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On Moufang A-loops

J.D. Phillips

Abstract. In a series of papers from the 1940's and 1950's, R.H. Bruck and L.J. Paige developed a provocative line of research detailing the similarities between two important classes of loops: the diassociative A-loops and the Moufang loops ([1]). Though they did not publish any classification theorems, in 1958, Bruck's colleague, J.M. Osborn, managed to show that diassociative, commutative A-loops are Moufang ([5]). In [2] we relaunched this now over 50 year old program by examining conditions under which general — not necessarily commutative — diassociative A-loops are, in fact, Moufang. Here, we finish part of the program by characterizing Moufang A-loops. We also investigate simple diassociative A-loops as well as a class of centrally nilpotent diassociative A-loops. These results, *in toto*, reveal the distinguished positions two familiar classes of diassociative A-loops — namely groups and commutative Moufang loops–play in the general theory.

Keywords: diassociative, A-loop, Moufang

Classification: Primary 20N05; Secondary 20A

1. Basic notions

A *loop* is a set with a single binary operation, denoted by juxtaposition, such in $xy = z$, knowledge of any two of x, y, and z specifies the third uniquely, and with a unique two-sided identity element, denoted by 1. A *diassociative loop* is a loop in which the subloop generated by any pair of elements is a group. A *Moufang loop* is a loop satisfying the identity $((xy)x)z = x(y(xz))$. Moufang loops are diassociative ([4]).

The multiplication group, $\mathrm{Mlt}(L)$, of a loop L is the subgroup of the group of all bijections on L generated by right and left translations. That is, $\mathrm{Mlt}(L) := \langle R(x), L(x) : x \in L\rangle$, where $R(x)$ (respectively, $L(x)$) is right (respectively, left) translation by x. Clearly, $\mathrm{Mlt}(L)$ acts as a permutation group on L. The subgroup of $\mathrm{Mlt}(L)$ which fixes the identity element in L is called the *inner mapping group.* An *A-loop* is a loop L for which every inner mapping is an automorphism of L. There are A-loops that are not diassociative, hence not Moufang ([1]). Thus, the focus of the Bruck-Paige program, and our focus here, is on diassociative A-loops. The class of diassociative A-loops is a proper subvariety of the variety of all loops ([1]). Two familiar subvarieties of the variety of diassociative A-loops are the variety of all groups and the variety of all commutative Moufang

This paper is in final form and no version of it will be submitted for publication elsewhere.

loops ([5]). The results in this paper underscore the central role assumed by these two subvarieties.

Let L be either a Moufang loop or a diassociative A-loop. The *nucleus*, $\mathrm{Nuc}(L)$, of L is the normal subloop of all elements that associate with all pairs of elements from L. That is, $\mathrm{Nuc}(L) := \{x \in L : \forall\, y, z \in L, (xy)z = x(yz)\}$. The *Moufang center*, $C(L)$, of L is the subloop of those elements that commute with every element in L. That is, $C(L) := \{x \in L : \forall\, y \in L, xy = yx\}$. The Moufang center of an A-loop is normal, while the Moufang center of a Moufang loop is not necessarily normal. The *center*, $Z(L)$, of L is the normal subloop of those nucleus elements that commute with each element in L. That is, $Z(L) = \mathrm{Nuc}(L) \cap C(L)$. Finally, we remind the reader of the standard notation for the inner mapping $T(x) := L(x^{-1})R(x)$.

2. Simple diassociative A-loops

Identifying the simple algebras of a given variety is a fundamentally important part of any serious investigation of that variety. We will see that many of the simple diassociative A-loops have a surprisingly "simple" and familiar structure. Toward that end, we recall a useful technical result.

Theorem 1. *Let L be a diassociative A-loop.*

1. *There is a homomorphism f from L to a group G given by $f(x) = K^*T(x)$, where K^* is a certain normal subgroup of the inner mapping group.*
2. *If L is Moufang, then $K^* = 1$, and hence $T(x)T(y) = T(xy)$ for each $x, y \in L$, $\ker(f) = C(L)$, and $L/C(L)$ is a group.*

PROOF: [1, Theorem 3.4]. □

Corollary 2. *If L is a finite, Moufang A-loop, and if $C(L)$ is 2-divisible, then $\mathrm{Nuc}(L)$ contains all those elements in L whose orders are coprime with $|C(L)|$ (in addition to all cubes and commutators, as guaranteed by Theorem 5 below).*

PROOF: Given $x, y \in L$, let $h = R(x)R(y)R(xy)^{-1}$. Since $L/C(L)$ is a group, given $z \in L$, we must have $zh = zc$ for some $c \in C(L)$. Since h is an automorphism, $|z| = |zh| = |zc|$. Thus, since $c \in C(L)$, $|c|$ divides $|z|$. So if $|z|$ is coprime with $|C(L)|$, then since $C(L)$ satisfies the Lagrange property ([3, Theorem 2]), c must be trivial and $zh = z$, and hence $z \in \mathrm{Nuc}(L)$. □

For the balance of this paper, $\ker(f)$ will refer to the kernel of the homomorphism f given in Theorem 1. For an arbitrary diassociative A-loop L, clearly $C(L) \leq \ker(f)$. If L is Moufang, Theorem 1 guarantees that $\ker(f) \leq C(L)$. We are interested in generalizing this condition. For p a prime, let $C(L_p) = \{x \in L : \forall\, y \in L, xy^p = y^p x\}$. That is, the set $C(L_p)$ consists of all those elements of L that commute with all pth powers. Since clearly $C(L)$ is contained in $C(L_p)$, we generalize the setting of Theorem 1 by investigating diassociative A-loops for which $\ker(f)$ is contained in $C(L_p)$.

Theorem 3. *If L is a simple diassociative A-loop with* $\ker(f)$ *contained in* $C(L_p)$*, then either L has exponent p or L is, in fact, a group.*

Proof: Since L is simple, $\ker(f)$ is either trivial or all of L. If $\ker(f)$ is trivial, then by Theorem 1, L is a group. Otherwise, $L = \ker(f)$ is contained in $C(L_p)$. That is, for each $x \in L$, we have $x^p \in C(L)$. Thus, L^p, the subloop generated by the pth powers of elements in L, is contained in $C(L)$. Since L is an A-loop, L^p is normal in L. Thus, L^p is either trivial or all of L. If L^p is trivial, L has exponent p. Otherwise $L^p = L \leq C(L)$, i.e., L is commutative, and hence by Osborn's result, Moufang. And of course, simple commutative Moufang loops are groups. □

Corollary 4. *If L is a simple diassociative A-loop with* $\ker(f)$ *contained in* $C(L_2)$*, then L is, in fact, a group.*

Proof: Continuing from above, if L^2 is trivial, then L is commutative (since $abab = 1$, and this implies that $ba = a^{-1}b^{-1} = ab$) and as above, a group. □

Note: Compare Corollary 4 with [2, Theorem 7].

3. Moufang A-loops

We recall two necessary conditions for a diassociative A-loop to be Moufang:

Theorem 5. *If L is a Moufang A-loop, then*

1. *$L/\operatorname{Nuc}(L)$ is a commutative Moufang loop of exponent three, and*
2. *T is a homomorphism, i.e., $T(x)T(y) = T(xy)$.*

Proof: 1. [2, Theorem 5].
2. Theorem 1 (2) above. □

We adopt the notation of Bruck and Paige, and let $U(x,y) := R(x)R(y)R(x)R(xyx)^{-1}$. Clearly a diassociative A-loop is Moufang if $U(x,y) = 1$ for all x and y. Bruck and Paige ([1, 3.62]) managed to establish the following useful identity involving $U(x,y)$:

$$T(x)T(y)T(x) = U(x,y)^2 T(xyx). \tag{3.1}$$

While they were able to exploit this identity in proving only one theorem ([1, Theorem 3.7]), we now use (3.1) both in the proof of the sufficiency of the two conditions in Theorem 5, as well as in generalizing Bruck's and Paige's above-mentioned result ([1, Theorem 3.7]).

Theorem 6. *If L is a diassociative A-loop for which both*

1. *$L/\operatorname{Nuc}(L)$ is a commutative Moufang loop of exponent three, and*
2. *T is a homomorphism,*

then L is Moufang.

Proof: To simplify notation in the proof, we adopt the shorthand notation $U = U(x,y)$. Since T is a homomorphism, $L/C(L)$ is a group, hence Moufang. Thus, since $L/\operatorname{Nuc}(L)$ is also Moufang, given $z \in L$, we must have $zU = zc$ for some c in both $\operatorname{Nuc}(L)$ and $C(L)$. Since all cubes are nuclear, we have $z^3 = z^3U = (zU)^3 = (zc)^3 = z^3c^3$. So $c^3 = 1$. Notice that $zU^3 = (zc)U^2 = (zc^2)U = zc^3 = z$, and so $U^3 = 1$. But since T is a homomorphism, by (3.1) we have $U^2 = 1$. And thus $U = 1$ and L is Moufang. □

Clearly Theorems 5 and 6 combine to characterize Moufang A-loops:

Theorem 7. *A diassociative A-loop L is Moufang if and only if both*

1. *$L/\operatorname{Nuc}(L)$ is a commutative Moufang loop of exponent three, and*
2. *T is a homomorphism.*

If we weaken the requirement that T is a homomorphism, and balance this by adding a condition introduced in §2, we obtain a second characterization of Moufang A-loops.

Theorem 8. *A diassociative A-loop L is Moufang if and only if*

1. *$L/\operatorname{Nuc}(L)$ is a commutative Moufang loop of exponent three,*
2. *T is a semihomomorphism, i.e., $T(x)T(y)T(x) = T(xyx)$, and*
3. *$\ker(f)$ is contained in $C(L_2)$.*

Proof: Necessity follows from Theorem 5. For sufficiency note that since both $L/\operatorname{Nuc}(L)$ and $L/\ker(f)$ are Moufang, given $z \in L$, we must have $zU = zn$ for some n in both $\operatorname{Nuc}(L)$ and $\ker(f)$. Since T is a semihomomorphism, by (3.1) we have $U^2 = 1$, and thus $z = zU^2 = (zn)U = zn^2$ and $n^2 = 1$. Moreover, since all cubes are nuclear, we have $z^3 = z^3U = (zU)^3 = znznzn$. Of course, this implies $z^2 = nznzn$. Since $\ker(f)$ is contained in $C(L_2)$, and since $n^{-1} = n$, we have $z^2n = nz^2 = znzn$. This in turn implies $z = nz$. So $n = 1$. And thus $U = 1$ and L is Moufang. □

4. Central nilpotence

In this section we offer a generalization of Bruck's and Paige's theorem about centrally nilpotent diassociative A-loops ([1, Theorem 3.7]), the only other theorem on centrally nilpotent diassociative A-loops in the literature. First, a preparatory lemma.

Lemma 9. *If L is a 2-divisible, diassociative A-loop such that both T is a semihomomorphism and $L/Z(L)$ is Moufang, then L is Moufang.*

Proof: Given $z \in L$, and with the shorthand notation U, we have $zU = zc$ for some $c \in Z(L)$. Thus $z = zU^2 = (zc)U = zc^2$, and hence $c^2 = 1$. Finally, since L is 2-divisible, $c = 1$ and $U = 1$. □

Theorem 10. *If L is a centrally nilpotent 2-divisible diassociative A-loop, and if T is a semihomomorphism, then L is Moufang.*

Proof: We proceed by induction on n, the nilpotence class of L. If $n = 1$, then L is an abelian group. Assume $n \geq 2$. Then $L/Z(L)$ is a centrally nilpotent 2-divisible diassociative A-loop of nilpotency class $n-1$. By induction, $L/Z(L)$ is Moufang. By Lemma 9, L is Moufang. □

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