Angel Tamariz-Mascarúa; H. Villegas-Rodríguez

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Spaces of continuous functions, box products and almost-ω-resolvable spaces

A. Tamariz-Mascarúa, H. Villegas-Rodríguez

Abstract. A dense-in-itself space X is called $C_{\square}$*-discrete* if the space of real continuous functions on X with its box topology, $C_{\square}(X)$, is a discrete space. A space X is called *almost-ω-resolvable* provided that X is the union of a countable increasing family of subsets each of them with an empty interior. We analyze these classes of spaces by determining their relations with κ-resolvable and almost resolvable spaces. We prove that every almost-ω-resolvable space is $C_{\square}$-discrete, and that these classes coincide in the realm of completely regular spaces. Also, we prove that almost resolvable spaces and almost-ω-resolvable spaces are two different classes of spaces if there exists a measurable cardinal. Finally, we prove that it is consistent with ZFC that every dense-in-itself space is almost-ω-resolvable, and that the existence of a measurable cardinal is equiconsistent with the existence of a Tychonoff space without isolated points which is not almost-ω-resolvable.

Keywords: box product, κ-resolvable space, almost resolvable space, almost-ω-resolvable space, Baire irresolvable space, measurable cardinals

Classification: 54C35, 54F65, 54A35

Introduction

The spaces of continuous functions defined on a topological space X and with values in $\mathbb{R}$, $C(X)$, have been widely studied as a purely algebraic structure ([GJ]), and with a topological (topologico-algebraic) structure ([BNS], [DH]).

One of the natural topologies associated with $C(X)$ is the pointwise convergence topology, which is the topology in $C(X)$ inherited from the Tychonoff topology of $\mathbb{R}^X$. This space is usually denoted by $C_p(X)$. A classical general problem on C_p-spaces consists of determining the relations between the topological properties of space X with the topological properties of $C_p(X)$ ([Ar]).

A generalization of the Tychonoff topology for a product of topological spaces, is the box topology (see definition in Section 1) which was introduced by Tietze in [T]. The study of the box product of an infinite family of topological spaces has been useful to construct some interesting topological spaces ([R], [V]).

So, it seems natural to ask about the relations between the topological properties of a space X and those of $C(X)$ with its box topology, which we denote by $C_{\square}(X)$. The purpose of this article is to analyze some of these relations when

X is dense-in-itself. In particular, we determine for which dense-in-themselves spaces X, $C_\Box(X)$ is a discrete space. This analysis will lead us to consider the resolvable and irresolvable spaces and the measurable cardinals. Resolvable and irresolvable spaces were studied extensively first by Hewitt [H]. Later, El'kin and Malykhin published a number of papers on this subject and its connection with various topological problems. One of the problems considered by Malykhin [M2] refers to the existence of irresolvable spaces satisfying the Baire Category Theorem. He proved that there is such a space if and only if there is a space X on which every real-valued function is continuous at some point. The question about the existence of a Hausdorff space on which every real-valued function is continuous at some point was posed by M. Katětov in [K]. Bolstein introduced in [B] the spaces X on which it is possible to define a real-valued function f with countable range and such that f is discontinuous at every point of X (he called these spaces *almost resolvable*), and proved that every resolvable space satisfies this condition. Here we introduce the almost-ω-resolvable spaces defined as spaces on which it is possible to define a real-valued function f with countable range, and such that $r \circ f$ is discontinuous in every point of X, for every real-valued finite-to-one function r. We prove in Section 2 that, for completely regular spaces, $C_\Box(X)$ is discrete if and only if X is almost-ω-resolvable. Section 3 is devoted to obtaining the basic properties of $C_\Box$-discrete and almost-ω-resolvable spaces, and we determine their relations with spaces having resolvable-like properties. In Section 4 we prove that the existence of a measurable cardinal is equiconsistent with the existence of a Tychonoff space without isolated points which is not almost-ω-resolvable, and that, on the contrary, if $V = L$ then every dense-in-itself space is almost-ω-resolvable. Also, we prove that almost-ω-resolvable spaces and almost resolvable spaces are two different classes of spaces if there exists a measurable cardinal. Finally, in the last section we list some unsolved questions concerning these subjects.

1. Definitions and generalities

Let $\mathcal{F} = \{X_s : s \in S\}$ be a family of topological spaces. By $\Box_{s\in S}X_s$ we denote the cartesian product of the family $\mathcal{F}$ considered with the so called box topology $\tau_\Box$, which is that generated by the sets of the form $\prod_{s\in S} A_s$ where, for each $s \in S$, A_s is an open subset of X_s. It is obvious that the Tychonoff topology is contained in the box topology, and they coincide when $|S| < \aleph_0$.

It is well known that, for an infinite family $\{X_s : s \in S\}$ of non-trivial topological spaces, $\Box_{s\in S}X_s$ is neither first countable nor locally compact, and it never is a topological vector space over $\mathbb{R}$, but it is a topological group if each of the spaces X_s is a topological group. A good survey of the characteristics of the box topology can be found in [Wi].

For a topological space X we denote by $\mathbb{R}^X$ the set of functions from X to $\mathbb{R}$. The subset of $\mathbb{R}^X$ whose elements are the continuous functions is denoted by $C(X)$. The space $\Box\mathbb{R}^X$ will be the set $\mathbb{R}^X$ with the box topology, and $C_\Box(X)$ is the set $C(X)$ considered as a subspace of $\Box\mathbb{R}^X$. Of course, $C_\Box(X)$ is a topological subgroup of $\Box\mathbb{R}^X$.

For spaces X and Y and for a function $f : X \to Y$, $D(f)$ will denote the set of points in X in which f is not continuous.

A point x in X is an *isolated point* of X if $\{x\}$ is open in X, and a space X is *dense-in-itself* if it does not contain isolated points. A space X is *resolvable* if it is the union of two disjoint dense subsets. We say that X is *irresolvable* if it is dense-in-itself and it is not resolvable. For a cardinal number $\kappa > 1$, we say that X is *κ-resolvable* if X is the union of κ pairwise disjoint dense subsets.

The *dispersion character* $\Delta(X)$ of a space X is the minimum of the cardinalities of nonempty open subsets of X.

If X is $\Delta(X)$-resolvable, then we say that X is *maximally resolvable*.

A space X is *hereditarily irresolvable* if it is dense-in-itself and every subspace of X is irresolvable. And X is *open-hereditarily irresolvable* if it is dense-in-itself and every open subspace of X is irresolvable.

A space (X, t) is *maximal* if (X, t) is dense-in-itself and (X, t') contains at least an isolated point when t' strictly contains the topology t. And a space X is *submaximal* if it is dense-in-itself and every dense subset of X is open.

A space X is called *almost resolvable* if X is the union of a countable collection of subsets each of them with an empty interior.

Of course, every κ-resolvable space is τ-resolvable if $\tau \leq \kappa$. Besides, every maximal space is submaximal, and these are hereditarily irresolvable spaces, which in turn are open hereditarily irresolvable.

The concept of resolvability and related topics were introduced by Hewitt in [H]; and Bolstein [B] proved that a space X is resolvable if it is the union of a finite collection of subsets each of them with an empty interior. That is, every resolvable space is almost resolvable.

The class of resolvable spaces includes spaces with well known properties:

Theorem 1.1 ([E2]). *If X is dense-in-itself and has a π-network $\mathcal{N}$ such that $|\mathcal{N}| \leq \Delta(X)$ and each $N \in \mathcal{N}$ satisfies $|N| \geq \Delta(X)$, then X is maximally resolvable.*

Theorem 1.2. *All the spaces considered in this theorem are dense-in-themselves.*

1. *The locally compact Hausdorff spaces are maximally resolvable* [H].
2. *First countable spaces are maximally resolvable* [E1].
3. *Hausdorff k-spaces are maximally resolvable* [P] (*in particular, metrizable spaces are maximally resolvable* [H]).
4. *Countably compact regular T_1 spaces are ω-resolvable* [CGF].
5. *Arc connected spaces are ω-resolvable* (*in particular, every topological vector space over $\mathbb{R}$ is ω-resolvable*).
6. *Every biradial space* (*in particular, every linearly orderable topological space*) *is maximally resolvable* [Vi2].

(7) *Every homogeneous space containing a non-trivial convergent sequence is ω-resolvable* [Vi1].
(8) *If G is a non-countable T_0 $\aleph_0$-bounded topological group, then G is $\aleph_1$-resolvable* [Vi2].

The terms not defined in this paper are considered as in [W].

2. $C_\square$-discrete and almost-ω-resolvable spaces

As we have already mentioned, we are interested in knowing when $C_\square(X)$ is a discrete subspace of $\square\mathbb{R}^X$; if this happens, we say that X is *$C_\square$-discrete*. The finite spaces are not $C_\square$-discrete because if $|X| = n+1$, $n \in \omega$, and $r \in \bigcap_{0\leq i\leq n} A_i$ where A_i is an open set in $\mathbb{R}$ with $0 \in A_i$ for every $0 \leq i \leq n$, then $f \equiv 0$ and $g \equiv r$ are two different continuous functions in $A_0 \times \cdots \times A_n$. *Because of this, from now on we will only consider spaces with cardinality* $\geq \aleph_0$. We also observe that if X is a T_1 or a regular space and $a \in X$ is an isolated point, then $X = \{a\} \oplus X \setminus \{a\}$; thus $C_\square(X) \cong \mathbb{R} \times C_\square(X \setminus \{a\})$. Therefore, in this case, $C_\square(X)$ is not discrete.

Furthermore, if $C(X)$ is equal to the set of constant functions, then $C_\square(X)$ is discrete in $\square\mathbb{R}^X$. Indeed, let $N = \{x_0, \ldots, x_n, \ldots\} \subset X$ with $x_i \neq x_j$ if $i \neq j$; and let $f(x) = r \in \mathbb{R}$ for every $x \in X$. Then, $\{f\} = C_\square(X) \cap \prod_{x\in X} A_x$ where $A_{x_n} = (r - \frac{1}{n}, r + \frac{1}{n})$ and $A_x = \mathbb{R}$ for every $x \in X \setminus N$. Thus, we can find a T_0 $C_\square$-discrete space having isolated points: take an infinite set X, and let $a \in X$. The collection $\tau = \{A \subset X : a \in A\} \cup \{\emptyset\}$ is a T_0 topology in X, $\{a\}$ is open and every continuous real-valued function on X is constant.

Proposition 2.1. *If X is $C_\square$-discrete and regular or T_1, then X does not have isolated points.*

Observe also that for every space X, $C_\square(X)$ is a closed subspace of $\square\mathbb{R}^X$. This can be proved as follows: Let $f \in \square\mathbb{R}^X$, and let $x_0 \in X$ such that f is not continuous in x_0. Then, there is an open neighborhood U of $f(x_0)$ such that, for every neighborhood V of x_0, there is $x_V \in V$ satisfying $f(x_V) \notin \operatorname{cl} U$. For each $f(x_V)$ we take a neighborhood W_V of $f(x_V)$ with $W_V \cap U = \emptyset$. Let $G = \prod_{x\in X} U_x$ where $U_x = U$ if $x = x_0$, $U_x = W_V$ if $x = x_V$, and $U_x = \mathbb{R}$ in all other cases. It is clear that $f \in G \subset \square\mathbb{R}^X \setminus C_\square(X)$.

Theorem 2.2. *Let (X, τ) be a topological space, and let $\tau^* = \tau \setminus \{\emptyset\}$. The following properties are equivalent.*

(1) *There is a partition $\mathcal{X} = \{X_n : n < \omega\}$ of X such that each $A \in \tau^*$ has a nonempty intersection with each of the elements of an infinite subcollection of $\mathcal{X}$.*
(2) *$X = \bigcup_{n<\omega} Y_n$ with $Y_n \subset Y_{n+1}$ and $\operatorname{int}(Y_n) = \emptyset$ for every $n < \omega$.*
(3) *There is a decreasing sequence of dense subsets of X with an empty intersection.*

(4) *There is a function* $f : X \to \omega$ *for which* $D(r \circ f)$ *is equal to* X*, for every finite-to-one function* $r : \omega \to \omega$*.*
(5) *There is a function* $h : X \to \mathbb{R}^+ \setminus \{0\}$ *such that, for each* $A \in \tau^*$*, there is a sequence* $(x_n)_{n<\omega}$ *in* A *such that* $(h(x_n))_{n<\omega}$ *converges to* 0*.*
(6) *There is a set* $S \neq \emptyset$ *and, for each* $s \in S$*, there exists* $C_s = \{x_n^s : n < \omega\}$*, such that* (i) $X = \bigcup_{s\in S} C_s$*;* (ii) *for each* $s_0 \in S$ *and each* $n_0 < \omega$*,* C_{s_0} *is not contained in* $\bigcup\{x_k^s : s \in S,\ k \leq n_0\}$*;* (iii) *for each* $A \in \tau^*$*, there is* $s \in S$ *such that* $C_s \subset A$*.*

Proof: (1) $\Rightarrow$ (2): The set $Y_n = \bigcup_{0\leq i\leq n} X_n$ has an empty interior for every $n < \omega$.

(2) $\Rightarrow$ (1): If we define $X_0 = Y_0$ and $X_{n+1} = Y_{n+1} \setminus \bigcup_{0\leq i\leq n} Y_n$, then $\{X_n : n < \omega\}$ satisfies that which is required.

(2) $\Leftrightarrow$ (3): Define $D_n = X \setminus Y_n$. Then, $D_{n+1} \subset D_n$ if and only if $Y_n \subset Y_{n+1}$; D_n is dense in X if and only if $\operatorname{int} Y_n = \emptyset$; and $\bigcap_{n\in\omega} D_n = \emptyset$ if and only if $X = \bigcup_{n<\omega} Y_n$.

(2) $\Rightarrow$ (4): The function $f : X \to \omega$ defined by $f(x) = 0$ if $x \in Y_0$, and $f(x) = n + 1$ if $x \in Y_{n+1} \setminus Y_n$, satisfies the required conditions.

(4) $\Rightarrow$ (5): Let $h(x) = \frac{1}{f(x)+1}$ for each $x \in X$, and let $A \in \tau^*$. If for some $n < \omega$, $A \subset \bigcup_{0\leq i\leq n} f^{-1}(i)$ and $x_0 \in A$, then $r \circ f$ is a continuous function in x_0, where $r : \omega \to \omega$ is defined by $r(i) = 0$ if $0 \leq i \leq n$, and $r(i) = i$ if $i > n$. But this is not possible, thus, there is a cofinal sequence $(n_k)_{k<\omega}$ in ω such that $A \cap f^{-1}(n_k) \neq \emptyset$ for every $k < \omega$. If $x_k \in A \cap f^{-1}(n_k)$, then $\lim_{k\to\infty} h(x_k) = 0$.

(5) $\Rightarrow$ (6): Fix a point $x_0 \in X$. For each $A \in \tau^*$, let $(x_n^A)_{n<\omega}$ be a sequence in A such that $(h(x_n^A))_{n<\omega}$ converges to 0. We take $y_n^A \in h^{-1}([\frac{1}{n+1}, \frac{1}{n})) \cap \{x_n^A : n < \omega\} = T_n^A$ if $T_n^A \neq \emptyset$, and $y_n^A = x_0$ if $T_n^A = \emptyset$. Let $C_A = \{y_n^A : n < \omega\}$. Since $h(x) > 0$ for every $x \in X$ and $h(x_n^A) \to 0$, then $|C_A| = \aleph_0$. Besides, because of the election of each y_n^A, for each $A_0 \in \tau^*$ and each $n_0 < \omega$, C_{A_0} is not contained in $\bigcup\{y_n^A : A \in \tau^*, k \leq n_0\}$.

If $X \setminus \bigcup_{A\in\tau^*} C_A = K$ is finite, we add the points in K to some C_A. In this case, we define $T = \emptyset$. If, on the contrary, $|K| \geq \aleph_0$, let $\{C_t : t \in T\}$ be a partition of K where $|C_t| = \aleph_0$ for each $t \in T$. Then, the collection $\{C_A : A \in \tau\} \cup \{C_t : t \in T\}$ satisfies the conditions in (6).

(6) $\Rightarrow$ (2): For each $s \in S$ we enumerate C_s: $\{c_n^s : n < \omega\}$ such that $c_n^s \neq c_m^s$ if $n \neq m$. We define, $Y_0 = \{c_0^s : s \in S\}$, and $Y_{n+1} = \bigcup_{0\leq i\leq n} Y_i \cup \{c_{n+1}^s : s \in S\}$. Now, we have that $X = \bigcup_{n<\omega} Y_n$ and $\operatorname{int} Y_n = \emptyset$ for every $n < \omega$. $\square$

Definition 2.3. A space X is *almost-ω-resolvable* if X satisfies one (thus all) of the properties (1)–(6) enlisted in Theorem 2.2. We will call the partition $\{X_n : n < \omega\}$, which satisfies the conditions in (1) of this theorem, an *almost-ω-resolvable partition on* X; and the sequence $\{Y_n : n < \omega\}$ in (2) will be called an *almost-ω-resolvable sequence*.

Observe that every almost-ω-resolvable space is an infinite dense-in-itself space. Besides, it is evident that every ω-resolvable space is almost-ω-resolvable, and this class of spaces is contained in the class of almost resolvable spaces.

Theorem 2.4. *Let X be an almost-ω-resolvable space. Then $C_\square(X)$ is a discrete subspace of $\square\mathbb{R}^X$.*

Proof: Let $f \in C(X)$, and let $\mathcal{X} = \{X_n : n \in \omega\}$ be a partition on X for which each nonempty open set of X intersects $\aleph_0$ elements in $\mathcal{X}$. We take the following open subset of $\square\mathbb{R}^X$: $G = \prod_{x\in X} G_x$, where $G_x = (f(x) - 1/n, f(x) + 1/n)$ if $x \in X_n$. Of course, $f \in G$. Assume that g is also a continuous function from X to $\mathbb{R}$ which belongs to G. Let x_0 be a fix point in X, and let n_0 be a natural number. Since g and f are continuous functions in x_0, there is an open set V of X containing x_0 such that $f(V) \subset (f(x_0)-\frac{1}{n_0}, f(x_0)+\frac{1}{n_0})$ and $g(V) \subset (g(x_0)-\frac{1}{n_0}, g(x_0)+\frac{1}{n_0})$. Because of the definition of $\mathcal{X}$, there is $k \geq n_0$ such that $V \cap X_k \neq \emptyset$. If $y \in V \cap X_k$, we have that $d(f(x_0), g(x_0)) \leq d(f(x_0), f(y)) + d(f(y), g(y)) + d(g(y), g(x_0)) \leq \frac{1}{n_0} + \frac{1}{k} + \frac{1}{n_0} \leq \frac{3}{n_0}$. Since x_0 is fixed and n_0 is arbitrary, we conclude that $f = g$. That is, $G \cap C(X) = \{f\}$. □

The converse of the last result is true if we consider completely regular spaces:

Theorem 2.5. *Let X be a completely regular space. Then X is an almost-ω-resolvable space if and only if $C_\square(X)$ is a discrete subspace of $\square\mathbb{R}^X$.*

Proof: Assume that $C_\square(X)$ is a discrete subspace of $\square\mathbb{R}^X$. So there is an open set $G = \prod_{x\in X} G_x$ in $\square\mathbb{R}^X$ which only contains the continuous function $f \equiv 0$. Let $d : X \to \omega$ be the function defined by $d(x) =$ the least natural number n such that $(\frac{-1}{n}, \frac{1}{n}) \subset G_x$. Let $Y_n = \{x \in X : d(x) \leq n+1\}$.

Assume that there exists a nonempty open set A of X satisfying $A \subset Y_{n_0}$ for an $n_0 \in \omega$. We pick a point x_0 in A. Since X is a completely regular space, there is a continuous function $h : X \to [0, \frac{1}{2(n_0)}]$ such that $h(x_0) = \frac{1}{2(n_0)}$ and $h(y) = 0$ if $y \in X \setminus A$. It happens that $h \in G \cap C(X)$ and, of course, $h \neq f$. This contradiction implies that $\operatorname{int} Y_n = \emptyset$ for every $n < \omega$. So, $\{Y_n : n < \omega\}$ is an almost-ω-resolvable sequence on X. □

We are going to see that almost resolvable spaces, almost ω-resolvable spaces and $C_\square$-discrete spaces are very similar classes of spaces, and, in fact, we will prove that the proposition "every dense-in-itself topological space is almost-ω-resolvable" is consistent with *ZFC*.

3. Basic properties of $C_\square$-discrete and almost ω-resolvable spaces

Since a T_0 topological space is dense-in-itself if and only if its dispersion character is infinite, we obtain:

Proposition 3.1. *Every T_0 dense-in-itself space with cardinality $\aleph_0$ is an almost-ω-resolvable space.*

Proposition 3.2. *If X contains an almost-ω-resolvable (resp., $C_\Box$-discrete) dense subspace, then X is an almost-ω-resolvable (resp., $C_\Box$-discrete) space.*

PROOF: Assume that D is a dense subspace of X, and let $\mathcal{D} = \{D_n : n < \omega\}$ be an almost-ω-resolvable partition of D. The partition of X determined by $\mathcal{X} = \{X \setminus D\} \cup \mathcal{D}$ witnesses that X is almost-ω-resolvable.

Now, assume that D is a dense subset of X and D is $C_\Box$-discrete. Let $f \in C(X)$. There exists an open set $B = \prod_{d\in D} B_d$ of $\Box\mathbb{R}^D$ such that $f|_D$ is the only continuous function contained in B. Then, $\{f\} = G \cap C(X)$ where $G = \prod_{x\in X} G_x$ and $G_x = B_x$ if $x \in D$ and $G_x = \mathbb{R}$ if $x \notin D$. □

Corollary 3.3. *Every T_0 dense-in-itself space X such that* $\min\{|D| : D \subset X,$ *D is dense in X and dense-in-itself* $\} = \aleph_0$, *is an almost-ω-resolvable space. In particular, every T_1 dense-in-itself separable space is almost-ω-resolvable.*

Observe that for every topological space X there is a dense subset of X, $D_X = F_X \cup G_X$, such that every point in F_X (if F_X is not empty) is an isolated point of D_X and G_X is dense-in-itself (G_X can be an empty set). It is possible to prove that every T_0 space X without isolated points for which $|G_X| = \aleph_0$ and $\mathrm{cl}_X F_X \cap \mathrm{cl}_X G_X = \emptyset$, is a $C_\Box$-discrete space. A similar result does not hold for almost-ω-resolvable spaces as the examples in Section 4 testify.

We have already mentioned that every ω-resolvable space is almost-ω-resolvable, thus by Theorems 1.1 and 1.2 we obtain:

Theorem 3.4. *Dense-in-themselves spaces with the following properties are almost-ω-resolvable: locally compact Hausdorff spaces, first countable spaces, Hausdorff k-spaces (in particular, metrizable spaces), countably compact regular T_1 spaces, arc connected spaces, biradial spaces (in particular, linearly orderable topological spaces), homogeneous spaces with a non-trivial convergent sequence, non-countable T_0 $\aleph_0$-bounded groups, and spaces X with a π-network $\mathcal{N}$ such that $|\mathcal{N}| \leq \Delta(X)$ and $|N| \geq \Delta(X)$ for every $N \in \mathcal{N}$.*

We present now some subclasses of almost-ω-resolvable spaces which are not contained in the class of ω-resolvable spaces.

Theorem 3.5. *Let X be a dense-in-itself space. If X is*

(i) *of the first category, or*
(ii) *T_0 and σ-discrete, or*
(iii) *a T_0 $\aleph_0$-bounded topological group,*

then X is an almost-ω-resolvable space.

PROOF: Let X be of the first category. Since $|X| \geq \aleph_0$, we can split X into a sequence $\{X_n : n < \omega\}$ such that $X_n \neq \emptyset$ and $\mathrm{int}_X(\mathrm{cl}_X X_n) = \emptyset$ for each $n < \omega$. Then, $\{X_n : n < \omega\}$ is an almost-ω-resolvable partition on X.

Assume now that $X = \bigcup_{n<\omega} X_n$, where each X_n is a discrete subspace of X. Since X is dense-in-itself and T_0, X must be the union of an infinite countable collection of different discrete subspaces. So, we can assume, without loss of generality, that $X_n \neq X_m$ if $n \neq m$. Let $Z_0 = X_0$, $Z_1 = X_1 \setminus X_0$, ... ,$Z_{n+1} = X_{n+1} \setminus \bigcup_{0\leq i\leq n} X_n, \dots$. Each Z_n is a discrete subspace of X, and if $Z_n = \emptyset$ for every $n > n_0$, then X is the union of a finite collection of discrete subspaces. But this implies that X has an isolated point, which is not possible. Therefore, there exists an increasing sequence $\{n_k : k < \omega\}$ of natural numbers, such that $Z_n \neq \emptyset$ if and only if $n \in \{n_k : k < \omega\}$. We now have that the collection $\{Z_{n_k} : k < \omega\}$ is an almost-ω-resolvable partition on X.

The last assertion of the theorem is a consequence of Proposition 3.1 and Theorem 3.4. □

For each infinite cardinal number κ, there are dense-in-themselves first category and σ-discrete spaces with cardinality κ, which are not resolvable. Indeed, van Douwen constructed a countable irresolvable Hausdorff zero-dimensional space X in [vD] using a maximal independent family of subsets of ω. The free topological sum of κ copies of X is a first category σ-discrete space with cardinality κ and it is not resolvable. Besides, Malykhin in [M1] constructed an extremally disconnected group topology on the countable Boolean group $G = \bigoplus_{n<\omega}\{0,1\}$, in the system [*ZFC* + $P(c)$], which is maximal among regular Hausdorff topologies without isolated points (so, G is an irresolvable $\aleph_0$-bounded topological group).

The following two propositions are easy to prove.

Proposition 3.6. *If $\{X_s : s \in S\}$ is a family of almost-ω-resolvable (resp., $C_\Box$-discrete) spaces, then the free topological sum of the elements of this family, $\bigoplus_{s\in S} X_s$, is almost ω-resolvable (resp., $C_\Box$-discrete).*

Proposition 3.7. *Every open subspace of an almost-ω-resolvable space is almost-ω-resolvable.*

It is not possible to get, in *ZFC*, a proposition like Proposition 3.7 for dense or G_δ closed subsets. Indeed, assume that (T, τ) is a dense-in-itself Tychonoff non almost-ω-resolvable space T (see Theorem 4.16.1 below), then $\beta(T)$, the Stone-Čech compactification of T, is maximally resolvable (Theorem 1.2) (so, it is almost-ω-resolvable) and T is dense in $\beta(T)$. Besides, $Y = T \times \prod_{n<\omega} X_n$, where each X_n is a non-trivial T_1 space with countable pseudo-character, is an almost-ω-resolvable space (see Theorem 3.9 below), and T is homeomorphic to a G_δ closed subset of Y. On the other hand, it is not possible to prove in *ZFC* that $C_\Box$-discreteness is inherited by open subspaces: Consider the space $Z = T \cup \{p\}$ with $p \notin T$, and such that $A \subset Z$ is open if and only if $A \in \tau$ or $p \in A$ and $|X \setminus A| < \aleph_0$. We have that $C(Z)$ is equal to the set of constant functions, and, hence, $C_\Box(Z)$ is discrete. Moreover, T is open in Z and it is not $C_\Box$-discrete.

The proof of the following result follows the pattern of the proof of the Theorem in [CLF] (see also [CGF], Theorem 2.2).

Theorem 3.8. *If X is the union of almost-ω-resolvable (resp., almost resolvable, κ-resolvable) subspaces, then X is almost-ω-resolvable (resp., almost resolvable, κ-resolvable).*

PROOF: We prove the theorem for almost-ω-resolvability. For almost resolvable or κ-resolvable spaces the proof is analogous. Assume that $X = \bigcup_{j\in J} X_j$ where each X_j is almost-ω-resolvable. Let $\mathcal{A} = \{A_s : s \in S\}$ be a maximal family of pairwise disjoint, nonempty almost-ω-resolvable subspaces of X. For $s \in S$, let $\mathcal{C}_s = \{A(s,n) : 0 < n < \omega\}$ be an almost-ω-resolvable partition of A_s. If there is $j \in J$ for which $B = X_j \setminus \mathrm{cl}_X \bigcup \mathcal{A} \neq \emptyset$, then, by Proposition 3.7, B is almost-ω-resolvable and its intersection with each element $A \in \mathcal{A}$ is equal to the empty set. This contradicts the maximality of $\mathcal{A}$; so, $\bigcup \mathcal{A}$ must be dense in X. Now, the family $\{X_n : n < \omega\}$, where $X_0 = X \setminus \bigcup \mathcal{A}$ and $X_n = \bigcup_{s\in S} A(s,n)$ for every $0 < n < \omega$, is an almost-ω-resolvable partition of X. □

If X is a Tychonoff product with an infinite number of non-trivial T_1 factors, then a σ-product in X is dense, has no isolated points and is a first category set. Then, by Proposition 3.2 and Theorem 3.5, X is almost-ω-resolvable. The following theorem and corollary seem not to have been noted before. The maximal resolvability of a product of maximal resolvable spaces is treated in [C] and [CP].

Theorem 3.9. *The Tychonoff product X of an infinite collection $\{X_\lambda : \lambda < \kappa\}$ of non-trivial T_1 topological spaces is, at least, 2^κ-resolvable. In particular, if $\sup_{\lambda<\kappa} |X_\lambda| \leq 2^\kappa$, then X is maximally resolvable.*

PROOF: The space X can be covered by copies of the Cantor cube of weight κ, $C = \{0,1\}^\kappa$, and C is maximally resolvable; that is, C is 2^κ-resolvable. We finish the proof by applying Theorem 3.8.

The last part of the Theorem is obtained because $\Delta(X) \leq (\sup_{\lambda<\kappa} |X_\lambda|)^\kappa$. □

Corollary 3.10. *Every T_2 locally path connected space (in particular, every topological vector space over $\mathbb{R}$) is, at least, 2^ω-resolvable.*

PROOF: If X is a T_2 locally path connected space, then X can be covered by copies of the real line, and $\mathbb{R}$ is 2^ω-resolvable. It remains to refer to Theorem 3.8. □

Corollary 3.11. *Let X be a homogeneous space. If X contains an almost-ω-resolvable subspace, then X is almost-ω-resolvable.*

Proposition 3.12. *Let X and Y be two spaces, and $f : X \to Y$ be an onto function.*

(1) *If $\mathrm{int}\, f(A) \neq \emptyset$ for every nonempty open subset A of X (in particular, if f is open) and Y is almost-ω-resolvable, then X is almost-ω-resolvable.*

(2) *If f is a one-to-one continuous function and X is almost-ω-resolvable (resp., $C_\Box$-discrete), then Y is almost-ω-resolvable (resp., $C_\Box$-discrete).*

Proof: (1) Let $\{Y_n : n < \omega\}$ be an almost-ω-resolvable partition of Y. Then, $\{f^{-1}(Y_n) : n < \omega\}$ is an almost-ω-resolvable partition of X.

(2) If $\{X_n : n < \omega\}$ is an almost-ω-resolvable partition of X. Then, $\{f(X_n) : n < \omega\}$ is an almost-ω-resolvable partition of Y.

Now assume that X is $C_\Box$-discrete. So, $C_\Box(X)$ is a discrete subspace of $\Box\mathbb{R}^X = \Box\mathbb{R}^Y$. Moreover, the function $f^\# : C_\Box(Y) \to C_\Box(X)$ defined by $f^\#(g) = g \circ f$ is an embedding. Then, $C_\Box(Y)$ is discrete in $\Box\mathbb{R}^Y$. □

The following result is a consequence of Proposition 3.12.1.

Corollary 3.13. *Let $\{X_s : s \in S\}$ be a family of topological spaces, and let $X = \prod_{s\in S} X_s$. If for an $s_0 \in S$, X_{s_0} is almost-ω-resolvable, and t is a topology on X contained in the box topology of X, then (X, t) is almost-ω-resolvable.*

Remarks 3.14.

(1) With regard to Proposition 3.12, we give the following counterexamples which prove that this proposition cannot be ameliorated in various senses: Let $Z = [0, 1] \oplus X$ where X is the non almost-ω-resolvable space in Example 4.2 below. Let Y be the quotient space obtained from Z by identifying 0 in $[0, 1]$ with p in X; and let $q : Z \to Y$ be the quotient map. We have that q is a continuous finite-to-one quotient map, Z is not almost-ω-resolvable and Y is almost-ω-resolvable. In addition, the function $g : Z \to [0, 1]$ defined as $g|_{[0,1]} = \text{id}$ and $g|_X \equiv 0$ is continuous and closed. On the other hand, the projection $\pi : X \times \mathbb{Q} \to X$, where $\mathbb{Q}$ is the space of rational numbers with its usual topology, is a continuous open map with countable fibers from an almost-ω-resolvable space (Corollary 3.13) and with values in a space which does not satisfy this property. Finally, let W be a space which is not $C_\Box$-discrete, $V = W \oplus X$ and $f : V \to X$ which sends every point in W to p, and coincides with the identity function in X; then, f is a continuous function from a space which is not $C_\Box$-discrete onto a $C_\Box$-discrete space.

(2) In a similar way as in Theorem 3.8, several results presented in this section can be formulated by almost resolvable spaces and for κ-resolvable spaces. Indeed, observe that Propositions 3.2, 3.6, 3.7, and 3.12, and Corollaries 3.11 and 3.13 remain true if we put "almost resolvable" or "κ-resolvable" instead of "almost-ω-resolvable".

4. Measurable cardinals, almost resolvable spaces and almost-ω-resolvable spaces

As we have already mentioned, every resolvable space X and every almost-ω-resolvable space is almost resolvable. The spaces mentioned in the paragraph following Theorem 3.5 are examples of irresolvable almost-ω-resolvable spaces of cardinality κ, where κ is an arbitrary infinite cardinal number. Now we are going to give examples of almost-ω-resolvable hereditarily irresolvable Tychonoff spaces with an arbitrarily large dispersion character. Also, we are going to see that the existence of a measurable cardinal number provides us with an example of a resolvable $C_{\square}$-discrete space which is not almost-ω-resolvable. A simple modification of this space produces a $C_{\square}$-discrete space which is not even almost resolvable:

Example 4.1. *For each cardinal number κ, there are Tychonoff spaces X which are almost-ω-resolvable, hereditarily irresolvable and $\Delta(X) \geq \kappa$.*

Proof: Let α be a cardinal number such that $\kappa \leq \alpha$ and $\mathrm{cof}(\alpha) = \aleph_0$. Let H, G and τ be the topological groups and the topology in G, respectively, defined in [LF, pages 33 and 34], with $|H| = \alpha$. L. Feng proved there that $(H, \tau|_H)$ is an irresolvable card-homogeneous (every open subset of H has the same cardinality as H) Tychonoff space, and each subset $S \subset H$ with cardinality $< \alpha$ is a nowhere dense subset of H.

Let $(\alpha_n)_{n<\omega}$ be a sequence of cardinal numbers such that $\alpha_n < \alpha_{n+1}$ for every $n < \omega$ and $\sup\{\alpha_n : n < \omega\} = \alpha$. We take subsets H_n of H with the properties $H_n \subset H_{n+1}$ and $|H_n| = \alpha_n$ for each $n < \omega$, and $H = \bigcup_{n<\omega} H_n$. We have that each H_n is nowhere dense in H; so $\{H_n : n < \omega\}$ is an almost-ω-resolvable sequence on H. That is, H is almost-ω-resolvable.

By the Hewitt Decomposition Theorem (see Lemma 4.7 below), there exists a nonempty open subset U of H which is hereditarily irresolvable. Besides, $\Delta(U) = \Delta(H) \geq \kappa$ and it is almost-ω-resolvable (Proposition 3.7). $\square$

Example 4.2. *If there is a measurable cardinal α, then there is a T_0 resolvable $C_{\square}$-discrete Baire space X which is not almost-ω-resolvable and $\Delta(X) = \alpha$.*

Proof: Let α be a non-countable Ulam-measurable cardinal, and let p be a free ultrafilter on α w^+-complete. Let $X = \alpha \cup \{p\}$. We define a topology t for X as follows: $A \in t \setminus \{\emptyset\}$ if and only if $p \in A$ and $A \cap \alpha \in p$. This space is resolvable because α and $\{p\}$ are two disjoint dense subsets, and $\Delta(X) = \alpha$ because the cardinality of each element of p is α. Assume now that $X = \bigcup_{n<\omega} Y_n$, and $Y_n \subset Y_{n+1}$ for every $n < \omega$. Let n_0 be a natural number such that $p \in Y_{n_0}$, and let $n_1 < \omega$ such that $Y_{n_1} \cap \alpha \in p$ (this is because p is ω^+-complete). Thus, if $k \geq \max\{n_0, n_1\}$, then $\mathrm{int}\, Y_k \neq \emptyset$. This means that (X, t) is not almost-ω-resolvable. Furthermore, X is $C_{\square}$-discrete because the only continuous functions from X to $\mathbb{R}$ are constant functions. Finally, X is a Baire space because if $\{U_n : n < \omega\}$ is a sequence of open dense subsets of X, then $U_n \cap \alpha \in p$ for each $n < \omega$. Since p is ω^+-complete, $D = \bigcap_{n<\omega}(U_n \cap \alpha) \in p$. Therefore, D is a dense subset of X. $\square$

Example 4.3. *If there is a measurable cardinal α, then there is a T_1 submaximal $C_\square$-discrete Baire space X which is not almost resolvable and $\Delta(X) = \alpha$.*

Proof: Let α be a non-countable Ulam-measurable cardinal, and let p be a free ultrafilter on α w^+-complete. Let $X = \alpha \cup \{p\}$. We define a topology σ for X as follows: $A \in \sigma \setminus \{\emptyset\}$ if and only if $A \cap \alpha \in p$. This space is not almost resolvable because if $X = \bigcup_{n<\omega} X_n$, at least for a $n_0 \in \omega$, $X_{n_0} \cap \alpha \in p$; that is $\operatorname{int} X_{n_0} \neq \emptyset$. Because of the same argument of the former example, (X, σ) is $C_\square$-discrete, Baire and $\Delta(X) = \alpha$.

Also, observe that a subset D of X is dense in (X, σ) if and only if $D \cap \alpha \in p$. So, D is open. This means that (X, σ) is submaximal. □

Example 4.4. *If there is a measurable cardinal α, then there is a T_1 compact $C_\square$-discrete Baire space Y which is not almost resolvable (in particular, it is irresolvable), and $\Delta(Y) = \alpha$.*

Proof: Let (X, σ) be as in Example 4.3. Now, let $q \notin X$ and let $\mathcal{F} = \{\{q\} \cup A : A \subset \alpha \text{ and } |\alpha \setminus A| < \aleph_0\}$. Let τ be the topology in $Y = X \cup \{q\}$ generated by the base $\sigma \cup \mathcal{F}$. Then, (Y, τ) is compact, T_1 and non almost resolvable. It is not difficult to prove that (Y, τ) is a Baire space and $C_\square(Y)$ coincides with the set of constant functions. □

It is not possible to get a T_2 irresolvable Baire space from measurable cardinals. Indeed, Theorem 3.8 in [KST] states that there are models of *ZFC* with measurable cardinals in which all Baire T_2 spaces are resolvable.

We will prove that the existence of a measurable cardinal number is equiconsistent with the existence of a Tychonoff space which is not $C_\square$-discrete. In order to achieve this goal we need to establish some previous statements.

The following lemma is Fact 1.13 in [vD].

Lemma 4.5. *A dense-in-itself space X is open-hereditarily irresolvable if and only if for every $A \subset X$, $\operatorname{int} A = \emptyset$ implies $\operatorname{int}(\operatorname{cl} A) = \emptyset$.*

Then we conclude:

Proposition 4.6. *In the class of open-hereditarily irresolvable (completely regular) spaces, almost resolvability and almost-ω-resolvability (and $C_\square$-discreteness) coincide.*

Proof: These are consequences of Lemma 4.5 and Theorems 2.5 and 3.5. □

We are going to prove that every Baire irresolvable space is not almost-ω-resolvable by using the following two lemmas. (The referee pointed out to the authors that the proof of Corollary 4.9 below, was already given in [Ma].)

Lemma 4.7 ([H]). *Every dense-in-itself topological space X is the union of a resolvable subspace X_1, and a hereditarily irresolvable open subspace X_2, and $X_1 \cap X_2 = \emptyset$.*

Lemma 4.8 ([FL]). *For a dense-in-itself space X the following statements are equivalent.*

(1) *X is an almost resolvable space.*
(2) *$X = X_1 \cup X_2$ where X_1 is closed (with a nonempty interior if $X_1 \neq \emptyset$) and it is resolvable, and X_2 is a first category open set.*
(3) *There is a function $f : X \to \omega$ such that $D(f) = X$.*

Corollary 4.9. *If X is a Baire irresolvable space, then X is not almost resolvable; in particular, X is not almost-ω-resolvable. Besides, if X is completely regular, $C_\Box(X)$ is not discrete in $\Box\mathbb{R}^X$.*

Proof: If X is a Baire irresolvable space, then, by Lemma 4.7, it contains a nonempty open hereditarily irresolvable subspace Y, and Y is a Baire space. If X is almost resolvable, then Y is almost resolvable. But this means that Y is a set of the first category (Lemma 4.8). This is a contradiction. □

Even more, it is consistent with *ZFC* that every Baire dense-in-itself space is resolvable (see Theorem 4.11 below). Indeed:

Proposition 4.10. *Assume $V = L$. Then every space without isolated points is almost resolvable.*

Proof: Proposition 3.9 in [KST] says that $V = L$ implies that every space of regular cardinality without isolated points is almost resolvable.

Now, let X be a space without isolated points such that $|X| = \kappa$ is a singular cardinal. κ^+ is regular and $Z = \bigoplus_{\lambda<\kappa^+} X_\lambda$, with $X_\lambda = X$ for every $\lambda < \kappa^+$, is a topological space with regular cardinality and without isolated points. So, since we are assuming that $V = L$, Z is almost resolvable; that is Z can be written as $\bigcup_{n<\omega} J_n$ where $\operatorname{int}_Z J_n = \emptyset$. Then $X_0 = \bigcup_{n<\omega} L_n$ where $L_n = X_0 \cap J_n$ for every $n < \omega$. We have that $\operatorname{int}_{X_0} L_n = \emptyset$ and X_0 is homeomorphic to X, so X is almost resolvable. □

Then, by Corollary 4.9 and Proposition 4.10, we obtain:

Theorem 4.11. *Assume $V = L$. Then every Baire space (in particular, every Tychonoff pseudocompact space) without isolated points is resolvable.*

The authors of [A] prove their Theorem 3.1 assuming that every space is Hausdorff. By a slight modification of that proof we obtain the same result without assuming any separation axiom:

Lemma 4.12. *The following assertions are equivalent in ZFC.*

(1) *There is a Baire irresolvable space.*
(2) *There is a maximal space which is not σ-discrete.*
(3) *There is a submaximal space which is not σ-discrete.*

We remind the reader that a dense-in-itself Tychonoff space (X, t) is *maximal Tychonoff* if for every Tychonoff topology t' on X with $t \subset t'$ and $t \neq t'$, (X, t') has an isolated point.

Remarks 4.13.

(1) If $\mathcal{C}$ is a chain of Tychonoff topologies on a set X, then the topology τ generated by $\bigcup \mathcal{C}$ is a Tychonoff topology in X; if in addition, (X, t) is dense-in-itself for every $t \in \mathcal{C}$, so is (X, τ).

(2) Every submaximal space is T_1. Moreover, every σ-discrete T_0 space is almost-ω-resolvable (Theorem 3.5), and every maximal Tychonoff space is hereditarily irresolvable ([vD, Theorem 1.8 and Fact 1.11]).

Theorem 4.14. *The following assertions are equivalent in ZFC.*

(1) *There is a Baire irresolvable (Tychonoff) space.*

(2) *There is a maximal (maximal Tychonoff) space which is not almost resolvable.*

(3) *There is a submaximal space which is not almost resolvable.*

(4) *There is a maximal (maximal Tychonoff) space which is not almost-ω-resolvable.*

(5) *There is a submaximal space which is not almost-ω-resolvable.*

Proof: (1) $\Rightarrow$ (2): Let (X, t) be a Baire irresolvable (Tychonoff) space. Because of Corollary 4.9, X is not almost resolvable. Let (X, t') be a maximal (maximal Tychonoff) topology containing t (The existence of t' is guaranteed by Remark 4.13.1 and Zorn Lemma). Assume that (X, t') is almost resolvable. It happens that the direct image of an almost resolvable space under a continuous bijective function is again almost resolvable; so, (X, t) must be almost resolvable, which is a contradiction by Corollary 4.9. So, (X, t') is not almost resolvable.

(2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5): Every maximal space is submaximal.

(2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5): Every almost-ω-resolvable space is almost resolvable.

(5) $\Rightarrow$ (1): The statement in (5) implies that there is a submaximal space which is not σ-discrete (Remark 4.13.2), and this assertion implies that there is a Baire irresolvable space (Lemma 4.12).

Tychonoff case of (4) $\Rightarrow$ Tychonoff case of (1): Assume that (X, t) is a maximal Tychonoff space which is not almost-ω-resolvable. Let $\mathcal{S} = \{U \in t : U \neq \emptyset$ and U is almost-ω-resolvable $\}$. Using Theorem 3.8 and Proposition 3.2, we obtain that $Y = \operatorname{cl} \bigcup \mathcal{S}$ is almost-ω-resolvable. Therefore, $Z = X \setminus Y$ is an open dense-in-itself nonempty subspace of X which has no open almost-ω-resolvable subsets. Since X is a maximal Tychonoff space, it is hereditarily irresolvable (Remark 4.13.2), hence Z is irresolvable. Besides, Z is a Baire space: indeed, if U_n is an open dense subset of Z for every $n \in \omega$, then $T = \bigcup_{n<\omega}(Z \setminus U_n)$ is of the first category, hence, it is almost-ω-resolvable; so T cannot contain a nonempty open subset of Z. This means that $\bigcap\{U_n : n < \omega\}$ is dense in Z, and hence Z is a Baire space. □

In the article of Kunen, Szymanski and Tall [KST], Theorem 3.3, they established the following result which contrasts with Theorem 4.11.

Theorem 4.15. *ZFC is consistent with the existence of a measurable cardinal if and only if ZFC is consistent with the existence of an irresolvable (zero-dimensional, Tychonoff) Baire space.*

As a consequence of Theorems 4.14 and 4.15 we obtain:

Theorem 4.16.

(1) *ZFC is consistent with the existence of a measurable cardinal if and only if ZFC is consistent with the existence of a Tychonoff dense-in-itself space X for which $C_\square(X)$ is not discrete in $\square\mathbb{R}^X$.*
(2) *ZFC is consistent with "every space X without isolated points is almost-ω-resolvable and $C_\square(X)$ is a discrete subspace of $\square\mathbb{R}^X$".*
(3) *Con (ZFC) $\rightarrow$ Con (ZFC + for every T_1 or regular space X, $C_\square(X)$ is a discrete subspace of $\square\mathbb{R}^X$ if and only if X does not have isolated points).*

Proof: (1) This is a consequence of Theorems 4.14 and 4.15.

(2) In the system $[ZFC + V = L]$ every maximal space X is almost-ω-resolvable because of Theorems 4.11 and 4.14. Let (X, t) be a topological space (dense-in-itself). Then, we can consider a maximal topology t' of X such that $t \subset t'$. So, (X, t') is almost-ω-resolvable; then (X, t) is also almost-ω-resolvable, because id : $(X, t') \rightarrow (X, t)$ is bijective and continuous (Proposition 3.12). Moreover, every almost ω-resolvable space is $C_\square$-discrete.

(3) Here we need to apply (2) and Proposition 2.1. □

Examples 4.1, 4.2, 4.3 and 4.4 and this last theorem gives us a good picture of what we can expect about almost-ω-resolvable spaces when we assume the existence of measurable cardinals or $V = L$. We finish this section by giving, in Theorem 4.19, one more detail of this panorama.

The following lemma was proved in [FL].

Lemma 4.17. *Suppose X is a dense-in-itself topological space. Then X can be written as the union of mutually disjoint subsets F, B and N (resp., R, I, and M) such that F, B, R and I are open, N and M are nowhere dense, F is of the first category, R is resolvable, B is Baire or $B = \emptyset$, and I is irresolvable or $I = \emptyset$.*

Corollary 4.18. *A dense-in-itself topological space X is almost resolvable if and only if $X = X_0 \cup X_1 \cup X_2 = Y_0 \cup Y_1$, where X_0 is open and of the first category, X_1 is open, Baire and resolvable or $X_1 = \emptyset$, X_2 is a nowhere dense set, Y_0 is of the first category and Y_1 is open Baire and resolvable or $Y_1 = \emptyset$. Besides, X_0, X_1 and X_2 (resp., Y_0 and Y_1) are mutually disjoint.*

Proof: Assume that X is almost resolvable and put $X = F \cup B \cup N = R \cup I \cup M$ as in Lemma 4.16.

We have that $B \cap I$ is empty or is an open Baire almost resolvable space. By Corollary 4.9, if $B \cap I \neq \emptyset$, then $B \cap I$ must be resolvable.

Therefore $X = F \cup (B \cap R) \cup (B \cap I) \cup (B \cap M) \cup N$. The set $(B \cap R) \cup (B \cap I)$ is empty or is open, Baire and resolvable; and the set $(B \cap M) \cup N$ is nowhere dense, so $F \cup (B \cap M) \cup N$ is of the first category. Take $X_0 = F$, $X_1 = (B \cap R) \cup (B \cap I)$, $X_2 = (B \cap M) \cup N$, $Y_0 = X_0 \cup X_2$ and $Y_1 = X_1$.

The converse is a consequence of Lemma 4.8. □

Theorem 4.19. *The following properties are equivalent.*

(1) *There exists a resolvable Baire (Tychonoff) space which is not almost-ω-resolvable.*
(2) *There exists a resolvable (Tychonoff) space which is not almost-ω-resolvable.*
(3) *There exists an irresolvable almost resolvable (Tychonoff) space which is not almost-ω-resolvable.*
(4) *There exists an almost resolvable (Tychonoff) space which is not almost-ω-resolvable.*

Proof: (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are trivial.

(2) $\Rightarrow$ (3): Assume that X is resolvable (Tychonoff) and it is not almost-ω-resolvable. Let Y be a countable irresolvable Tychonoff space (see for example [vD]), then $X \oplus Y$ is irresolvable (Tychonoff) and it is not almost-ω-resolvable. Besides, since X and Y are almost resolvable (X is resolvable and Y is countable), then $X \oplus Y$ is almost resolvable.

(4) $\Rightarrow$ (2): In fact, let X be an almost resolvable (Tychonoff) space which is not almost-ω-resolvable. By Lemma 4.8, $X = X_1 \cup X_2$, where X_1 is resolvable and X_2 is of the first category. If X_1 were empty or almost-ω-resolvable, then X would be almost-ω-resolvable. Therefore, X_1 is a resolvable non almost-ω-resolvable (Tychonoff) space.

(3) $\Rightarrow$ (1): Because of Corollary 4.18, the irresolvable almost resolvable and non almost-ω-resolvable (Tychonoff) space X can be written as the union of two disjoint subspaces Y_0 and Y_1, where Y_0 is of the first category and Y_1 is open, Baire and resolvable. Since X is not almost-ω-resolvable, then Y_1 is a nonempty Baire resolvable (Tychonoff) space which is not almost-ω-resolvable. □

By Example 4.2, if there is a measurable cardinal, then each of the equivalent conditions in Theorem 4.19 holds.

5. Questions

The results obtained in Section 4 produce several questions:

Questions 5.1.

(1) Is the existence of a measurable cardinal a consequence of the existence of a Tychonoff space without isolated points which is not almost-ω-resolvable?

(2) Is it possible to construct, assuming the existence of a measurable cardinal, a regular (or a completely regular) space without isolated points which is not almost-ω-resolvable?

Questions 5.2.

(1) Are almost resolvability and almost-ω-resolvability equivalent in the class of irresolvable spaces?
(2) Is every Baire irresolvable space $C_\square$-discrete?
(3) Is the existence of an irresolvable Baire space equivalent to the existence of a maximal non $C_\square$-discrete space?
(4) Does *Con* (*ZFC*) imply *Con* (*ZFC* + every dense-in-itself space is $C_\square$-discrete + there exists a dense-in-itself non almost-ω-resolvable space)?
(5) Is every non almost resolvable irresolvable space a Baire space?
(6) Is every maximal space $C_\square$-discrete?

W.W. Comfort and S. García-Ferreira asked in [CGF] whether every Tychonoff pseudocompact space without isolated points is resolvable. Theorem 4.11 implies that there is a model of *ZFC* in which the answer to this question is in the affirmative. Besides, Malykhin asked if every Lindelöf Tychonoff space without isolated points with $\Delta(X) > \omega$ was resolvable [CGF]. This leads us to the following problems:

Questions 5.3.

(1) Is every pseudocompact Tychonoff dense-in-itself space almost-ω-resolvable?
(2) Is every Lindelöf Tychonoff dense-in-itself space almost-ω-resolvable?

The following problem is motivated by Corollary 3.10, Examples 4.2, 4.3 and 4.4, and Question 8.13 in [CGF].

Question 5.4. Is every connected Tychonoff dense-in-itself space almost-ω-resolvable?

Of course, the more general question regarding topological groups is:

Question 5.5. Is every dense-in-itself topological group almost-ω-resolvable?

In [CM] it is proved that every Baire topological group is resolvable, thus a more restricted but still reasonable question is:

Question 5.6. Is every dense-in-itself Baire topological group almost-ω-resolvable?

Theorem 3.9 and Corollary 3.10 produce the following:

Questions 5.7.

(1) Is every Tychonoff product of an infinite collection of non-trivial T_1 topological spaces a maximally resolvable space?
(2) Is every arc connected T_2 space maximally resolvable?

We finish this list of problems with the following natural questions:

Questions 5.8.

(1) (O. Okunev) Let $\{\tau_s : s \in S\}$ be a chain of almost-ω-resolvable topologies on X. Is, then, the topology generated by $\bigcup_{s\in S} \tau_s$ almost-ω-resolvable?
(2) Is X almost-ω-resolvable if X^2 satisfies this property?

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Departamento de Matemáticas, Facultad de Ciencias, Universidad Nacional Autónoma de México, Ciudad Universitaria, Mexico D.F., 04510, México

E-mail: atamariz@servidor.unam.mx

Escuela de Ciencias Físico Matemáticas, Universidad Autónoma de Sinaloa, Ciudad Universitaria, Ave. de las Américas Culiacan, Sinaloa, 80010 México

E-mail: vrhi@uas.uasnet.mx

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