

Kyriakos Keremedis; Eleftherios Tachtsis

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On the extensibility of closed filters in T_1 spaces and the existence of well orderable filter bases

KYRIAKOS KEREMEDIS, ELEFThERIOS TACHTSIS

Abstract. We show that the statement $CCFC =$ “the character of a maximal free filter F of closed sets in a T_1 space (X, T) is not countable” is equivalent to the *Countable Multiple Choice Axiom* CMC and, the axiom of choice AC is equivalent to the statement $CFE_0 =$ “closed filters in a T_0 space (X, T) extend to maximal closed filters”. We also show that AC is equivalent to each of the assertions:

“every closed filter \mathcal{F} in a T_1 space (X, T) extends to a maximal closed filter with a well orderable filter base”,

“for every set $A \neq \emptyset$, every filter $\mathcal{F} \subseteq \mathcal{P}(A)$ extends to an ultrafilter with a well orderable filter base” and

“every open filter \mathcal{F} in a T_1 space (X, T) extends to a maximal open filter with a well orderable filter base”.

Keywords: closed filters, bases for filters, characters of filters, ultrafilters

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1. Definitions

Let (X, T) be a topological space and $\mathcal{E} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$. A non empty collection $\mathcal{F} \subseteq \mathcal{E}$ is an \mathcal{E} -filter iff

- (i) if $F_1, F_2 \in \mathcal{F}$ then $F_1 \cap F_2 \in \mathcal{F}$;
- (ii) if $F \in \mathcal{F}$, $F \subset F'$ and $F' \in \mathcal{E}$, then $F' \in \mathcal{F}$.

In particular if \mathcal{E} is the collection of all non empty closed sets then we say that \mathcal{F} is a *closed filter*. Likewise, if \mathcal{E} is the collection of all non empty open sets then we say that \mathcal{F} is an *open filter*. If $\mathcal{E} = \mathcal{P}(X) \setminus \{\emptyset\}$ then an \mathcal{E} -filter is called simply a *filter* on X i.e. a filter on X is an open (closed) filter of the space X carrying the discrete topology. An \mathcal{E} -filter \mathcal{F} is *free* iff $\bigcap \mathcal{F} = \emptyset$.

A non empty collection $\mathcal{B} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$ is a *filter base* for some filter \mathcal{F} iff for every $B_1, B_2 \in \mathcal{B}$ there exists $B_3 \in \mathcal{B}$ with $B_3 \subseteq B_1 \cap B_2$.

The *character* of a filter (open, closed, etc.,) \mathcal{F} is the minimum cardinality (if it exists) of a filter base \mathcal{B} for \mathcal{F} .

A family \mathcal{U} of subsets of X is *locally finite* if every point of X has a neighborhood meeting a finite number of elements of \mathcal{U} .

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The *Axiom of choice* AC is the statement:

For every family $\mathcal{A} = \{A_i : i \in k\}$ of disjoint non empty sets there exists a set C which consists of one and only one element from each element of \mathcal{A} .

The *Axiom of multiple choice* MC is the statement:

For every family $\mathcal{A} = \{A_i : i \in k\}$ of disjoint non empty sets there exists a set $\mathcal{F} = \{F_i : i \in k\}$ of finite non empty sets such that $F_i \subseteq A_i$ for all $i \in k$.

CAC and CMC stand for AC and MC respectively restricted to countable sets. The *Boolean Prime Ideal Theorem* BPI is the proposition:

Every Boolean algebra has a prime ideal.

For notation and terminology used but not defined here the reader is referred to any standard text of General Topology such as [10].

2. Introduction and some preliminary results

If one goes carefully through the proof that AC implies Tychonoff's compactness theorem, see [10], he will realize that, in order to get that proof go through, he only needs two facts:

1. *in Tychonoff products of compact T_1 spaces every closed filter extends to a maximal closed filter, and*
2. *in Tychonoff products of compact T_1 spaces the projections are closed maps.*

Motivated by (1) and (2), it is plausible to define, as the authors in [6] did, the axiom of *closed filter extensibility* CFE by requiring:

- *closed filters in a T_1 space (X, T) extend to maximal closed filters.*

In [6] it has been proved that

- Proposition 1.** (i) *AC iff CFE + CAC.*
(ii) *CFE implies but it is not equivalent to BPI.*

There remains the question:

- (A) *Does CFE imply AC?*

Since the statement CFE_0 given by

closed filters in a T_0 space (X, T) extend to maximal closed filters

clearly implies CFE, one may expect that CFE_0 implies AC. Indeed, we have:

Theorem 2. CFE_0 is equivalent to AC.

PROOF: An easy application of Zorn's Lemma, shows that AC implies CFE_0 . To see the converse let $\mathcal{A} = \{A_i : i \in k\}$ be a disjoint family of infinite sets. Topologize $X_i = A_i \cup \{\infty_i\}$ by declaring basic neighborhoods of points $x \in X_i$ to be all cofinite supersets of $\{x, \infty_i\}$. Let X be the Tychonoff product of the X_i 's. CFE_0 implies that X has a maximal closed filter \mathcal{G} and one more application of CFE_0 to X with the discrete topology extends \mathcal{G} to an ultrafilter \mathcal{F} . We show first that $\bigcap \{\overline{F} : F \in \mathcal{F}\} \neq \emptyset$ (and consequently $\bigcap \mathcal{G} \neq \emptyset$). To this end it suffices to find, in view of the proof of the Tychonoff's compactness theorem given in [10], $c \in X$, with $c(i) \in B_i = \bigcap \{\overline{\pi_i(F)} : F \in \mathcal{F}\}$ for all $i \in k$. As X_i is compact and $\{\pi_i(F) : F \in \mathcal{F}\}$ is a filter of X_i , it follows that $B_i \neq \emptyset$. Without loss of generality we may assume that $B_i \neq X_i$ for all $i \in k$. If $B_i = X_i$ choose $c_i = \infty_i$ ($\{\overline{\infty_i}\} = X_i$). Put $B = \{B_i : i \in k\}$. Since CFE_0 clearly implies the axiom of choice restricted to families of finite sets, see [5], it follows that B has a choice function c as required.

\mathcal{G} being a closed ultrafilter implies that c cannot be the element $c(i) = \infty_i$ for all $i \in k$ for the only closed set including c is X itself. In order to complete the proof of the theorem it suffices to show:

Claim. $c_i \neq \infty_i$ for all $i \in k$.

PROOF OF THE CLAIM: Indeed, if $c_i = \infty_i$ for some $i \in k$ then letting

$$Y = \prod_{i \in k} Y_i, Y_i = \begin{cases} \{c_i\} & \text{if } c_i \neq \infty_i \\ X_i & \text{if } c_i = \infty_i \end{cases}$$

carry the subspace topology, we see that $Y \in \mathcal{G}$ (\mathcal{G} is a closed ultrafilter and Y is a closed subset of X meeting each member of \mathcal{G}). Now, the only closed subset of Y including c is Y itself. Thus $Y \subseteq \bigcap \mathcal{G}$ and we have reached a contradiction. (If $c_{i^*} = \infty_{i^*}$ then we pick a finite non empty subset K of A_{i^*} and let

$$H = \prod_{i \in k} H_i, H_i = \begin{cases} \{c_i\} & \text{if } c_i \neq \infty_i \\ X_i & \text{if } c_i = \infty_i \text{ and } i \neq i^* \\ K & \text{if } i = i^* \end{cases}.$$

It follows that $H \notin \mathcal{G}$ is a closed set included in every member of \mathcal{G} , contradicting the fact that \mathcal{G} is a maximal closed filter.) Thus, c is a choice function finishing the proof of the claim and of the theorem. □

If X is a countable set, say $X = \omega$, and \mathcal{F} is a maximal free filter on ω then one can verify, in the Zermelo-Fraenkel set theory without the axiom of choice, ZF-AC for abbreviation, that \mathcal{F} cannot have a countable filter base. More generally, if X is any infinite set and \mathcal{F} is a maximal free filter on X then one can easily verify in ZF-AC that \mathcal{F} cannot have a well ordered nested filter base $\mathcal{B} = \{B_i : i \in \mathbb{N}\}$. (If

$C_i = B_i \setminus B_{i+1}$, $i \in \mathbb{N}$ and $S \subseteq \mathbb{N}$ a set of size \aleph whose complement has also size \aleph then $G = \bigcup\{C_i : i \in S\}$ meets, but does not include, all members of \mathcal{B} . As \mathcal{B} is a filter base and the filter \mathcal{F} is maximal, it follows that $G \in \mathcal{F}$. Hence, G includes a member of \mathcal{B} and this is a contradiction). So, there remains the question what happens if X carries some topology other than the discrete one and the members of the filter are maximal with respect to some property, say closedness. That is:

(B) *Can maximal closed filters have a countable filter base?*

The research in this paper is motivated by questions (A) and (B). We show in Sections 3 and 4 that the answer to (B) depends on AC and the answer to (A) is related to (B).

Let CCFC, CCFC2 and COFC stand for the statements

- *no maximal closed free filter in a T_1 space has countable character,*
- *no maximal closed free filter in a T_2 space has countable character,*
- *no maximal open free filter in a T_2 space has countable character*

respectively.

Proposition 3. *In CCFC the T_1 separation axiom cannot be replaced by the T_0 axiom.*

PROOF: $\mathcal{T} = \{\emptyset, \omega, [0, n) : n \in \omega\}$ is a T_0 topology on ω and $\mathcal{F} = \{[n, \infty) : n \in \omega\}$ is a countable free closed ultrafilter of (ω, \mathcal{T}) . □

Proposition 4. *In COFC the requirement that the space X be T_2 cannot be dropped out.*

PROOF: Let $A = \{A_i : i \in \omega\}$ be a disjoint family of non empty sets. Topologize $X = \bigcup A$ by declaring basic neighborhoods of points $x \in X$ to be all sets of the form $V_{xn} = \{x\} \cup (\bigcup\{A_i : i \geq n\})$. It can be readily verified that X is a first countable T_1 space having a countable filter base $B = \{\bigcup\{A_i : i \geq n\} : n \in \omega\}$ for the open ultrafilter \mathcal{F} of all non empty open sets independently of AC. □

Remark 1. *If each member of A is finite then X is a compact space. It follows that if no infinite subset of A has a choice function then X is not separable. Furthermore, as each sequence in X has a finite range, we see that even though X is first countable, the closure operator cannot be described sequentially. ($W \subseteq X$ is closed iff whenever $(w_n)_{n \in \omega} \subseteq W$ converges to w then $w \in W$).*

Proposition 5. *COFC can be proved in ZF-AC.*

PROOF: Let (X, \mathcal{T}) be a T_2 space and \mathcal{F} a free maximal filter of open sets having a countable filter base $B = \{B_n : n \in \omega\}$. Without loss of generality we may assume that B is strictly descending. Put $b = \bigcap \bar{B}$, $\bar{B} = \{\bar{B}_n : n \in \omega\}$. Then any open set O meeting b is in \mathcal{F} . Hence, b can have at most one point. (If $x, y \in b$ then since X is a T_2 space it follows that there exist open disjoint sets O_x and O_y including x and y . Then both O_x, O_y are in \mathcal{F} and this is a contradiction.) For every $n \in \omega$, put $O_n = B_n \setminus \bar{B}_{n+1}$.

Claim. There exists $m \in \omega$ such that $\forall n \geq m, O_n = \emptyset$.

PROOF OF THE CLAIM: If not then there exists an infinite subset $S \subseteq \omega$ such that $O_s \neq \emptyset$ for all $s \in S$. Without loss of generality we may assume that $S = \omega$. Put $O = \bigcup \{O_{2n} : n \in \omega\}$. Clearly O is an open set meeting but not including each B_n . Hence $O \in \mathcal{F}$ and consequently $B_{n^*} \subseteq O$ for some $n^* \in \omega$. This contradiction establishes the claim.

By the claim we conclude that $\bar{B}_m = \bar{B}_n$ for all $n \geq m$. Thus, $\bar{B}_m \subseteq b$ and we have arrived at a contradiction. \square

Remark 2. Since COFC is provable in ZF-AC it follows that CCFC2 is of interest only in case where some member of the filter is a closed nowhere dense set.

In what follows we shall make use of the following results.

Proposition 6 ([8]). CMC iff PCMC (= for every countable family B of disjoint infinite sets, some infinite subfamily \bar{B} of B has a multiple choice).

PROOF: Mimic the proof of Lemma 1 (iv) given in [4]. \square

Lemma 7 (Levy's Lemma [9]). MC iff every set can be written as a well ordered union of disjoint finite sets.

Below we give a list of the statements (all provable in $ZF^0 + AC$) that will be studied in the paper.

- (1) Every closed filter \mathcal{F} in a T_1 space (X, T) has a well orderable filter base.
- (2) Every open filter \mathcal{F} in a T_1 space (X, T) has a well orderable filter base.
- (3) Every open filter \mathcal{F} in a dense-in-itself T_1 space (X, T) has a well orderable filter base.
- (4) Every open ultrafilter \mathcal{F} in a T_1 space (X, T) has a well orderable filter base.
- (5) For every set $A \neq \emptyset$, every filter $\mathcal{F} \subseteq \mathcal{P}(A)$ has a well orderable filter base.
- (6) Every closed filter \mathcal{F} in a dense-in-itself T_1 space (X, T) has a well orderable filter base.
- (7) Every closed filter \mathcal{F} in a dense-in-itself T_2 space (X, T) has a well orderable filter base.
- (8) If (X, T) is a T_2 topological space and \mathcal{B} is a lattice of closed sets then every maximal \mathcal{B} -filter \mathcal{F} has a well orderable filter base.
- (9) Every closed filter \mathcal{F} in a T_1 space (X, T) extends to a maximal closed filter with a well orderable filter base.
- (10) For every set $A \neq \emptyset$, every filter $\mathcal{F} \subseteq \mathcal{P}(A)$ extends to an ultrafilter with a well orderable filter base.
- (11) Every open filter \mathcal{F} in a T_1 space (X, T) extends to a maximal open filter with a well orderable filter base.
- (12) Every closed filter \mathcal{F} in a T_1 space (X, T) has a well orderable base and is included in a maximal closed filter \mathcal{G} .
- (13) Every closed ultrafilter \mathcal{F} in a T_1 space (X, T) has a well orderable filter base.

(14) For every set $A \neq \emptyset$, every ultrafilter $\mathcal{F} \subseteq \mathcal{P}(A)$ has a well orderable filter base.

3. Existence of well ordered filter bases

We begin this section with a list of equivalent forms of MC.

Theorem 8. *The following are equivalent: MC.*

- (1) Every closed filter \mathcal{F} in a T_1 space (X, T) has a well orderable filter base.
- (2) Every open filter \mathcal{F} in a T_1 space (X, T) has a well orderable filter base.
- (3) Every open filter \mathcal{F} in a dense-in-itself T_1 space (X, T) has a well orderable filter base.
- (4) Every open ultrafilter \mathcal{F} in a T_1 space (X, T) has a well orderable filter base.
- (5) For every set $A \neq \emptyset$, every filter $\mathcal{F} \subseteq \mathcal{P}(A)$ has a well orderable filter base.
- (6) Every closed filter \mathcal{F} in a dense-in-itself T_1 space (X, T) has a well orderable filter base.
- (7) Every closed filter \mathcal{F} in a dense-in-itself T_2 space (X, T) has a well orderable filter base.
- (8) If (X, T) is a T_2 topological space and \mathcal{B} is a lattice of closed sets then every maximal \mathcal{B} -filter \mathcal{F} has a well orderable filter base.

PROOF: MC \rightarrow (1), (2), (3), (4), (5), (6), (7) and (8). We prove MC \rightarrow (1). All the other implications can be proved similarly. Fix (X, T) and \mathcal{F} as in (1). By Levy's Lemma, there exists a disjoint family $G = \{G_n : n \in k\}$, k a cardinal, of finite sets covering \mathcal{F} . Clearly, $\mathcal{B} = \{B_n = \bigcap G_n : n \in k\}$ is a well ordered base for \mathcal{F} . (As G covers \mathcal{F} it follows that each member of \mathcal{F} includes a member of G . Furthermore, as \mathcal{F} is a filter, we see that $\mathcal{B} \subseteq \mathcal{F}$ and consequently \mathcal{B} is a base.)

(2) \rightarrow MC, (5) \rightarrow MC. Fix $A = \{A_i : i \in k\}$ a disjoint family of infinite sets. Put $X = \bigcup A$ and let $\mathcal{F} = \{f \subseteq X : |A_i \setminus f| < \omega \text{ for all } i \in k\}$. Clearly, \mathcal{F} is a filter of X and an open filter of X taken with the discrete topology. Let $\mathcal{B} = \{B_n : n \in \mu\}$ be a well ordered base for \mathcal{F} . For every $i \in k$ we let $n_i = \min\{n \in \mu : |A_i \setminus B_n| \neq 0\}$. It is straightforward to verify that $D = \{D_i = A_i \setminus B_{n_i} : i \in k\}$ is a multiple choice for A finishing the proof of (2) \rightarrow MC and (5) \rightarrow MC.

(3) \rightarrow MC, (4) \rightarrow MC. Fix A as in (2) \rightarrow MC and put on $X = \bigcup A$ the cofinite topology T . Let $\mathcal{F} = T \setminus \{\emptyset\}$. Clearly, \mathcal{F} is an open ultrafilter of the dense-in-itself T_1 space (X, T) . Let $\mathcal{B} = \{B_n : n \in \mu\}$ be a well ordered base for \mathcal{F} . It is straightforward to see that the set D as defined in (2) \rightarrow MC is a multiple choice for A as required.

(6) \rightarrow MC. Fix A an infinite set. It suffices, in view of Levy's Lemma to show that A can be covered by a well ordered family of finite sets. Put $X = [A]^{<\omega}$ (i.e. the set of all finite subsets of A) and let T be the topology on X in which basic neighborhoods of points $x \in X$ are all sets of the form

$$B(x, z) = \{x\} \cup \{y \in X : y \cap z = \emptyset\}, \quad z \in X, \quad x \subseteq z.$$

Claim 1. (X, T) is a T_1 space. Indeed, fix $x, y \in X, x \neq y$. We consider the following cases.

- (a) If $x \cap y = \emptyset$ or if $x \setminus y \neq \emptyset$ and $y \setminus x \neq \emptyset$, then $B(x, x \cup y)$ and $B(y, x \cup y)$ are neighborhoods of x and y avoiding y and x respectively.
- (b) If $x \subseteq y$ then $B(x, y)$ and $B(y, y)$ are neighborhoods of x and y avoiding y and x respectively.
- (c) If $y \subseteq x$ then case (b) applies.

For every $x \in X$, put

$$(3.1) \quad G_x = \{y \in X : x \subseteq y\}.$$

Claim 2. G_x is closed in (X, T) . Fix $z \notin G_x$ then $B(z, x \cup z)$ is a neighborhood of z avoiding G_x . Hence G_x is closed as required.

Let \mathcal{F} be the closed (necessarily free) filter which is generated by the collection $\mathcal{G} = \{G_x : x \in X\}$. Let $\mathcal{D} = \{D_i : i \in k\}$ be a well ordered filter base for \mathcal{F} . Without loss of generality we may assume that each D_i is included in a G_x .

For every $i \in k$, we let $z_i = \bigcup Z_i, Z_i = \{x \in X : D_i \subseteq G_x\}$.

Claim 3. $|z_i| < \omega$. It suffices to show that $|Z_i| < \omega$. Assume on the contrary that Z_i is infinite. Clearly z_i is an infinite set and any member of $D_i = \bigcap \{G_x : D_i \subseteq G_x\}$ includes z_i . Thus, $D_i = \emptyset$. On the other hand, as $D_i \subseteq D_i$ we have $D_i \neq \emptyset$. This contradiction establishes Claim 3.

Put $Z = \{z_i : i \in k\}$. Clearly Z is a well ordered cover of A consisting of finite sets. Since we can always disjointify Z , Levy's Lemma follows and the proof of (6) \rightarrow MC is complete.

(1) \rightarrow (7). This is straightforward.

(7) \rightarrow MC. Fix A be an infinite set and let $X = [A]^{<\omega}$. N. Brunner ([3]) has shown that the collection \mathcal{B} of all sets of the form $B(y, z) = \{x \in X : y \subseteq x \wedge x \cap z = \emptyset\}, y, z \in X, y \cap z = \emptyset$ generates a dense in-itself T_2 topology on X . Now, the sets G_x given in (3.1) are closed in X . (If $z \in X, z \notin G_x$, then $B(z, x \setminus z)$ is a neighborhood of z avoiding G_x .) We can finish the proof of (7) \rightarrow MC as in (6) \rightarrow MC.

(8) \rightarrow MC. Let $A, (X, T)$ and $\mathcal{G} = \{G_x : x \in X\}$ be as in (7) \rightarrow MC. It is easy to see that $\mathcal{B} = \{\bigcup Q : Q \in [\mathcal{G}]^{<\omega}\}$ is a lattice of closed sets which is also a \mathcal{B} -filter. Thus \mathcal{B} is a maximal \mathcal{B} -filter and by the hypothesis it has a well orderable filter base \mathcal{D} . Continue as in the proof of (6) \rightarrow MC to write A as a well ordered union of finite sets finishing the proof of (8) \rightarrow MC and of the theorem. \square

In the next corollary we give a list of equivalents of AC. We would like to stress out the resemblance of (9) and (12).

Corollary 9. *The following are equivalent: AC.*

- (9) Every closed filter \mathcal{F} in a T_1 space (X, T) extends to a maximal closed filter with a well orderable filter base (= CFE + (13)).
- (10) For every set $A \neq \emptyset$, every filter $\mathcal{F} \subseteq \mathcal{P}(A)$ extends to an ultrafilter with a

well orderable filter base ($= BPI + (14)$).

(11) Every open filter \mathcal{F} in a T_1 space (X, T) extends to a maximal open filter with a well orderable filter base ($=$ every open filter \mathcal{F} in a T_1 space extends to a maximal open filter $+ (4)$) ($= BPI + (4)$).

(12) Every closed filter \mathcal{F} in a T_1 space (X, T) has a well orderable base and is included in a maximal closed filter \mathcal{G} ($= (1) + CFE$).

PROOF: $AC \rightarrow (9)$, (11), (12), (9) \rightarrow (10) and (11) \rightarrow (10). These are straightforward.

(12) $\rightarrow AC$. This follows from Proposition 1 and Theorem 8.

In order to complete the proof of the theorem it suffices to show that (10) $\rightarrow AC$. Clearly, (10) implies BPI which in turn implies AC_{fin} , the axiom of choice for families of finite sets. In order to complete the proof, it suffices to show that (10) also implies MC. The proof of (6) \rightarrow MC of Theorem 8 goes through even in case \mathcal{D} is a base for some ultrafilter \mathcal{H} extending the filter \mathcal{F} . Thus, (10) implies MC, finishing the proof of the corollary. \square

4. Maximal closed filters in T_1 spaces do not have countable characters

In this section we give an equivalent form of CMC.

Theorem 10. *CMC iff CCFC (i.e. no maximal closed free filter in a T_1 space has countable character).*

PROOF: $CMC \rightarrow CCFC$. Assume that $B = \{B_n : n \in \omega\}$ is a countable filter base for a free filter \mathcal{G} of closed sets in the T_1 space (X, T) . Without loss of generality we may assume that B is strictly descending. Put $A_n = B_n \setminus B_{n+1}$ for all $n \in \omega$ and $A = \{A_n : n \in \omega\}$. Let $\mathcal{F} = \{F_n : n \in \omega\}$ satisfy CMC for A . Then \mathcal{F} is a locally finite family of closed sets in X . Indeed let $x \in X$. We consider the following two cases:

(i) $x \notin \bigcup \mathcal{F}$. Since \mathcal{G} is a free filter there exists an $n \in \omega$ such that $x \notin B_n$. Then $W = (B_n)^c \setminus (F_0 \cup \dots \cup F_{n-1})$ is a neighborhood of x which avoids every element of \mathcal{F} .

(ii) $x \in F_n$ for some $n \in \omega$. Then $W = (B_{n+1})^c \setminus (F_0 \cup \dots \cup F_{n-1})$ is a neighborhood of x which meets only one element of \mathcal{F} , namely F_n .

Thus \mathcal{F} is locally finite and consequently the family $\mathcal{H} = \{F_{2n} : n \in \omega\}$ is also locally finite. It is a well known fact (see [10]) that the union of a locally finite family of closed sets is a closed set, consequently, $c = \bigcup \mathcal{H}$ is a closed set meeting each but not including properly any member of B . Thus, the closed filter generated by $\{c\} \cup \mathcal{G}$ extends properly \mathcal{G} meaning that \mathcal{G} is not an ultrafilter.

$CCFC \rightarrow CMC$. In view of Proposition 6 it suffices to show that $CCFC \rightarrow PCMC$. Fix $B = \{B_n : n \in \omega\}$ a family of disjoint infinite sets. Topologize $X = \bigcup B$ by declaring basic neighborhoods of points $x \in X$, $x \in B_n$ to be all supersets of $\{x\}$ whose complements in $X_n = \bigcup \{B_m : m \leq n\}$ are finite. Clearly,

- (1) X is a T_1 space,
- (2) if a set $U \neq X$ is closed in X , then for all $n \in \omega$, $|U \cap B_n| < \omega$ or $B_n \subseteq U$,
- (3) $A = \{A_i = \bigcup\{B_n : n \geq i\} : i \in \omega\}$ is a descending family of closed (nowhere dense-except A_0) sets with empty intersection.

Let \mathcal{F} be the closed filter (necessarily free) which is generated by A . As A is countable, it follows from CCFC that \mathcal{F} is not maximal. Thus, there exists a non empty closed set Q meeting non trivially but not including any member of A . Hence, there is a set $\bar{B} = \{B_{n_i} : i \in \omega\} \subseteq B$ such that $F_i = Q \cap B_{n_i} \neq \emptyset$. Since $Q \neq X$, we see that F_i is finite and consequently $F = \{F_i : i \in \omega\}$ is a multiple choice for \bar{B} finishing the proof of the theorem. \square

5. Independence results

Lemma 11. *MC (and consequently (1) through (8)) implies (14) but the converse is not true.*

PROOF: By Theorem 8, we have that MC implies (14). On the other hand, A. Blass has shown in [1] that there exists a ZF model (\mathcal{M}, \in) without free ultrafilters. Thus, in \mathcal{M} (14) holds but AC, and consequently MC, fails. \square

Lemma 12. (i) *MC does not imply CFE.*

- (ii) *CAC does not imply CFE.*
- (iii) *CMC does not imply (14).*
- (iv) *CAC does not imply (14).*
- (v) *CAC does not imply (13).*
- (vi) *Neither (14) nor (13) imply CAC.*

PROOF: (i) By Proposition 1, $CFE + CMC \leftrightarrow AC$. Now, in model $\mathcal{N}2$ (the Second Fraenkel Model) in [5] MC and consequently CMC holds but AC fails. Therefore CFE fails in $\mathcal{N}2$.

(ii) By Proposition 1, $CFE + CAC \leftrightarrow AC$. There are both Cohen models and permutational models (M, \in) where CAC holds but AC fails. See, for example Solovay's model, Model $\mathcal{M}5(\aleph)$ in [5]. Thus, in $\mathcal{M}5(\aleph)$ CAC holds but CFE fails.

(iii) It is known, see model $\mathcal{N}38$ in [5], that $CMC+BPI$ does not imply AC. ($\mathcal{N}38$ is a permutational model satisfying CMC, BPI and the negation of AC.) On the other hand, in view of Corollary 9, $(14) + BPI$ does imply AC. Thus (14) fails in $\mathcal{N}38$.

(iv) CAC holds in $\mathcal{N}38$ but as we have seen in (iii), (14) fails.

(v) Clearly, (13) implies (14). On the other hand CAC, in view of (iv), holds in $\mathcal{N}38$ but (14) fails. Hence (13) fails also in $\mathcal{N}38$.

(vi) In the Second Fraenkel Model, Model $\mathcal{N}2$ in [5], MC and consequently (13) and (14) hold but CAC fails. \square

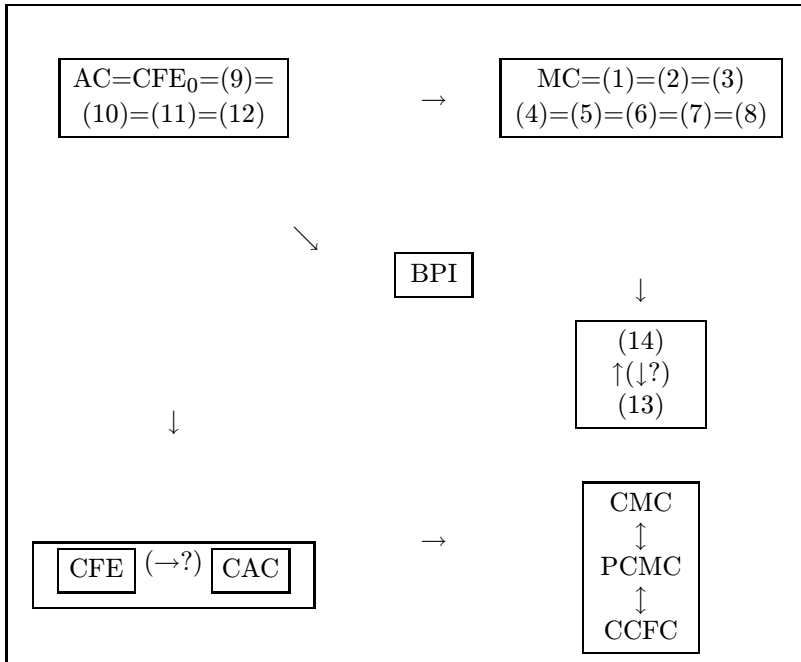
Lemma 13. (i) *BPI does not imply neither (14) nor (13).*
 (ii) *None of (13) and (14) implies BPI.*

PROOF: (i) The statements $BPI+(14)$ and $BPI+(13)$ are, in view of Corollary 9, equivalent to AC. Now, in Cohen’s original model, Model $\mathcal{M}1$ in [5], BPI is true but AC is false. Thus, in $\mathcal{M}1$, BPI is true but (14) and (13) are false.

(ii) As seen in (vi) of Lemma 12, (13) and (14) are true in model $\mathcal{N}2$. On the other hand, BPI is false in $\mathcal{N}2$ (BPI is equivalent to Form 14J (= the product of compact T_2 spaces is compact) in [5] and there exists a family in $\mathcal{N}2$ of compact T_2 spaces such that their Tychonoff product is not compact).

In Blass’ model, in [1], (14) is true whereas BPI is false (see [12]). □

6. Summary



Questions. (i) Does CFE imply CMC?
 (ii) Is CCFC2 provable in ZF-AC?
 (iii) Does (13) imply MC?

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF THE AEGEAN, KARLOVASSI 83200, SAMOS, GREECE

E-mail: kker@aegean.gr
ltah@math.aegean.gr

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