Lukáš Krump

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# Construction of BGG sequences for AHS structures

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*Abstract.* This paper gives a description of a method of direct construction of the BGG sequences of invariant operators on manifolds with AHS structures on the base of representation theoretical data of the Lie algebra defining the AHS structure. Several examples of the method are shown.

*Keywords:* Hermitian symmetric spaces, standard operators, BGG sequence, Hasse diagram, weight graph

*Classification:* 22E46, 43A85, 53A55, 53C30

## 1. Introduction

Invariant operators on manifolds have been studied recently by many authors. The basic and best understood case is that of conformally invariant operators, studied by Baston, Branson, Eastwood, Fegan, Jakobsen, Slovák, Wünsch (see [2], [5], [12], [13], [14], [20], [30]) and others. A broader concept of so-called AHS structures generalizing the conformal structure was introduced and studied by Baston, Gindikin, Goncharov, Čap, Slovák, Souček and others (see [1], [17], [18], [9], [10], [11]).

It turned out that there exists a class of so-called standard operators for AHS structures on a manifold that can be described in a constructive way. This construction is described in [11]. A natural question is a systematization of standard operators. It is known (see e.g. [1]) that one can consider sequences of operators called BGG (Bernstein-Gelfand-Gelfand) sequences. These sequences are studied by many authors (originally [3], [29]) and it turns out that they contain a lot of information. A standard way of computing a BGG sequence uses iterated action of the Weyl group. The present paper shows basic ideas of constructing the BGG sequence all at once, directly from the so-called weight graph of the positive part $\mathfrak{g}_1$ of the $|1|$-grading. This gives in fact no new information about particular sequences that are known, but it rather shows a surprising connection between complexes of operators (which is a geometrical notion) and purely representation theoretical properties of certain Lie algebra.

For technical reasons, the method is developed for the cases of AHS structures for which all the weights of $\mathfrak{g}_1$ are extremal. This excludes the odd conformal and symplectic structures, where certain technical modifications are necessary.

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**1.1 Definitions.** Consider a complex simple Lie group $G$ with a Lie algebra $\mathfrak{g}$ which is $|1|$-graded, i.e.

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. There is a parabolic subgroup $P$ of $G$ corresponding to the algebra $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ (see e.g. [23]).

Throughout this paper, we consider always complex Lie algebras. In the real case, more complicated structures occur that should be dealt separately. A $|1|$-graded Lie algebra has several important properties, especially:

(1) $\mathfrak{g}_0 = \mathfrak{g}_0^s \oplus \mathbb{C}E$, where $\mathfrak{g}_0^s$ is semisimple and $\mathbb{C}E$ is a one-dimensional center generated by an element $E$ characterized by the property that it acts on every $\mathfrak{g}_i$ by the multiplication by $i$,

(2) $\mathfrak{g}_{\pm 1}$ are mutually dual irreducible representations of $\mathfrak{g}_0$.

If $V$ is an irreducible representation of $\mathfrak{g}_0$, its highest weight will be denoted $(\lambda, w)$ where $\lambda$ is the highest weight of $V$ as a representation of $\mathfrak{g}_0^s$ and the eigenvalue $w \in \mathbb{C}$ of the action of $E$ on $V$ is the so-called **generalized conformal weight**. Then write $V = V(\lambda, w)$.

An **AHS (almost Hermitian symmetric) structure** on a manifold $M$ is given by a principal bundle $\mathcal{G}$ on $M$ with structure group $P$ together with a normal Cartan connection $\omega$ on $\mathcal{G}$.

In [11], a construction of a broad class of invariant differential operators

$$D : \Gamma(M, \mathcal{V}(\lambda, w)) \to \Gamma(M, \mathcal{V}(\lambda', w'))$$

on a manifold $M$ with AHS-structure is described. More precisely, let $\Gamma(M, \mathcal{V}(\lambda, w))$ denote the space of sections of the associated bundle $\mathcal{V}(\lambda, w) = \mathcal{G} \times_P V(\lambda, w)$; then it is shown that if $\lambda$ and $\lambda' = \lambda + k\beta$ are dominant weights for $\mathfrak{g}_0^s$, where $k$ (the order of the operator) is a positive integer and $\beta$ is a weight of the representation $\mathfrak{g}_1$ of $\mathfrak{g}_0^s$, then there exists a value $w$ (see Theorem 2.4) of the generalized conformal weight such that there exists a so-called standard invariant operator $D : \Gamma(M, \mathcal{V}(\lambda, w)) \to \Gamma(M, \mathcal{V}(\lambda', w'))$, $w' = w + k$. These operators correspond to projections $\pi : \otimes^k \mathfrak{g}_1 \otimes V_\lambda \to V_{\lambda'}$.

This construction is independent of the manifold $M$ and what really matters in the classification of operators are the representation spaces $V(\lambda, w)$. That is why we only write arrows

$$D : (\lambda, w) \to (\lambda', w')$$

to denote standard operators. If $\lambda' = \lambda + k\beta$, $w' = w + k$, denote this arrow $D = D(\beta, k) = D(\beta, k, \lambda)$.

For a given Lie algebra $\mathfrak{g}$, consider the set of all invariant differential operators — in our notation

$$\begin{aligned}\{D(\beta, k, \lambda) : (\lambda, w) \to (\lambda + k\beta, w + k);\\ \beta \text{ is a weight of } \mathfrak{g}_1, k \in \mathbb{N},\\ \lambda, \lambda + k\beta \text{ are dominant weights for } \mathfrak{g}_0^s\}.\end{aligned}$$

This set may be considered as a graph whose vertices are couples $(\lambda, w)$ and arrows are operators $D(\beta, k, \lambda)$. It is known (see [1]) and will be shown here again that this graph decomposes into connected components, the most of which, so called regular ones, have the same underlying graph structure, which will be called the **Hasse diagram**. These components are called **BGG sequences** and every one is characterized by the weight $\lambda^0$ of the initial vertex $(\lambda^0, w^0)$ and the order $k^0$ of the initial operator (see Theorem 3.11). The singular components are not considered in this paper.

The main aim of the paper is to use the information contained in the weight structure of $\mathfrak{g}_1$ for definition of Hasse diagram and to show that the BGG sequences are isomorphic graphs ("have the same shape") to the Hasse diagram.

## 2. Computations

Let $\mathfrak{g}$ be a $|1|$-graded Lie algebra with the property that all weights of $\mathfrak{g}_1$ are extremal. It follows from the classification of $|1|$-graded Lie algebras that then all roots of $\mathfrak{g}$ are of the type $e_i \pm e_j$ and thus have the same length.

**2.1 Inner products.** Denote by $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{g}$, by $\mathfrak{h}^*$ its dual. For $\mathfrak{g}$ $|1|$-graded, denote by $\mathfrak{h}^0$ the Cartan subalgebra of $\mathfrak{g}_0^s$, then obviously $\mathfrak{h}^0 \subset \mathfrak{h}$ and it has codimension one in $\mathfrak{h}$. We define an inclusion of dual spaces $\mathfrak{h}^{0*} \subset \mathfrak{h}^*$ by the condition $\lambda \in \mathfrak{h}^{0*} \mapsto \tilde{\lambda} \in \mathfrak{h}^*$, where $\tilde{\lambda} = \lambda$ on $\mathfrak{h}^{0*}$ and 0 on $\mathbb{C}E$.

All invariant inner products on a Lie algebra are multiples of the Killing form $B(.,.)$. We introduce two new invariant inner products on $\mathfrak{g}$. The first one will be denoted by $(.,.)$ and it is defined so that $(E, E) = 1$. More exactly,

$$(X, Y) = \frac{B(X, Y)}{B(E, E)}$$

for all $X, Y \in \mathfrak{g}$. This definition restricts onto $\mathfrak{h}$ and it defines an inner product on the dual space $\mathfrak{h}^*$, where we have dually

$$(\alpha, \beta) = B(E, E)B(\alpha, \beta)$$

for all $\alpha, \beta \in \mathfrak{h}^*$. This inner product is then defined on $\mathfrak{h}^{0*}$ by restriction and is also denoted here by $(.,.)$.

The norm on $\mathfrak{h}^*$ and also on $\mathfrak{h}^{0*}$ will be denoted

$$|\alpha|^2 = (\alpha, \alpha).$$

The other inner product on $\mathfrak{g}$ will be denoted by $\langle .,. \rangle$ and it is defined by the condition that, for the dual product on $\mathfrak{h}^*$, all roots in this product have length 2. It is known that then for every root $\alpha$ and every weight $\beta$, the product $\langle \alpha, \beta \rangle$ is an integer.

We have

$$(X, Y) = \frac{\langle X, Y \rangle}{\langle E, E \rangle}$$

for all $X, Y \in \mathfrak{g}$ and
$$(\alpha, \beta) = \langle E, E\rangle\langle \alpha, \beta\rangle$$
for all $\alpha, \beta \in \mathfrak{h}^*$.

From now on, a weight will mean a weight from the weight lattice for $\mathfrak{g}_0^s$, if not specified otherwise.

**Definition 2.1.** *For a fixed representation $V$ of $\mathfrak{g}_0^s$ and for any its weight $\beta$, define the number*
$$r_\beta = \frac{|\beta|^2 + 1}{2}\,.$$
*If all weights of $V$ are extremal (have the same length), then denote by $r = r_V$ the common value of $r_\beta$.*

**Lemma 2.2.** *Let $\mathfrak{g}$ be $|1|$-graded, suppose that all weights of the representation $\mathfrak{g}_1$ of $\mathfrak{g}_0^s$ are extremal. Then $r = r_{\mathfrak{g}_1} = \langle E, E\rangle$ and therefore $(.,.) = r\langle .,.\rangle$ on $h^*$.*

Proof: It is known (see e.g. [23]) that if $\theta$ is the highest root of $\mathfrak{g}$, then $\theta(E) = 1$ and the highest weight $\beta = \beta_{max}$ of $\mathfrak{g}_1$ is equal to the orthogonal projection of $\theta$ to $\mathfrak{h}^{0*}$. Denote $\gamma = \theta - \beta$. Since $\gamma$ is orthogonal to $\beta$, we have $|\theta|^2 = |\beta|^2 + |\gamma|^2$. $\theta$ is a root, hence $\langle\theta, \theta\rangle = 2$. Therefore
$$2\langle E, E\rangle = |\beta|^2 + |\gamma|^2.$$

Now it is enough to show that $|\gamma|^2 = 1$. We know that $\gamma$ is perpendicular to $\mathfrak{h}^{0*}$, $E$ is perpendicular to $\mathfrak{h}^0$ (all with respect to the Killing form) and that $\gamma(E) = \theta(E) = 1$, i.e. $\gamma$ and $E$ are dual elements of dual bases of $\mathfrak{h}^*$ and $\mathfrak{h}$, respectively. Therefore $(\gamma, \gamma) = B(\gamma, \gamma)B(E, E) = 1$. $\square$

This will allow us to use both inner products $(.,.)$ (that appears naturally in the value of the conformal weight) and $\langle .,.\rangle$ (whose advantage is that it gives integral results for a root and a weight).

**Notation 2.3.** *The weight $\delta$ is defined as half the sum of all positive roots, it equals also to the sum of the fundamental weights. Therefore if $\alpha_i$ is a simple root, then $\langle\delta, \alpha_i\rangle = 1$.*

The following theorem is the Corollary 5.3 of [11].

**Theorem 2.4.** *Let $V_\lambda$ be an irreducible representation of $\mathfrak{g}_0^s$ and let $\beta_{max}$ be the highest weight of the representation $\mathfrak{g}_1$. Suppose that an extremal weight $\beta$ of $\mathfrak{g}_1$ and a positive integer is chosen in such a way that $\lambda + k\beta$ is dominant. Let $\pi : \otimes^k\mathfrak{g}_1 \otimes V_\lambda \to V_{\lambda'}$ be the projection onto the unique irreducible component of the product with highest weight $\lambda'$.*

*Then there is a unique value for the generalized conformal weight $w$ such that $\pi$ defines an invariant operator $D(\beta, k) : (\lambda, w) \to (\lambda', w + k)$. The value of the generalized conformal weight is given by*
$$w = w_{\lambda,\beta} - kr,$$

*where*

$$w_{\lambda,\beta} = (\beta_{max}, \delta) - (\lambda + \delta, \beta) + r. \tag{2.1}$$

From now on, we extend the notation — for any weights $\lambda$, $\lambda'$ (i.e. not only for dominant ones) write $D(\beta, k) : (\lambda, w) \to (\lambda', w + k)$ if $\lambda' = \lambda + k\beta$, $w = w_{\lambda,\beta} - kr$ and $w' = w + k$.

We prove now two auxiliary lemmas that will be used later in the proof of Theorem 3.11.

**Lemma 2.5.** *If $\beta^1 \le \beta^2$, then $w_{\lambda,\beta^1} \ge w_{\lambda,\beta^2}$ for any weight $\lambda$.*

PROOF: It follows directly from the formula (2.1). □

**Lemma 2.6.** *If $(\lambda^1, w^1) \xrightarrow{\beta^1, k^1} (\lambda^2, w^2)$, then $w^2 = w_{\lambda^2,\beta^1} + k^1 r$.*

PROOF:

$$\begin{aligned} w^2 &= w^1 + k^1 = (\beta_{max}, \delta) - (\lambda^1 + \delta, \beta^1) + r - k^1 r + k^1 = \\ &= (\beta_{max}, \delta) - (\lambda^2 + \delta, \beta^1) + r + k^1(1 - r + (\beta^1, \beta^1)) = \\ &= w_{\lambda^2,\beta^1} + k^1 r, \end{aligned}$$

since $1 - r + (\beta^1, \beta^1) = r$. □

Let us look for the conditions saying when two operators can be composed. Suppose that there are two operators, the image space of the first one being the source space of the second one (with $\lambda^i$ not necessarily dominant):

$$(\lambda^1, w^1) \xrightarrow{D(\beta^1,k^1,\lambda^1)} (\lambda^2, w^2) \xrightarrow{D(\beta^2,k^2,\lambda^2)} (\lambda^3, w^3). \tag{2.2}$$

**Lemma 2.7.** *Two operators can be composed as in the situation* (2.2) *if and only if*

$$(\lambda^2 + \delta, \beta^1 - \beta^2) = (k^1 + k^2)r. \tag{2.3}$$

PROOF: We know by Theorem 2.4:

$$w^1 = w_{\lambda^1,\beta^1} - k^1 r,$$
$$\lambda^2 = \lambda^1 + k^1\beta^1$$

and

$$w^2 = w_{\lambda^2,\beta^2} - k^2 r.$$

The two operators can be composed if and only if

$$w^2 = w^1 + k^1. \tag{2.4}$$

Compute

$$\begin{aligned} w_{\lambda^1,\beta^1} - k^1 r + k^1 &= w_{\lambda^2,\beta^2} - k^2 r \\ (\lambda^2 - k^1\beta^1 + \delta, \beta^1) + k^1(r - 1) &= (\lambda^2 + \delta, \beta^2) + k^2 r \\ (\lambda^2 + \delta, \beta^1 - \beta^2) &= -k^1(r - 1 - (\beta^1, \beta^1)) + k^2 r. \end{aligned}$$

But $r - 1 - (\beta^1, \beta^1) = -r$ and so we have proved the lemma. □

**Remark 2.8.** We in fact proved the equivalence between the equalities 2.4 and 2.3 without the assumption that any of $\lambda^i$ is dominant (i.e. that any of these arrows really defines an operator). This will be needed in the proof of Theorem 3.11 when the dominance of $\lambda^3$ will be to be proved.

**Corollary 2.9.** *A version of the formula* (2.3) *from the preceding lemma expressed using the inner product* $\langle .,. \rangle$*:*

$$\langle \lambda^2 + \delta, \beta^1 - \beta^2 \rangle = k^1 + k^2. \tag{2.5}$$

**Definition 2.10.** *Define the relation* $\leq$ *on the weight lattice. If* $\beta^1$, $\beta^2$ *are two weights, write* $\beta^1 \leq \beta^2$ *if and only if there exists an element of the positive root lattice* $\alpha$ *such that* $\beta^1 + \alpha = \beta^2$. *This defines the standard (partial) ordering of the weight lattice.*

*Weights* $\beta^1$, $\beta^2$ *are incomparable if neither* $\beta^1 \leq \beta^2$ *nor* $\beta^2 \leq \beta^1$*, then denote* $\beta^1 \not\approx \beta^2$.

**Corollary 2.11.** *If two operators are composed as in the situation* (2.2)*, then* $\beta_2 \leq \beta_1$ *or* $\beta^1 \not\approx \beta^2$.

Proof: For $\beta_1 \leq \beta_2$ is $(\lambda^2 + \delta, \beta^1 - \beta^2) \leq 0$ since $\lambda + \delta$ is a dominant weight, but $k^1$, $k^2$, $r$ are all positive. □

Corollary 2.11 implies that in the situation (2.2), the element $\alpha = \beta^1 - \beta^2$ of the root lattice is not negative (it may be positive or a combination of positive and negative simple roots). This is an important property that gives arise to the following method giving the general rule of constructing BGG sequences directly from the information contained in the weight graph of the representation $\mathfrak{g}_1$.

## 3. Construction of the BGG sequence

### 3.1 Weight graphs and Hasse diagrams.

**Definition 3.1.** *If* $B$ *is a set, then a* **graph with** $B$**-labeled arrows** *or a* **graph labeled by** $B$ *or just a* $B$**-graph** *is a finite oriented graph* $G = (V, A)$ *with no cycles. Equivalently, it is a set of vertices* $V$ *such that the set of all arrows* $A \subset V \times V$*, if we denote* $u \geq v \iff u \to v$*, generates a partial ordering on* $V$*. Moreover, there is a mapping* $\psi : A \longrightarrow B$.

*If* $u, v \in V, a = (u, v) \in A$ *and* $\psi(a) = b \in B$*, then write* $u \xrightarrow{b} v$.

If the set $B$ is known, we often call a $B$-graph simply a graph.

**Definition 3.2.** *Let* $\mathfrak{g}$ *be a semisimple Lie algebra and* $\rho$ *its irreducible representation with highest weight* $\beta_{max}$*. The* **weight graph** *of the representation* $\rho$ *is the following graph* $W$ *labeled by the set of simple roots of* $\mathfrak{g}$*:*

- *the set of vertices is the set of all weights of* $\rho$*,*
- *there is an arrow* $\beta_1 \to \beta_2$ *labeled by* $\alpha_i$ *if and only if there exists a simple root* $\alpha_i$ *of* $\mathfrak{g}_0^s$ *such that* $\beta_2 = \beta_1 - \alpha_i$.

The partial ordering generated in the weight graph corresponds to the standard ordering of the weight lattice, and the highest weight is its greatest element.

The weight graph of a given representation $\rho$ corresponds to the usual construction of the weights of the representation when the highest weight $\beta_{max}$ is known. If $\beta$ is a weight, find all simple roots $\alpha_i$ such that the integer $m = 2\frac{(\beta,\alpha_i)}{(\alpha_i,\alpha_i)}$ is positive. Then for every $k = 1, \ldots, m$, the element $\beta_k = \beta - k\alpha_i$ is also a weight of $\rho$ and there is an arrow from $\beta_{k-1}$ to $\beta_k$ labeled by $\alpha_i$ and there is always $(\beta_{k-1}, \alpha_i) - (\beta_k, \alpha_i) = (\alpha_i, \alpha_i)$. Doing the same for all weights that occur we get the whole weight graph.

If all weights of the representation are extremal, then the number $m$, if positive, may take only the value 1.

**Definition 3.3.** *A subgraph $V$ of the weight graph $W$ of a representation $\rho$ is called* **acceptable** *if the following condition is satisfied: whenever $\gamma$ is a vertex of $V$ then every vertex $\beta$, such that there exists an arrow that starts in $\beta$ and ends in $\gamma$, is also contained in $V$.*

*Equivalently, an acceptable subgraph regarded as a partially ordered set contains with every $\gamma$ all elements $\beta$ such that $\beta > \gamma$.*

For this definition, see also [22].

**Definition 3.4.** *Let $W$ be a weight graph of a representation $\rho$ of a Lie algebra $\mathfrak{g}$. Then the* **Hasse diagram** *for $\rho$ is a graph labeled by the vertices of the weight graph, defined by:*

- *vertices are acceptable subgraphs of $W$,*
- *if $U \neq V$ are acceptable subgraphs of $W$ and $\beta \in V$ such that $U \cup \{\beta\} = V$ then there is an arrow $d(\beta) : U \xrightarrow{\beta} V$.*

The Hasse diagram is obviously a well-defined graph — it is in fact a partially ordered set of (some) subsets of $W$.

**Lemma 3.5.** *The definition of the Hasse diagram implies the following fact: if $U$ is an acceptable subgraph, then the arrows $d(\beta)$ ending at $U$ correspond exactly to the minimal elements $\beta$ of $U$ and the arrows $d(\gamma)$ starting at $U$ correspond exactly to the maximal elements $\gamma$ of $W - U$.*

PROOF: The statements follow directly from the definition of the Hasse diagram. □

**Definition 3.6.** *We say that a graph $F$ labeled with a set $B$ has the* **square completing property** *if whenever* 
$$\begin{tikzcd} & v^0 \arrow[dl, "b^1"'] \arrow[dr, "b^2"] & \\ v^1 & & v^2 \end{tikzcd}$$
*is a subgraph of $F$ then there exists a vertex $v^3$ such that* 
$$\begin{tikzcd} & v^0 \arrow[dl, "b^1"'] \arrow[dr, "b^2"] & \\ v^1 \arrow[dr, "b^2"'] & & v^2 \arrow[dl, "b^1"] \\ & v^3 & \end{tikzcd}$$
*is also a subgraph of $F$. We say that*

*$F$ has the* **dual square completing property** *if whenever*
$$\begin{tikzcd} v^1 \arrow[dr, "b^2"] & & v^2 \arrow[dl, "b^1"'] \\ & v^3 & \end{tikzcd}$$
*is a subgraph of $F$ then there exists a vertex $v^0$ such that*
$$\begin{tikzcd} & v^0 \arrow[dl, "b^1"'] \arrow[dr, "b^2"] & \\ v^1 \arrow[dr, "b^2"] & & v^2 \arrow[dl, "b^1"'] \\ & v^3 & \end{tikzcd}$$
*is a subgraph of $F$.*

**Remark 3.7.** *For any given $\mathfrak{g}$ and its representation $\rho$, both the weight graph and the Hasse diagram have the square completing property and dual square completing property.*

**Definition 3.8. Isomorphism of $B$-labeled graphs** *$G$, $H$ is a mapping $\varphi : G \to H$ which is a one-to-one mapping of both vertices and arrows with the condition that if $g \xrightarrow{b} g'$, $b \in B$, then $\varphi(g) \xrightarrow{b} \varphi(g')$.*

**3.2 Construction of the BGG sequence.** The idea of construction of the BGG sequence deals with the weight graph $W$ of the representation $\mathfrak{g}_1$ of $\mathfrak{g}_0^s$. Before we state the main theorem, let us prove an important technical lemma about properties of $W$. Recall that we restrict ourselves to the case that all weights of $\mathfrak{g}_1$ are extremal.

**Lemma 3.9.** *If*
$$\begin{tikzcd} & \beta^0 \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2"] & \\ \beta^1 & & \beta^2 \end{tikzcd}$$
*is a subgraph of $W$ with $\alpha_1$, $\alpha_2$ simple, then $\langle \alpha_1, \alpha_2 \rangle = 0$ and $\langle \beta^i, \alpha_j \rangle = -1$ if $i = j$ and $1$ if $i \neq j$.*

Proof: Recall that every weight $\beta$ of $\mathfrak{g}_1$ is an orthogonal projection of a root $\theta$ of $\mathfrak{g}$ onto $\mathfrak{h}^{0*}$ (see Lemma 2.2, and [23]). Therefore
$$\langle \beta^0, \alpha_i \rangle = \langle \theta^0, \alpha_i \rangle$$
for every simple root $\alpha_i$. By assumption, $\langle \beta^0, \alpha_i \rangle = 1$ for $i = 1, 2$, hence $\theta^0, \alpha_1, \alpha_2$ are three linearly independent roots of $\mathfrak{g}$ ($\alpha_1, \alpha_2$ are independent in $\mathfrak{h}^{0*}$ and $\theta^0$ is not in $\mathfrak{h}^{0*}$).

But we know that all roots of $\mathfrak{g}$ have the same length and the only possible values of the inner product $\langle \alpha_1, \alpha_2 \rangle$ are $0, -1$ (this follows from the Dynkin diagram of $\mathfrak{g}$). If $\langle \alpha_1, \alpha_2 \rangle = -1$, then necessarily $\theta^0 = \alpha_1 + \alpha_2$ which is a contradiction with the linear independence of these three roots.

The relation $\langle \beta^1 \alpha_1 \rangle = \langle \beta^2, \alpha_2 \rangle = -1$ follows from the definition of the weight graph and $\langle \beta^1 \alpha_2 \rangle = \langle \beta^2, \alpha_1 \rangle = 1$ is obtained by
$$\langle \beta^1, \alpha_2 \rangle = \langle \beta^0, \alpha_2 \rangle - \langle \alpha_1, \alpha_2 \rangle = 1$$
and vice versa. $\square$

**Corollary 3.10.** *Suppose that $\beta^i = \beta^0 - \alpha^i$ are two incomparable weights, $i = 1, 2$, and $\alpha^1, \alpha^2$ are positive roots such that each of them, expressed as a linear combination of simple roots, has the least possible number of summands. In other words, $\beta^0$ is the supremum of $\beta^1$, $\beta^2$, i.e. the least weight greater than both $\beta^1$, $\beta^2$. Then also $\langle \alpha^1, \alpha^2 \rangle = 0$ and $\langle \beta^i, \alpha^j \rangle = -1$ if $i = j$ and $1$ if $i \neq j$.*

PROOF: This follows by applying the preceding lemma to the summands of $\alpha^1$, $\alpha^2$. □

Recall that a BGG sequence $B$ is defined as a connected component of the graph of all invariant operators. Call $B'$ the underlying graph of $B$, i.e. the $W$-labeled graph obtained by forgetting the orders of operators and keeping just the directions $\beta \in W$.

**Theorem 3.11.** *Let $\mathfrak{g}$ be a $|1|$-graded Lie algebra, suppose that all weights of $\mathfrak{g}_1$ are extremal and denote its highest weight $\beta_{max}$. Let $\lambda^0$ be a dominant weight for $\mathfrak{g}_0^s$ and let $k^0 \in \mathbb{N}$. Let $w^0 = -(\lambda, \beta_{max}) + (1 - k^0)r$. Denote by $B$ the BGG sequence containing the subgraph $D(\beta_{max}, k^0) : (\lambda^0, w^0) \longrightarrow (\lambda^0 + k^0\beta_{max}, w^0 + k^0)$ and by $B'$ the underlying $W$-labeled graph. Denote by $W$ the weight graph and by $H$ the Hasse diagram for the representation $\mathfrak{g}_1$.*

*Then there exists a unique $W$-graph isomorphism $\varphi : H \to B'$ such that $\varphi(\emptyset) = (\lambda^0, w^0)$.*

**Remark 3.12.** *The value of $w^0$ is defined as $w_{\lambda^0,\beta_{max}} - k^0 r$, which assures, by Theorem 2.4, the existence of an operator $D(\beta_{max}, k^0, \lambda^0)$.*

PROOF: For $l \in \mathbb{Z}_+$ denote by $H_l$ the full subgraph of $H$ containing all the acceptable subgraphs of $W$ the cardinality of which is at most $l$.

The mapping $\varphi$ will be constructed by induction on the cardinality $l$ of acceptable subgraphs $U$. On every step, the image of $H_l$ under $\varphi$ will be denoted $B_l$ and it will be proved that

(a) $\varphi|_{H_l} : H_l \to B_l$ is a $W$-graph isomorphism,

(b) all arrows leaving the vertices of $B_{l-1}$ are images of arrows leaving the vertices of $H_{l-1}$ under $\varphi$ and

(c) all arrows entering the vertices of $B_l$ are images of arrows entering the vertices of $H_l$ under $\varphi$.

(I.) First induction step ($l = 0$). Put $\varphi(\emptyset) = (\lambda^0, w^0)$. Then (a) is obvious, (b) is void and we only have to prove (c): the set of arrows of $B$ ending at $(\lambda^0, w^0)$ is the same as the set of arrows of $H$ ending at $\emptyset$, which is empty.

This is straightforward: if there is an arrow labeled $\beta$ entering the vertex $(\lambda^0, w^0)$, so necessarily $\beta \leq \beta_{max}$, and this is a contradiction with Corollary 2.11.

(II.) Second induction step. Suppose that $\varphi$ is constructed for all $U \in H_l$, $l \geq 0$. We know by induction that (a), (b), (c) hold and also that for every such $U$, the component $\lambda_U$ of $\varphi(U) = (\lambda_U, w_U)$ is a dominant weight.

Let an acceptable subgraph $V \in H_{l+1}$ be fixed. Choose $U \in H_l$ and $\beta \in W$ such that $V = U \cup \{\beta\}$. Define

$$\begin{aligned} k = k_\beta &= \frac{1}{r}(w_{\lambda_U,\beta} - w_U),\\ \lambda_V &= \lambda_U + k\beta,\\ w_V &= w_U + k,\\ \varphi(V) &= (\lambda_V, w_V). \end{aligned}$$

Define also the image of the arrow $U \xrightarrow{\beta} V$ by $\varphi(U) \xrightarrow{\beta} \varphi(V)$.

We must prove that $\varphi(V)$ is well defined, i.e. that $k$ is a positive integer, $\lambda_V$ is dominant and the value of $\varphi(V)$ does not depend on the choice of $U$, $\beta$.

This will be proved separately in two cases:

Case 1. There exists only one pair $U, \beta^2$ such that $V = U \cup \{\beta^2\}$, i.e. $\beta^2$ is the only minimal element of $V$ ("no branching").

Case 2. There are more such pairs ("branching").

1. First consider the "no branching" case.

In the subcase $l+1 = 1$ the only acceptable subgraph of cardinality 1 is $\{\beta_{max}\}$, and we have already $k^0 \in \mathbb{N}$ given and it is obvious that $\lambda_{\{\beta\}} = \lambda^0 + k^0\beta_{max}$ is dominant, since both $\lambda^0$ and $\beta_{max}$ are.

If $l+1 \geq 2$, then there exists $\beta^1$ minimal in $U$ such that $\alpha_i = \beta^1 - \beta^2$ is a simple root (there may exist more such weights $\beta^1$). Then necessarily $\langle \beta^1, \alpha_i \rangle = 1$ and $\langle \beta^2, \alpha_i \rangle = -1$. Denote $k^2 = k_{\beta^2}$, $\varphi(U) = (\lambda^2, w^2)$, $\varphi(U - \{\beta^1\}) = (\lambda^1, w^1)$ and write

$$(\lambda^1, w^1) \xrightarrow{D(\beta^1, k^1)} (\lambda^2, w^2)$$

with $\lambda^1$, $\lambda^2$ dominant and $k^1 \in \mathbb{N}$ (by induction). We prove the

**Lemma 3.13.** *If $\beta^1$ is such that $\alpha_i = \beta^1 - \beta^2$ is a simple root, then for $\lambda^1$ defined above there is $k^2 = \langle \lambda^1, \alpha_i \rangle + 1 \in \mathbb{N}$ and the weight $\lambda^3 = \lambda^2 + k^2\beta^2$ is dominant.*

Proof: The value of $k^2$ is defined so that 2.4 and therefore 2.5 hold. Hence

$$k^1 + k^2 = \langle \lambda^2 + \delta, \beta^1 - \beta^2 \rangle = \langle \lambda^2, \alpha_i \rangle + 1 = \langle \lambda^1, \alpha_i \rangle + k^1 \langle \beta^1, \alpha_i \rangle + 1,$$

and since $\langle \beta^1, \alpha_i \rangle = 1$, we get $k^2 = \langle \lambda^1, \alpha_i \rangle + 1 \in \mathbb{N}$, what we had to prove.

Dominance of $\lambda^3$: by definition, $\lambda^3$ is dominant if and only if

$$\langle \lambda^3, \alpha_j \rangle = \langle \lambda^2 + k^2\beta^2, \alpha_j \rangle \geq 0$$

for all $j \in \{1, \ldots, n\}$, where $\alpha_j$ are the simple roots. If $\langle \beta^2, \alpha_j \rangle \geq 0$ then $\langle \lambda^3, \alpha_j \rangle \geq 0$ as well. If $\langle \beta^2, \alpha_k \rangle < 0$ for some $k$, then necessarily $\langle \beta^2, \alpha_k \rangle = -1$

(this follows from the extremality of all weights), and the weight $\beta^1 = \beta^2 + \alpha_k$ is minimal in $U$. Then

$$\begin{aligned}\langle\lambda^3, \alpha_k\rangle &= \langle\lambda^2 + (\langle\lambda^1, \alpha_k\rangle + 1)\beta^2, \alpha_k\rangle\\ &= \langle\lambda^2, \alpha_k\rangle + (\langle\lambda^1, \alpha_k\rangle + 1)\langle\beta^2, \alpha_k\rangle\\ &= \langle\lambda^2 - \lambda^1, \alpha_k\rangle - 1 = k^1\langle\beta^1, \alpha_k\rangle - 1 = k^1 - 1 \geq 0.\end{aligned}$$

□

2. In the "branching" case, introduce the following notation: let $U^1, U^2$ be acceptable subgraphs and $\beta^1$, $\beta^2$ weights such that $V = U^1 \cup \{\beta^2\} = U^2 \cup \{\beta^1\}$. This means that $\beta^1, \beta^2$ are incomparable. Denote $T = U^1 \cap U^2$, $\varphi(T) = (\lambda, w)$ and $\varphi(U^i) = (\lambda^i, w^i)$, $i = 1, 2$, and denote by $k^i$ the order of the arrow $D(\beta^i, k^i, \lambda)$ : $(\lambda, w) \to (\lambda^i, w^i)$, $i = 1, 2$. Pictured, we have a subgraph

$$\begin{tikzcd} & T \arrow[dl, "\beta^1"'] \arrow[dr, "\beta^2"] & \\ U^1 \arrow[dr, "\beta^2"'] & & U^2 \arrow[dl, "\beta^1"] \\ & V & \end{tikzcd}$$

in $H$ and

$$\begin{tikzcd} & (\lambda, w) \arrow[dl, "\beta^1{,}k^1"'] \arrow[dr, "\beta^2{,}k^2"] & \\ (\lambda^1, w^1) & & (\lambda^2, w^2) \end{tikzcd}$$

in $B$ with $\lambda$, $\lambda^1$, $\lambda^2$ dominant. We want to prove that then

**Lemma 3.14.**

$$\begin{tikzcd} & (\lambda, w) \arrow[dl, "\beta^1{,}k^1"'] \arrow[dr, "\beta^2{,}k^2"] & \\ (\lambda^1, w^1) \arrow[dr, "\beta^2{,}k^2"'] & & (\lambda^2, w^2) \arrow[dl, "\beta^1{,}k^1"] \\ & (\lambda^3, w^3) & \end{tikzcd}$$

*is a subgraph of $B$, i.e. $B$ has the square completing property.*

Proof: Consider the decomposition $V = U^1 \cup \beta^2$, then $\varphi(U^1) = (\lambda^1, \omega^1)$, $\varphi(V) = (\lambda^3, w^3)$, where $\lambda^3 = \lambda^1 + k^2\beta^2$ and $w^3 = w^1 + k^2$ and we have $k^{2'} = k_{\beta^2}$ defined so that 2.5 holds. We prove the

**Lemma 3.15.** *$k^{2'} = k^2$ and $\lambda^3$ is dominant.*

Proof: Let $\beta = \sup(\beta^1, \beta^2)$, i.e. the least weight greater than both $\beta^1$, $\beta^2$. Then there exist positive roots $\alpha^1$, $\alpha^2$ such that $\beta = \beta^1 + \alpha^1 = \beta^2 + \alpha^2$, with the property that the expression of $\alpha^k$, $k = 1, 2$, as a linear combination of simple roots has the least possible number of summands; in other words no simple root appears in the expression for both $\alpha^1$, $\alpha^2$. We also know by 3.10 that $\langle \beta^i, \alpha^j \rangle = -1$ if $i = j$ and 1 if $i \neq j$.

We know

$$\begin{aligned} w &= w_{\lambda,\beta^2} - k^2 r, \\ w^1 &= w_{\lambda^1,\beta^2} - k^{2'} r = w + k^1. \end{aligned}$$

Therefore

$$\begin{aligned} w_{\lambda^1,\beta^2} - k^{2'} r &= w_{\lambda,\beta^2} - k^2 r + k^1, \\ -(\lambda^1 + \delta, \beta^2) - k^{2'} r &= -(\lambda + \delta, \beta^2) - k^2 r + k^1, \\ (k^2 - k^{2'}) r &= (\lambda^1 - \lambda, \beta^2) + k^1 = k^1 (\beta^1, \beta^2) + k^1. \end{aligned}$$

This vanishes if and only if $(\beta^1, \beta^2) = -1$. $\beta^1$, $\beta^2$ are incomparable, hence $\alpha^k \neq 0$, $k = 1, 2$. It follows that

$$(\beta^1, \beta^2) = (\beta^1, \beta^1) + (\beta^1, \alpha^1 - \alpha^2) = 2r - 1 - 2r = -1.$$

Prove now the dominance of $\lambda^3$. Let $\alpha_j$ be a simple root. Again, if $\langle \beta^2, \alpha_j \rangle \geq 0$, then $\langle \lambda^3, \alpha_j \rangle \geq 0$, too. If for some $j$, $\langle \beta^2, \alpha_j \rangle < 0$, then $\langle \beta^2, \alpha_j \rangle = -1$ again and

$$\begin{aligned} \langle \lambda^3, \alpha_j \rangle &= \langle \lambda + k^1 \beta^1 + k^2 \beta^2, \alpha_j \rangle \\ &= \langle \lambda^2, \alpha_j \rangle + k^1 \langle \beta^1, \alpha_j \rangle, \end{aligned}$$

but $\langle \beta^1, \alpha_j \rangle \geq 0$, hence $\langle \lambda^3, \alpha_j \rangle \geq 0$ as required.

It remains to show that $\langle \beta^1, \alpha_j \rangle \geq 0$. $\alpha_j$ appears in the expression of $\alpha^2$ hence it does not appear in the expression of $\alpha^1$. Therefore $\langle \beta^1, \alpha_j \rangle \geq 0$. □

The role of $\beta^1$ and $\beta^2$ may now be interchanged and so we proved that the value $(\lambda^3, w^3) = (\lambda + k^1 \beta^1 + k^2 \beta^2, w + k^1 + k^2)$ does not depend on the choice of $U^1$, $U^2$, and that there is a square in $B$ as required in 3.14. □

Then, when $\varphi(V)$ is defined for all $V \in H_{l+1}$, we see that $\varphi|_{H_{l+1}}$ is a $W$-graph isomorphism.

The next aim is to prove that the only arrows that start at points of $B_l - B_{l-1}$ (this proves (b)) and the only arrows that end at points of $B_{l+1} - B_l$ (this proves (c)) are images of arrows of $H$. Call a weight $\beta \in W$ admissible for $\varphi(U)$ if the arrow

$$(\lambda_U, w_U) \xrightarrow{D(\beta,k)} (\lambda_V, w_V)$$

has a nonnegative integral order $k$ and $\lambda_V = \lambda_U + k\beta$ is dominant.

We have to prove that the weights admissible for $\lambda_U$ are exactly the maximal vertices of $W - U$. Then, by Remark 3.5, the set of the arrows starting at $\varphi(U)$ is an isomorphic image under $\varphi$ of the set of the arrows starting at $U$.

First prove that no point from $U$ is admissible. This follows immediately from the Corollary 2.11: if $\beta \in U$, then there exists $\beta'$ minimal in $U$ and such that $\beta' \leq \beta$, then $d(\beta')$ points to $\varphi(U)$, which contradicts 2.11.

For the case $U = W$, i.e. $U$ is minimal in $H$, all weights are in $U$ hence no weight is admissible. Further suppose $U \neq W$.

Now, when we know that the maximal weights of $W - U$ are admissible for $\varphi(U)$, we have to prove that no other weights of $W - U$ are admissible. This follows immediately from the following

**Lemma 3.16.** *If $U \neq W$, $\varphi(U) = (\lambda, w)$ and $\beta^2 \in W - U$ is not maximal in $W - U$, then $\lambda^2 = k^2 + \beta^2$ is not dominant.*

Proof: $\beta^2$ is not maximal in $W - U$ hence there exists $\beta^1 \in W - U$ such that $\beta^1 = \beta^2 + \alpha_j$, $\alpha_j$ simple root, and

$$\begin{tikzcd} & (\lambda, w) \arrow[dl, "{\beta^1,k^1}"'] \arrow[dr, "{\beta^2,k^2}"] & \\ (\lambda^1, w^1) & & (\lambda^2, w^2) \end{tikzcd}$$

with $\lambda^1$ not necessarily dominant.

We know that $\langle \beta^1, \alpha_j \rangle = 1$ and $\langle \beta^2, \alpha_j \rangle = -1$. We show that then $\langle \lambda^2, \alpha_j \rangle < 0$.

By 2.1, we have

$$w = w_{\lambda,\beta^1} - k^1 r = w_{\lambda,\beta^2} - k^2 r,$$
$$-\langle \lambda + \delta, \beta^1 \rangle - k^1 = -\langle \lambda + \delta, \beta^2 \rangle - k^2,$$

hence

(3.1) $$k^2 - k^1 = \langle \lambda + \delta, \beta^1 - \beta^2 \rangle = \langle \lambda + \delta, \alpha_j \rangle.$$

Now compute

$$\begin{aligned} \langle \lambda^2, \alpha_j \rangle &= \langle \lambda^1 - k^1\beta^1 + k^2\beta^2, \alpha_j \rangle \\ &= \langle \lambda^1, \alpha_j \rangle - k^1 - k^2 \\ &= \langle \lambda^1, \alpha_j \rangle - 2k^1 - \langle \lambda + \delta, \alpha_j \rangle \\ &= k^1 \langle \beta^1, \alpha_j \rangle - 2k^1 - \langle \delta, \alpha_j \rangle = -k^1 - 1. \end{aligned}$$

It is hence enough to show that $k^1 \geq 0$. If $\beta^1$ is admissible for $\varphi(U)$, then $k^1$ is positive by definition. If $\beta^1$ is not admissible, then there exists $\beta^0$ maximal in $W - U$, hence admissible for $\varphi(U)$, so the operator $D(\beta^0, k^0)$ has order $k^0 > 0$. But by Lemma 2.5, $\beta^0 \geq \beta^1$ implies $w_{\lambda,\beta^0} \leq w_{\lambda,\beta^1}$, thus $k^1 \geq k^0 > 0$. □

This proves (b); the statement (c) is proved dually in the following sense. We know (see Lemma 2.6) that if $(\lambda^1, w^1) \xrightarrow{\beta^1,k^1} (\lambda, w)$ then $w = w_{\lambda,\beta^1} + k^1 r$. If

$$\begin{tikzcd}
(\lambda^1, w^1) \arrow[dr, "\beta^1{,}k^1"] & & (\lambda^2, w^2) \arrow[dl, "\beta^2{,}k^2"'] \\
& (\lambda, w) &
\end{tikzcd}$$

and $\beta^2 = \beta^1 + \alpha_j$ then we get a dual formula to (3.1), namely

$$k^1 - k^2 = \langle \lambda + \delta, \beta^1 - \beta^2 \rangle = \langle \lambda + \delta, \alpha_j \rangle,$$

and by repeating the computation in the proof of the Lemma 3.16 we get that $\lambda^2$ is not dominant. Therefore only minimal weights in $U$ can enter into $\varphi(U)$.

This finishes the proof of Theorem 3.11. □

**Corollary 3.17.** *It follows from the proof* (*Lemma* 3.15) *that for every weight* $\beta$ *of* $\mathfrak{g}_0^s$ *there exists a positive integer* $k_\beta$ *such that every operator labeled* $\beta$ *has the order* $k_\beta$.

## 4. Practical computations, examples

**4.1 Practical recipe.** The principle of constructing the Hasse diagram from the weight graph may be formulated as the following recipe. The idea here is to understand the weight graph as composed of elementary pieces (rules A, B). To every such piece there exists a corresponding block in the BGG sequence. The most important information is how to glue these blocks together (rules AA etc.).

**Recipe 4.1.** *For every following subgraph of the weight graph* (*on the left*) *there exists a subgraph of the Hasse diagram* (*on the right*)*:*

*Rule A: An elementary object: a point* $\Rightarrow$ *an arrow* (*by definition*).

$$\beta \qquad \Longrightarrow \qquad \begin{tikzcd} \bullet \arrow[d, "\beta"'] \\ \bullet \end{tikzcd}$$

*Rule B: Another elementary object: two incomparable points* $\Rightarrow$ *a square* (*by Case* 2 *of the proof of Theorem* 3.11).

$$\beta^1 \qquad \beta^2 \qquad \Longrightarrow \qquad \begin{tikzcd} & \bullet \arrow[dl, "\beta^1"'] \arrow[dr, "\beta^2"] & \\ \bullet \arrow[dr, "\beta^2"'] & & \bullet \arrow[dl, "\beta^1"] \\ & \bullet & \end{tikzcd}$$

*Rule AA: Two points connected by an arrow* $\Rightarrow$ *two arrows with a common point* (*by Case* 1).

$$\begin{tikzcd} \beta^1 \arrow[d] \\ \beta^2 \end{tikzcd} \qquad \Longrightarrow \qquad \begin{tikzcd} \bullet \arrow[d, "\beta^1"'] \\ \bullet \arrow[d, "\beta^2"'] \\ \bullet \end{tikzcd}$$

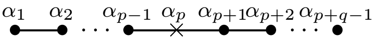
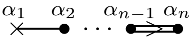
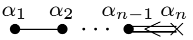
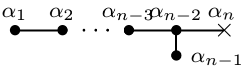
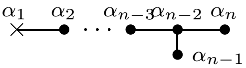
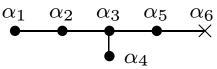
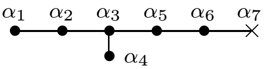
*Rule BB: Three points, two of them ordered, the third one incomparable with both of them* $\Rightarrow$ *two squares with a common edge.*

$$\begin{tikzcd} \beta^1 \arrow[d] & \beta^3 \\ \beta^2 & \end{tikzcd} \qquad \Longrightarrow \qquad \begin{tikzcd} & & \bullet \arrow[dl, "\beta^1"'] \arrow[dr, "\beta^3"] & \\ & \bullet \arrow[dl, "\beta^2"'] \arrow[dr, "\beta^3"] & & \bullet \arrow[dl, "\beta^1"] \\ \bullet \arrow[dr, "\beta^3"'] & & \bullet \arrow[dl, "\beta^2"] & \\ & \bullet & & \end{tikzcd}$$

*Rule AB: A point with arrows into two incomparable points* $\Rightarrow$ *an arrow and a square with a common point.*

$$\begin{tikzcd} & \beta^1 \arrow[dl] \arrow[dr] & \\ \beta^2 & & \beta^3 \end{tikzcd} \qquad \Longrightarrow \qquad \begin{tikzcd} & \bullet \arrow[d, "\beta^1"'] & \\ & \bullet \arrow[dl, "\beta^3"'] \arrow[dr, "\beta^2"] & \\ \bullet \arrow[dr, "\beta^2"'] & & \bullet \arrow[dl, "\beta^3"] \\ & \bullet & \end{tikzcd}$$

**4.2 Examples.** In the following table, all $|1|$-graded complex simple Lie algebras together with the Dynkin diagram with the crossed root, with the subalgebra $\mathfrak{g}_0^s$ (given by the non-crossed roots) and with the representation $\mathfrak{g}_1$ are listed. For more information about the classification of $|1|$-graded Lie algebras, see [23].

| $\mathfrak{g}$ | Dynkin diagram | $\mathfrak{g}_0^s$ | $\mathfrak{g}_1$ |
|---|---|---|---|
| $A_{p+q-1}$ |  | $A_{p-1} \oplus A_{q-1}$ | $\mathbb{C}^{pq}$ |
| $B_n$ |  | $B_{n-1}$ | $\mathbb{C}^{2n-1}$ |
| $C_n$ |  | $A_{n-1}$ | $\odot^2 \mathbb{C}^n$ |
| $D_n$ |  | $A_{n-1}$ | $\Lambda^2\mathbb{C}^n$ |
| $D_n$ |  | $D_{n-1}$ | $\mathbb{C}^{2n-2}$ |
| $E_6$ |  | $D_5$ | $S_{\frac{1}{2}}$ |
| $E_7$ |  | $E_6$ | $M_3^8$ |

(Here $S_{\frac{1}{2}}$ is a half-spin representation and $M_3^8$ is the space of $3 \times 3$ Hermitian Cayley matrices (see [21]).)

We restrict ourselves to the cases where all weights of $\mathfrak{g}_1$ are extremal — that are all cases but $B_n$ (odd conformal case) and $C_n$ (spinorial case) — the method must be proved independently for them. However, the remaining cases satisfy the extremality condition — the Grassmanian case ($A_{p+q-1}$) with special quaternionic subcase (for $p = 2$), the even conformal case ($\mathfrak{g} = D_n$, $\mathfrak{g}_0^s = D_{n-1}$), the spinorial case ($\mathfrak{g} = D_n$, $\mathfrak{g}_0^s = A_{n-1}$) and the two exceptional cases $E_6$, $E_7$.

We show examples of conformal case, of spinorial and Grassmanian cases in low dimensions and of the case $E_6$. The pictures consist of the weight graph on the left and the Hasse diagram with labeled arrows on the right. In more complicated diagrams, we draw the label only for one parallel arrow in a square, using the fact that parallel edges of a square have the same labels.

Even conformal case: $\mathfrak{g} = D_{n+1}$, $\mathfrak{g}_0^s = D_n$, $\mathfrak{g}_1 = \mathbb{C}^n$, highest weight $e_1$.

$$
\begin{tikzcd}[column sep=small]
 & e_1 \arrow[d] & \\
 & e_2 \arrow[d] & \\
 & e_3 & \\
 & \vdots & \\
 & e_{n-1} \arrow[dl] \arrow[dr] & \\
e_n \arrow[dr] & & -e_n \arrow[dl] \\
 & -e_{n-1} & \\
 & \vdots & \\
 & -e_3 \arrow[d] & \\
 & -e_2 \arrow[d] & \\
 & -e_1 &
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}[column sep=small]
 & \bullet \arrow[d, "e_1"'] & \\
 & \bullet \arrow[d, "e_2"'] & \\
 & \bullet \arrow[d, "e_3"'] & \\
 & \bullet & \\
 & \vdots & \\
 & \bullet \arrow[d, "e_{n-1}"'] & \\
 & \bullet \arrow[dl, "e_n"'] \arrow[dr, "-e_n"] & \\
\bullet \arrow[dr, "-e_n"] & & \bullet \arrow[dl, "e_n"'] \\
 & \bullet \arrow[d, "-e_{n-1}"'] & \\
 & \bullet & \\
 & \vdots & \\
 & \bullet \arrow[d, "-e_3"'] & \\
 & \bullet \arrow[d, "-e_2"'] & \\
 & \bullet \arrow[d, "-e_1"'] & \\
 & \bullet &
\end{tikzcd}
$$

Spinorial case: $\mathfrak{g} = D_{n+1}$, $\mathfrak{g}_0^s = A_n$, $\mathfrak{g}_1 = \Lambda^2\mathbb{C}^{n+1}$, highest weight $e_1 + e_2$.

$n = 1$

$$
e_1 + e_2 \quad \Longrightarrow \quad
\begin{tikzcd}
\bullet \arrow[d, "e_1+e_2"'] \\
\bullet
\end{tikzcd}
$$

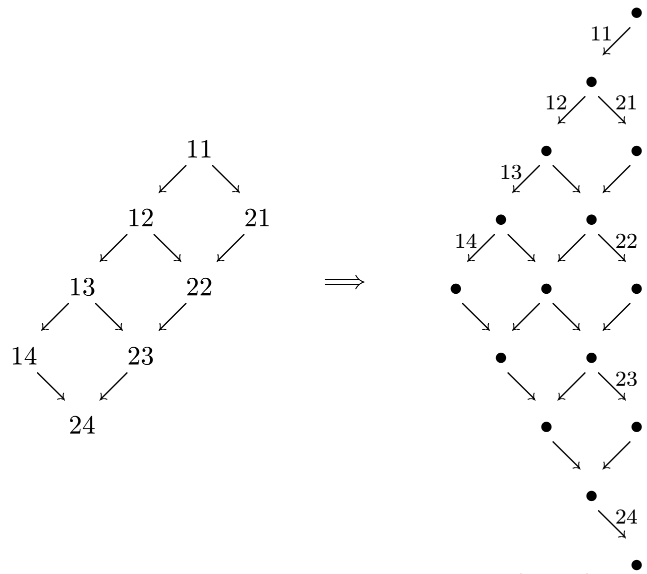
$n = 2$

$$
\begin{tikzcd}
 & e_1 + e_2 \arrow[dl] \\
e_1 + e_3 \arrow[dr] & \\
 & e_2 + e_3
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
\bullet \arrow[d, "e_1+e_2"'] \\
\bullet \arrow[d, "e_1+e_3"'] \\
\bullet \arrow[d, "e_2+e_3"'] \\
\bullet
\end{tikzcd}
$$

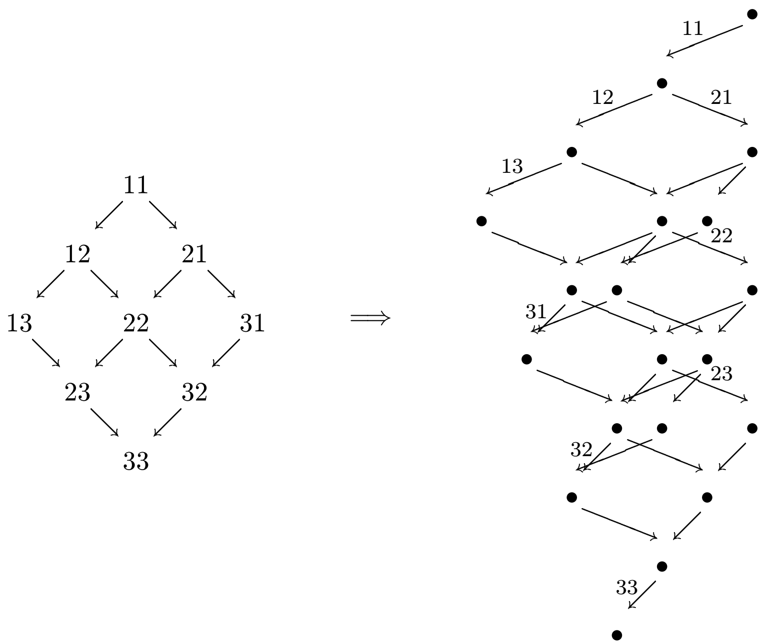
$n = 3$

$$
\begin{tikzcd}
 & & e_1 + e_2 \arrow[dl] \\
 & e_1 + e_3 \arrow[dl] \arrow[dr] & \\
e_1 + e_4 \arrow[dr] & & e_2 + e_3 \arrow[dl] \\
 & e_2 + e_4 \arrow[dr] & \\
 & & e_3 + e_4
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
 & \bullet \arrow[d, "e_1+e_2"'] & \\
 & \bullet \arrow[d, "e_1+e_3"'] & \\
 & \bullet \arrow[dl, "e_1+e_4"'] \arrow[dr, "e_2+e_3"] & \\
\bullet \arrow[dr, "e_2+e_3"'] & & \bullet \arrow[dl, "e_1+e_4"] \\
 & \bullet \arrow[d, "e_2+e_4"'] & \\
 & \bullet \arrow[d, "e_3+e_4"'] & \\
 & \bullet &
\end{tikzcd}
$$

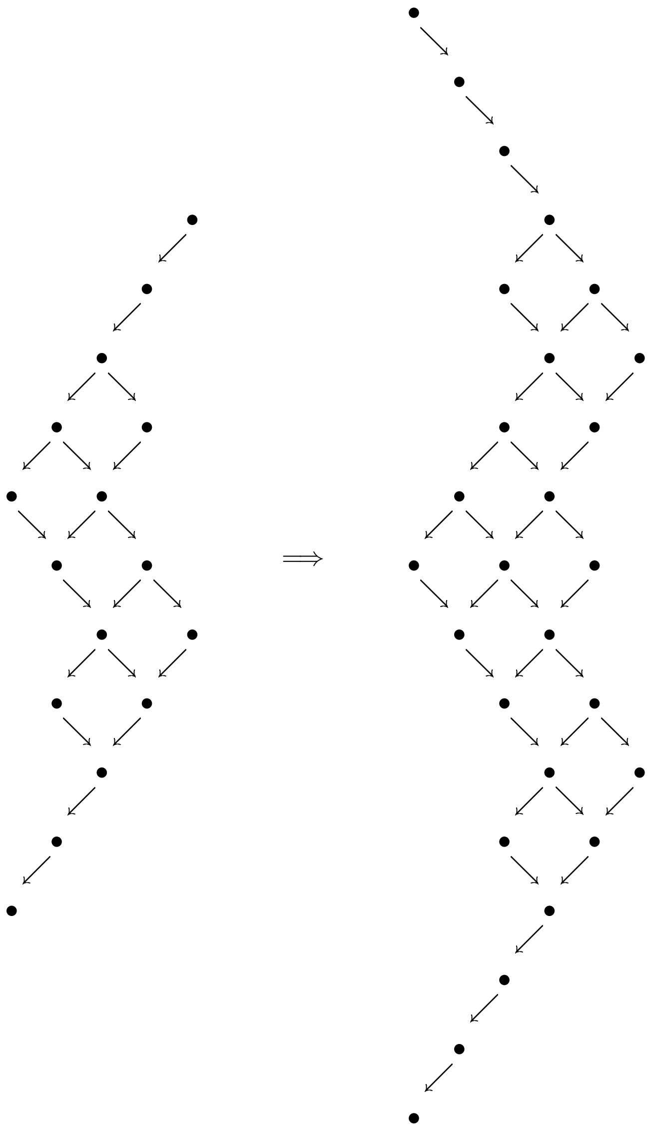
Grassmanian case $2,4$ (quaternionic): $\mathfrak{g} = A_5, \mathfrak{g}_0^s = A_1 \times A_3$, $\mathfrak{g}_1 = \mathbb{C}^2 \otimes \mathbb{C}^4$, highest weight $(e_1, e_1)$. For brevity, we use the notation $ij = (e_i, e_j)$.



Grassmanian case $3,3$: $\mathfrak{g} = A_5, \mathfrak{g}_0^s = A_2 \times A_2$, $\mathfrak{g}_1 = \mathbb{C}^3 \otimes \mathbb{C}^3$, highest weight $(e_1, e_1)$.



Case $E_6$: $\mathfrak{g} = E_6$, $\mathfrak{g}_0^s = D_5$, the representation $\mathfrak{g}_1$ is the half-spin representation with the highest weight $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$, all weights are of the form $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$ with even number of the sign $+$. For simplicity reasons, these weights are represented in the picture just by a dot.



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Mathematical Institute, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic

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