. To show that N is Haar null in H it suffices to verify that $\lambda(N) = 0$. Fix a universally measurable subset $M \supset TN$ of G and a probability measure μ on G such that $\mu(xMy) = 0$ for all $x, y \in G$. Consider the convolution $\lambda * \mu$ assigning to a Borel function $f : G \to \mathbb{R}$ the integral $\int_\lambda \int_\mu f(xy)\,dx\,dy$. It is standard to show that $\lambda * \mu(M) = 0$, see [THJ, 2.4.4]. Denote by $\chi_M : G \to \{0, 1\}$ the characteristic function of the set M and applying Fubini Theorem conclude that

$$0 = \lambda * \mu(M) = \int_\lambda \int_\mu \chi_M(xy)\,dx\,dy = \int_\mu \int_\lambda \chi_M(xy)\,dy\,dx = \int_\mu \lambda(x^{-1}M)\,dx.$$

Then $\lambda(x^{-1}M) = 0$ for some $x \in G$. Since $M \supset TN$, we get $0 = \lambda(x^{-1}TN) = \lambda(x^{-1}(s \circ \pi(x))N) = \lambda(N)$.

Therefore a subset $N \subset H$ is Haar zero if and only if $p^{-1}(N)$ is Haar null in G. Using this observation it is easy to show that $\mathrm{add}(\mathcal{HN}(G)) \le \mathrm{add}(\mathcal{HN}(H)) = \mathrm{add}(\mathcal{N})$, $\mathrm{cov}(\mathcal{HN}(G)) \le \mathrm{cov}(\mathcal{HN}(H)) = \mathrm{cov}(\mathcal{N})$, and $\mathrm{non}(\mathcal{HN}(G)) \ge \mathrm{non}(\mathcal{HN}(H)) = \mathrm{non}(\mathcal{N})$.

To show that $\mathrm{cof}(\mathcal{HN}(G)) \ge \mathrm{cof}(\mathcal{HN}(H)) = \mathrm{cof}(\mathcal{N})$ fix any family $\mathcal{F} \subset \mathcal{HN}(G)$ of size $|\mathcal{F}| = \mathrm{cof}(\mathcal{HN}(G))$ such that each Haar null subset of G lies in some $F \in \mathcal{F}$. For each set $F \in \mathcal{F}$ consider the subset $F' = H \setminus p(G \setminus F)$ of H which is Haar null in H since $p^{-1}(F') \subset F$. We claim that the family $\{F' : F \in \mathcal{F}\}$ is cofinal in $\mathcal{HN}(H)$. Indeed, for any Haar null set $N \subset H$ the set $p^{-1}(N)$ is Haar null in G. Then $p^{-1}(N) \subset F$ for some $F \in \mathcal{F}$ and hence $N \subset F'$. Therefore $\mathrm{cof}(\mathcal{N}) = \mathrm{cof}(\mathcal{HN}(H)) \le |\{F' : F \in \mathcal{F}\}| \le |\mathcal{F}| = \mathrm{cof}(\mathcal{HN}(G))$.

4. Assume that the center $Z = \{g \in G : \forall x \in G\ xg = gx\}$ of G is not locally compact. Then $\mathrm{cov}(\mathcal{HN}(Z)) \le \mathfrak{b} \le \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\} \le \mathrm{non}(\mathcal{HN}(Z))$ by the second statement of this theorem. So it rests to verify that $\mathrm{cov}(\mathcal{HN}(G)) \le \mathrm{cov}(\mathcal{HN}(Z))$ and $\mathrm{non}(\mathcal{HN}(G)) \ge \mathrm{non}(\mathcal{HN}(Z))$.

According to [Ke, 12.17] the quotient homomorphism $\pi : G \to G/Z$ admits a Borel section $s : G/Z \to G$. Let $T = s(G/Z)$ and consider the map $p : G \to Z$ defined by $p(x) = (s \circ \pi(x))^{-1}x$ for $x \in G$. In the preceding item we have shown that p is a Borel map with $p^{-1}(N) = TN$ for any subset $N \subset H$.

We claim that for any universally measurable Haar null set $N \subset Z$ the set TN is Haar null in G. First we note that the set $TN = p^{-1}(N)$, being the preimage of the universally measurable set N under the Borel map p, is universally measurable.

Since the set N is Haar null in Z, there is a Borel measure μ on Z such that $\mu(xNy) = 0$ for all $x, y \in H$. We claim that $\mu(xTNy) = 0$ for all $x, y \in G$. Given any points $x, y \in G$ let $t = s \circ \pi(x^{-1}y^{-1})$ and note that $xty \in Z$ and $xTNy \cap Z = xtNy = xtyN$. Then $\mu(xTNy) = \mu(xtyN) = 0$ which means that $TN = p^{-1}(N)$ is a Haar null subset of G.

Therefore for any Haar null subset $N \subset Z$ the preimage $p^{-1}(N) = TN$ is Haar null in G. Using this fact it is trivial to show that $\mathrm{cov}(\mathcal{HN}(G)) \le \mathrm{cov}(\mathcal{HN}(Z))$ and $\mathrm{non}(\mathcal{HN}(G)) \ge \mathrm{non}(\mathcal{HN}(Z))$.

5. If G admits a surjective continuous homomorphism onto a non-locally Polish compact group H with invariant metric, then Lemma 5 and the second item of this theorem imply that $\mathrm{cov}(\mathcal{HN}(G)) \le \mathrm{cov}(\mathcal{HN}(H)) \le \mathfrak{b}$, and $\mathrm{non}(\mathcal{HN}(G)) \ge \mathrm{non}(\mathcal{HN}(H)) \ge \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$.

Proof of Theorem 3

Suppose that a non-locally compact Polish group $G = \prod_{n \ge 0} G_n$ is the product of locally compact Polish groups G_n. Without loss of generality we can assume that the groups G_n are not trivial.

1. The estimates $\mathrm{cov}(\mathcal{HN}(G)) \le \mathrm{cov}(o\mathcal{B}(G)) \le \mathfrak{b}$ and

$$\begin{aligned}\max\{\mathrm{non}(\mathcal{N}), \mathfrak{d}\} &\le \max\{\mathrm{non}(\mathcal{UN}(G)), \mathrm{non}(o\mathcal{B}(G))\}\\ &\le \mathrm{non}(\mathcal{HN}(G)) \le \mathrm{cof}(\mathcal{HN}(G))\end{aligned}$$

follow from the inclusion $\mathcal{UN}(G) \cup o\mathcal{B}(G) \subset \mathcal{HN}(G)$ (see Theorem 1) and Lemma 1. To prove that $\mathrm{cof}(\mathcal{HN}(G)) > \min\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$ we consider separately two cases.

If $\mathrm{non}(\mathcal{N}) < \mathfrak{d}$, then $\mathrm{cof}(\mathcal{HN}(G)) \ge \mathfrak{d} > \min\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$. If $\mathrm{non}(\mathcal{N}) \ge \mathfrak{d}$, then $\mathrm{cof}(\mathcal{HN}(G)) > \mathfrak{d} \ge \min\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$ according to (the proof of) Theorem 2(2).

2. Suppose that all but finitely many groups G_n are amenable. Without loss of generality we can assume that the groups G_n are amenable for $n \ge 1$. For every $n \ge 0$ fix a left-invariant Haar measure λ_n on G_n. Each group G_n, $n \ge 1$, being amenable, satisfies the Følner condition. Using this condition, for every $n \ge 1$ we can construct an increasing sequence $(K_{n,m})_{m\ge 0}$ of compact subsets of the group G_n such that $\bigcup_{m\ge 0} K_{n,m} = G_n$, each $K_{n,m}$ lies in the interior of $K_{n,m+1}$ and $\lambda_n(xK_{n,m+1} \triangle K_{n,m+1}) < 2^{-m}\lambda(K_{n,m+1})$ for any $x \in K_{n,m}$.

For every $n \ge 0$ fix a probability measure $\tilde{\lambda}_n$ on G_n equivalent to the Haar measure λ_n (in the sense that they have the same null sets) and let $\lambda_{0,m} = \tilde{\lambda}_0$ for all $m \in \mathbb{N}$. For every $n, m \in \mathbb{N}$ define a probability measure $\lambda_{n,m}$ on the group G_n letting

$$\lambda_{n,m}(B) = \Big(1 - \frac{1}{2^m}\Big)\frac{\lambda_n(B \cap K_{n,m})}{\lambda_n(K_{n,m})} + \frac{1}{2^m}\tilde{\lambda}_n(B)$$

for any Borel subset $B \subset G_n$. For any function $f \in \mathbb{N}^\omega$, denote by μ_f the tensor product $\mu_f = \bigotimes_{n\ge 0} \lambda_{n,f(n)}$ which is a probability measure on G. In (the proof of) Theorem 4.1 [S_2] S. Solecki has shown that a universally measurable subset $N \subset G$ is Haar null in G if and only if there is a function $f \in \mathbb{N}^\omega$ such that $\mu_f(xNy) = 0$ for any $x, y \in G$ if and only if there is a function $f \in \mathbb{N}^\omega$ such that $\mu_g(xNy) = 0$ for any $x, y \in G$ and any $g \in \mathbb{N}^\omega$ with $f \le^* g$.

To estimate the cardinals $\mathrm{add}(\mathcal{HN}(G))$ and $\mathrm{cov}(\mathcal{HN}(G))$ fix any family $\mathcal{S} \subset \mathcal{HN}(G)$ of universally measurable Haar null subsets of G with $|\mathcal{S}| < \mathfrak{b}$. Using the mentioned result of S. Solecki, for any $S \in \mathcal{S}$ find a function $f_S \in \mathbb{N}^\omega$ such that $\mu_g(xSy) = 0$ for any $x, y \in G$ and any $g \in \mathbb{N}^\omega$ with $f_S \le^* g$. Since $|\mathcal{S}| < \mathfrak{b}$, the set $\{f_S : S \in \mathcal{S}\}$ is bounded in $(\mathbb{N}^\omega, \le^*)$. Consequently, there is a function $f \in \mathbb{N}^\omega$ such that $f_S \le^* f$ for all $S \in \mathcal{S}$. For this function f we get $\mu_f(xSy) = 0$ for all $x, y \in G$ and $S \in \mathcal{S}$. Now consider the union $\bigcup\mathcal{S}$. If $|\mathcal{S}| < \mathrm{add}(\mathcal{N})$, then $\bigcup\mathcal{S}$ is universally measurable by Lemma 3 and $\mu_f(x(\bigcup\mathcal{S})y) = 0$ for all $x, y \in G$. Applying Solecki's Theorem 4.1 [S_2] we conclude that the union $\bigcup\mathcal{S}$ is Haar null in G and hence $\mathrm{add}(\mathcal{HN}(G)) \ge \min\{\mathfrak{b}, \mathrm{add}(\mathcal{N})\} = \mathrm{add}(\mathcal{N})$. If $|\mathcal{S}| < \mathrm{cov}(\mathcal{N})$, then $\bigcup S \ne G$ (being the union of $< \mathrm{cov}(\mathcal{N})$ many μ_f-zero sets) and thus $\mathrm{cov}(\mathcal{HN}(G)) \le \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{N})\}$.

To prove that $\mathrm{non}(\mathcal{HN}(G)) \geq \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$, fix any dominating subset $D \subset \mathbb{N}^\omega$ of size $|D| = \mathfrak{d}$. For any $f \in D$ find a subset $N_f \subset G$ of size $|N_f| = \mathrm{non}(\mathcal{N})$ such that $\mu_f(N_f) \neq 0$. Then the union $N = \bigcup_{f \in D} N_f$ has size $|N| \leq \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$ and is not Haar null. Otherwise, using the Solecki's result we would find a function $f \in D$ such that $\mu_f(N) = 0$ which is not possible since $\mu_f(N_f) \neq 0$ and $N_f \subset N$.

3. If one of the groups G_n is non-discrete, then we may apply Theorem 2(3) to conclude that $\mathrm{add}(\mathcal{HN}(G)) \leq \mathrm{add}(\mathcal{N})$, $\mathrm{cov}(\mathcal{HN}(G)) \leq \mathrm{cov}(\mathcal{N})$, and $\mathrm{cof}(\mathcal{HN}(G)) \geq \mathrm{cof}(\mathcal{N})$. So assume that all groups G_n are discrete and infinitely many of them are almost kaleidoscopical. Without loss of generality we can assume that each group G_{2n}, $n \geq 0$, is almost kaleidoscopical.

Fix any sequence $(\varepsilon_n)_{n \geq 0}$ of positive real numbers such that

$$\frac{1}{2} < \prod_{n \geq 0} (1 - \varepsilon_n) < \prod_{n \geq 0} (1 + \varepsilon_n) < 2.$$

Since the groups G_{2n} are almost kaleidoscopical, for every $n \geq 0$ we can find a finitely supported probability measure μ_{2n} on G_{2n} and a nontrivial finite partition $\mathcal{P}_n$ of G_{2n} such that $|\mu_{2n}(xPy) - \frac{1}{|\mathcal{P}_n|}| < \frac{\varepsilon_n}{|\mathcal{P}|}$ for each $x, y \in G$ and $P \in \mathcal{P}_n$. Endow each set $\mathcal{P}_n$ with the discrete topology and consider the map $p_n : G_{2n} \to \mathcal{P}_n$ assigning to a point $g \in G$ a unique element $P \in \mathcal{P}_n$ containing g. Now consider the continuous map $\pi : \prod_{n \geq 0} G_n \to \prod_{n \geq 0} \mathcal{P}_n$ assigning to a sequence $(x_n)_{n \geq 0} \in \prod_{n \geq 0} G_n$ the sequence $(p_n(x_{2n}))_{n \geq 0}$ in $\prod_{n \geq 0} \mathcal{P}_n$. For every $n \geq 0$ fix any probability measure μ_{2n+1} on the group G_{2n+1} and endow the group $G = \prod_{n \geq 0} G_n$ with the measure μ equal to the tensor product $\bigotimes_{n \geq 0} \mu_n$ of the measures μ_n.

On the product $\mathbb{P} = \prod_{n \geq 0} \mathcal{P}_n$ consider the measure λ equal to the tensor product $\bigotimes_{n \geq 0} \lambda_n$ of uniformly distributed measures of $\mathcal{P}_n$'s. For every $m \geq 1$ let $\mathrm{pr}_m : \prod_{n \geq 0} \mathcal{P}_n \to \prod_{0 \leq n < m} \mathcal{P}_n$ be the projection onto the first m coordinates. Let us call a subset C of $\mathbb{P}$ *cylindrical* if $C = \mathrm{pr}_m^{-1}(A)$ for some $m \geq 1$ and some set $A \subset \prod_{0 \leq n < m} \mathcal{P}_n$. It follows from the choice of the measures μ_i that for any point $y \in \prod_{0 \leq n < m} \mathcal{P}_n$ the preimage $P = (\mathrm{pr}_m \circ \pi)^{-1}(y)$ has μ-measure satisfying

$$\prod_{0 \leq n < m} \frac{1 - \varepsilon_n}{|\mathcal{P}_n|} \leq \mu(P) \leq \prod_{0 \leq n < m} \frac{1 + \varepsilon_n}{|\mathcal{P}_n|}$$

and the same estimate is true for any shift xPy of P. Consequently, for any $x, y \in G$ and any cylindrical set $C = \mathrm{pr}_m^{-1}(A)$ we have

$$\lambda(C) = |A| \prod_{0 \leq n < m} \frac{1}{|\mathcal{P}_n|} \tag{1}$$

and for its preimage $\pi^{-1}(C)$ we get

$$|A| \prod_{0\le n<m} \frac{1-\varepsilon_n}{|\mathcal{P}_n|} \le \mu(x\pi^{-1}(C)y) \le |A| \prod_{0\le n<m} \frac{1+\varepsilon_n}{|\mathcal{P}_n|}. \tag{2}$$

Dividing (2) by (1) we get

$$\frac{1}{2} \le \prod_{0\le n<m} (1-\varepsilon_n) \le \frac{\mu(x\pi^{-1}(C)y)}{\lambda(C)} \le \prod_{0\le n<m} (1+\varepsilon_n) \le 2. \tag{3}$$

Next, we show that the same estimate

$$\frac{1}{2}\lambda(M) \le \mu(x\pi^{-1}(M)y) \le 2\lambda(M) \tag{4}$$

holds for any universally measurable subset M of $\mathbb{P}$. Assuming that $\mu(x\pi^{-1}(M)y) > 2\lambda(M)$ for some universally measurable set $M \subset \mathbb{P}$ and points $x, y \in G$, find a compact subset $K \subset \pi^{-1}(M)$ with $\mu(xKy) > 2\lambda(M)$. Express the compact set $\pi(K)$ as a countable intersection $\pi(K) = \bigcap_{n\ge 0} C_n$ of a decreasing sequence of cylindrical subsets of $\mathbb{P}$. Since $\mu(x\pi^{-1}(\pi(K))y) \ge \mu(xKy) > 2\lambda(M) \ge 2\lambda(\pi(K))$, the countable additivity of the measures μ and λ imply that $\mu(x\pi^{-1}(C_n)y) > 2\lambda(C_n)$ for a sufficiently large n, which is not possible because of (3). By a similar argument we can show that $\mu(x\pi^{-1}(M)y) \ge \frac{1}{2}\lambda(M)$ for any universally measurable set $M \subset \mathbb{P}$ and any points $x, y \in G$ and thus finish the proof of (4).

This estimate implies that for any universally measurable set $M \subset \mathbb{P}$ with $\lambda(M) = 0$ we get $\mu(x\pi^{-1}(M)y) = 0$ for all $x, y \in G$, which means that $\pi^{-1}(M)$ is Haar null in G.

Now we prove that the converse is also true, that is a subset $N \subset \mathbb{P}$ has zero λ-measure if its preimage $\pi^{-1}(N)$ is Haar null in G. Assuming that the set $\pi^{-1}(N)$ is Haar null in G, fix a universally measurable set $M \supset \pi^{-1}(N)$ of G and a probability measure ν on G such that $\nu(xMy) = 0$ for all $x, y \in G$. Now consider the convolution $\mu * \nu$ assigning to a bounded continuous function $f : G \to \mathbb{R}$ the integral $\int_\mu \int_\nu f(xy)\,dx\,dy$. It follows from the Fubini Theorem that $\mu * \nu(M) = 0$. Consequently, there is a σ-compact set $S \subset G \setminus M$ with $\mu * \nu(S) = 1$. Now consider the image $\pi(S) \subset \mathbb{P}$ and note that it is a σ-compact set disjoint with N. Let $Q = \pi^{-1}(\mathbb{P} \setminus \pi(S))$ and note that $S \cap Q = \emptyset$ and hence $\mu * \nu(Q) = 0$. Denote by $\chi_Q : G \to \{0, 1\}$ the characteristic function of the set Q. Using the Fubini Theorem and the estimate (4) we get

$$\begin{aligned} 0 = \mu * \nu(Q) &= \int_\mu \int_\nu \chi_Q(xy)\,dx\,dy = \int_\nu \int_\mu \chi_Q(xy)\,dy\,dx \\ &= \int_\nu \mu(x^{-1}Q)\,dx \ge \frac{1}{2} \int_\nu \lambda(\mathbb{P} \setminus \pi(S))\,dx = \frac{1}{2}\lambda(\mathbb{P} \setminus \pi(S)). \end{aligned}$$

Hence $\lambda(\mathbb{P} \setminus \pi(S)) = 0$ and $\lambda(N) = 0$ since $N \cap \pi(S) = \emptyset$. Therefore we have shown that a subset $N \subset \mathbb{P}$ has measure $\lambda(N) = 0$ if and only if its preimage $\pi^{-1}(N)$ is Haar null in G.

Now the estimations $\mathrm{add}(\mathcal{HN}(G)) \le \mathrm{add}(\mathcal{N})$, $\mathrm{cov}(\mathcal{HN}(G)) \le \mathrm{cov}(\mathcal{N})$ and $\mathrm{cof}(\mathcal{N}) \le \mathrm{cof}(\mathcal{HN}(G))$ can be derived by analogy with the proof of Theorem 2(3).

4. Finally assume that the group G is abelian. Then every group G_n, being abelian is amenable and kaleidoscopical, see Proposition 1. Applying Theorem 3(1),(2),(3) we get the required estimations.

Proof of Theorem 4

Let G be a Polish non-locally compact group. The estimate $\mathrm{cof}(\mathcal{HN}(G)) \ge \mathrm{cof}(\mathcal{UN}(G), \mathcal{Z}_2(G)) > \mathfrak{d}$ under $\mathrm{non}(\mathbb{N}) \ge \mathfrak{d}$ was proven in (the proof of) Theorem 2(2). Then under $\mathrm{non}(\mathcal{N}) = \mathfrak{d} = \mathfrak{c}$ we get $\mathrm{cof}(\mathcal{UN}(G), \mathcal{Z}_2(G)) > \mathfrak{c}$. Since the σ-algebra $\sigma\mathbf{P}$ has size $|\sigma\mathbf{P}| = \mathfrak{c}$, we conclude that there is a universally null subset of G, contained in no σ-projective 2-Zorn subset of G.

Acknowledgments. The author express his sincere thanks to Igor Protasov and Anatoly Plichko for valuable and stimulating discussions concerning (almost) kaleidoscopical groups and the small ball property.

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(*Received* January 9, 2003, *revised* October 13, 2003)