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## On the cardinality of Hausdorff spaces

ALESSANDRO FEDELI

*Abstract.* The aim of this paper is to show, using the reflection principle, three new cardinal inequalities. These results improve some well-known bounds on the cardinality of Hausdorff spaces.

*Keywords:* cardinal inequality, Hausdorff space

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Two of the most known inequalities in the theory of cardinal functions are the Hajnal-Juhász's inequality [7]: “For  $X \in T_2$ ,  $|X| \leq 2^{c(X)\chi(X)}$ ” and the Arhangel'skii's inequality [5]: “For  $X \in T_2$ ,  $|X| \leq 2^{L(X)t(X)\psi(X)}$ ”.

In this paper we will use the language of elementary submodels (see [4], [10], [1] and [2]) to establish three new cardinal inequalities which generalize the results mentioned above. We refer the reader to [3], [5], [7] for notations and terminology not explicitly given.  $\chi$ ,  $c$ ,  $\psi$ ,  $t$ ,  $L$  and  $\pi_\chi$  denote character, cellularity, pseudocharacter, tightness, Lindelöf degree and  $\pi$ -character respectively.

**Definitions.** (i) Let  $X$  be a Hausdorff space.

The closed pseudocharacter of  $X$ , denoted  $\psi_c(X)$ , is the smallest infinite cardinal  $\kappa$  such that for every  $x \in X$  there is a collection  $\mathcal{U}_x$  of open neighbourhoods of  $x$  such that  $\bigcap \{\overline{U} : U \in \mathcal{U}_x\} = \{x\}$  and  $|\mathcal{U}_x| \leq \kappa$  ([7]).

The Hausdorff pseudocharacter of  $X$ , denoted  $H\psi(X)$ , is the smallest infinite cardinal  $\kappa$  such that for every  $x \in X$  there is a collection  $\mathcal{U}_x$  of open neighbourhoods of  $x$  with  $|\mathcal{U}_x| \leq \kappa$  such that if  $x \neq y$ , there exist  $U \in \mathcal{U}_x, V \in \mathcal{U}_y$  with  $U \cap V = \emptyset$  ([6]).

Clearly  $\psi_c(X) \leq H\psi(X) \leq \chi(X)$  for every Hausdorff space  $X$ .

(ii) Let  $X$  be a topological space,  $ac(X)$  is the smallest infinite cardinal  $\kappa$  such that there is a subset  $S$  of  $X$  such that  $|S| \leq 2^\kappa$  and for every open collection  $\mathcal{U}$  in  $X$  there is a  $\mathcal{V} \in [\mathcal{U}]^{\leq \kappa}$  with  $\bigcup \mathcal{U} \subset S \cup \overline{\bigcup \{V : V \in \mathcal{V}\}}$ .

Observe that  $ac(X) \leq c(X)$  for every space  $X$ .

**Theorem 1.** *If  $X$  is a  $T_2$ -space then  $|X| \leq 2^{ac(X)H\psi(X)}$ .*

**PROOF:** Let  $\lambda = ac(X)H\psi(X)$ ,  $\kappa = 2^\lambda$ , let  $\tau$  be the topology on  $X$  and let  $S$  be an element of  $[X]^{\leq \kappa}$  witnessing that  $ac(X) \leq \lambda$ . For every  $x \in X$  let  $\mathcal{B}_x$  be a collection of open neighbourhoods of  $x$  with  $|\mathcal{B}_x| \leq \lambda$  such that if  $x \neq y$  then

there exist  $U \in \mathcal{B}_x, V \in \mathcal{B}_y$  such that  $U \cap V = \emptyset$ , and let  $f : X \rightarrow \mathcal{P}(\tau)$  be the map defined by  $f(x) = \mathcal{B}_x$  for every  $x \in X$ .

Let  $A = \kappa \cup \{S, X, \tau, \kappa, f\}$  and take a set  $\mathcal{M}$  such that  $\mathcal{M} \supset A, |\mathcal{M}| = \kappa$  and which reflects enough formulas to carry out our argument. To be more precise we ask that  $\mathcal{M}$  reflects enough formulas so that the following conditions are satisfied:

- (i)  $C \in \mathcal{M}$  for every  $C \in [\mathcal{M}]^{\leq \kappa}$ ;
- (ii)  $\mathcal{B}_x \in \mathcal{M}$  for every  $x \in X \cap \mathcal{M}$ ;
- (iii) if  $B \subset X$  and  $B \in \mathcal{M}$  then  $\overline{B} \in \mathcal{M}$ ;
- (iv) if  $\mathcal{A} \in \mathcal{M}$  then  $\bigcup \mathcal{A} \in \mathcal{M}$ ;
- (v) if  $B$  is a subset of  $X$  such that  $X \cap \mathcal{M} \subset B$  and  $B \in \mathcal{M}$  then  $X = B$ ;
- (vi) if  $E \in \mathcal{M}$  and  $|E| \leq \kappa$  then  $E \subset \mathcal{M}$ .

Observe that by (ii) and (vi)  $\mathcal{B}_y \subset \mathcal{M}$  for every  $y \in X \cap \mathcal{M}$ .

Claim:  $X \subset \mathcal{M}$  (and hence  $|X| \leq 2^{ac(X)H\psi(X)}$ ). Suppose not and take  $p \in X \setminus \mathcal{M}$ . Let  $\mathcal{B}_p = \{B_\alpha\}_{\alpha < \lambda}$ , clearly  $\bigcap \{\overline{B}_\alpha : \alpha < \lambda\} = \{p\}$ . Now for every  $\alpha < \lambda$  let  $(X \cap \mathcal{M})_\alpha = \{y \in X \cap \mathcal{M} : \exists B \in \mathcal{B}_y \text{ for which } B \cap B_\alpha = \emptyset\}$ . For every  $y \in (X \cap \mathcal{M})_\alpha$  choose a  $B_{y,\alpha} \in \mathcal{B}_y$  such that  $B_{y,\alpha} \cap B_\alpha = \emptyset$ , clearly  $\mathcal{U}_\alpha = \{B_{y,\alpha} : y \in (X \cap \mathcal{M})_\alpha\}$  covers  $(X \cap \mathcal{M})_\alpha$ . Since  $ac(X) \leq \lambda$  it follows that there is a  $\mathcal{V}_\alpha \in [\mathcal{U}_\alpha]^{\leq \lambda}$  such that  $(X \cap \mathcal{M})_\alpha \subset S \cup \bigcup \{\overline{V} : V \in \mathcal{V}_\alpha\}$ . Observe that  $p \notin S \cup \bigcup \{\overline{V} : V \in \mathcal{V}_\alpha\}$  ( $S \in \mathcal{M}$  and  $|S| \leq \kappa$  so by (vi)  $S \subset \mathcal{M}$ , moreover  $\bigcup \{\overline{V} : V \in \mathcal{V}_\alpha\} \subset X \setminus B_\alpha$ ). We have also  $\bigcup \{\overline{V} : V \in \mathcal{V}_\alpha\} \in \mathcal{M}$  ( $V \in \mathcal{M}$  for every  $V \in \mathcal{V}_\alpha$ , so by (iii)  $\overline{V} \in \mathcal{M}$ , therefore  $\{\overline{V} : V \in \mathcal{V}_\alpha\} \subset \mathcal{M}$  and  $\{\overline{V} : V \in \mathcal{V}_\alpha\} \in \mathcal{M}$  by (i), hence by (iv)  $\bigcup \{\overline{V} : V \in \mathcal{V}_\alpha\} \in \mathcal{M}$ , so  $\bigcup \{\overline{V} : V \in \mathcal{V}_\alpha\} \in \mathcal{M}$  by (iii)). Set  $C_\alpha = S \cup \bigcup \{\overline{V} : V \in \mathcal{V}_\alpha\}$  for every  $\alpha < \lambda$  and observe that  $C_\alpha \in \mathcal{M}$  (recall that  $S, \bigcup \{\overline{V} : V \in \mathcal{V}_\alpha\} \in \mathcal{M}$ ). Now  $X \cap \mathcal{M} \subset \bigcup \{C_\alpha : \alpha < \lambda\}$ , since  $\bigcup \{C_\alpha : \alpha < \lambda\} \in \mathcal{M}$  ( $\{C_\alpha : \alpha < \lambda\} \subset \mathcal{M}$ , so by (i)  $\{C_\alpha : \alpha < \lambda\} \in \mathcal{M}$ , hence by (iv)  $\bigcup \{C_\alpha : \alpha < \lambda\} \in \mathcal{M}$ ) it follows by (v) that  $\bigcup \{C_\alpha : \alpha < \lambda\} = X$ . This is a contradiction ( $p \notin \bigcup \{C_\alpha : \alpha < \lambda\}$ ).  $\square$

**Corollary 2** ([7]). *If  $X$  is a  $T_2$ -space then  $|X| \leq 2^{c(X)\chi(X)}$ .*

**Remark 3.** The above result of Hajnal and Juhász has been improved also by Hodel, in fact in [6] it is shown that  $|X| \leq 2^{c(X)H\psi(X)}$  for every Hausdorff space  $X$ . It is clear that Theorem 1 generalizes also this result of Hodel.

Now let  $X$  be the Michael line, i.e. let  $X$  be  $\mathbb{R}$  topologized by isolating the points of  $\mathbb{R} \setminus \mathbb{Q}$  and leaving the points of  $\mathbb{Q}$  with their usual neighbourhoods. Then  $X$  is a normal space such that  $|X| = 2^{ac(X)H\psi(X)} < 2^{c(X)H\psi(X)}$ .

Observe that in Theorem 1  $H\psi(X)$  cannot be replaced by  $\psi_c(X)$ , in fact for every infinite cardinal  $\kappa$  there is a  $T_3$ -space  $X$  with  $|X| = \kappa$  and  $\psi(X) = c(X) = ac(X) = \omega$  (see e.g. [5]).

**Definition 4.** Let  $X$  be a topological space,  $lc(X)$  is the smallest infinite cardinal  $\kappa$  such that there is a closed subset  $F$  of  $X$  such that  $|F| \leq 2^\kappa$  and for every open collection  $\mathcal{U}$  in  $X$  there is a  $\mathcal{V} \in [\mathcal{U}]^{\leq \kappa}$  with  $\bigcup \mathcal{U} \subset F \cup \bigcup \{\overline{V} : V \in \mathcal{V}\}$ .

Clearly  $ac(X) \leq lc(X) \leq c(X)$  for every space  $X$ .

**Theorem 5.** *If  $X$  is a Hausdorff space then  $|X| \leq 2^{lc(X)\pi_\chi(X)\psi_c(X)}$ .*

PROOF: Let  $\lambda = lc(X)\pi_\chi(X)\psi_c(X)$  and let  $\kappa = 2^\lambda$ , let  $\tau$  be the topology on  $X$  and let  $F$  be a closed subset of  $X$  with  $|F| \leq \kappa$  and witnessing that  $lc(X) \leq \lambda$ . For every  $x \in X$  let  $\mathcal{B}_x$  be a local  $\pi$ -base at  $x$  such that  $|\mathcal{B}_x| \leq \lambda$ , and let  $f : X \rightarrow \mathcal{P}(\tau)$  be the map defined by  $f(x) = \mathcal{B}_x$  for every  $x \in X$ . Let  $A = \kappa \cup \{F, X, \tau, \kappa, f\}$  and take a set  $\mathcal{M} \supset A$  such that  $|\mathcal{M}| = \kappa$  and which reflects enough formulas so that the conditions (i)-(vi) listed in Theorem 1 are satisfied.

Claim:  $X \subset \mathcal{M}$  (and hence  $|X| \leq 2^{lc(X)\pi_\chi(X)\psi_c(X)}$ ). Suppose not and take  $p \in X \setminus \mathcal{M}$ . Let  $\{G_\alpha : \alpha \in \lambda\}$  be a family of open neighbourhoods of  $p$  such that  $\bigcap \{G_\alpha : \alpha \in \lambda\} = \{p\}$ . Set  $V_\alpha = X \setminus \overline{G_\alpha}$  and  $S_\alpha = X \cap \mathcal{M} \cap V_\alpha$  for every  $\alpha \in \lambda$ . Now let  $\mathcal{W}_\alpha = \{B : B \in \mathcal{B}_y, y \in S_\alpha \wedge B \subset V_\alpha\}$ , since  $lc(X) \leq \lambda$  it follows that there is a  $\mathcal{V}_\alpha \in [\mathcal{W}_\alpha]^{\leq \lambda}$  such that  $\bigcup \mathcal{W}_\alpha \subset F \cup \bigcup \{\overline{V} : V \in \mathcal{V}_\alpha\}$ . Since  $S_\alpha \subset \bigcup \overline{\mathcal{W}_\alpha}$  (let  $y \in S_\alpha$  and  $U$  be an open neighbourhood of  $y$ ,  $y \notin \overline{G_\alpha}$  so there is an open neighbourhood  $V$  of  $y$  such that  $V \cap G_\alpha = \emptyset$ , let  $B \in \mathcal{B}_y$  such that  $B \subset U \cap V$ ,  $\emptyset \neq B \subset (\bigcup \mathcal{W}_\alpha) \cap U$  and  $y \in \bigcup \overline{\mathcal{W}_\alpha}$ ) it follows that  $S_\alpha \subset F \cup \bigcup \{\overline{V} : V \in \mathcal{V}_\alpha\}$ ; moreover  $\bigcup \{\overline{V} : V \in \mathcal{V}_\alpha\} \in \mathcal{M}$  and  $p \notin F \cup \bigcup \{\overline{V} : V \in \mathcal{V}_\alpha\}$ . Set  $C_\alpha = \bigcup \{\overline{V} : V \in \mathcal{V}_\alpha\}$ , since  $X \cap \mathcal{M} \subset F \cup \bigcup \{C_\alpha : \alpha < \lambda\}$  and  $F \cup \bigcup \{C_\alpha : \alpha < \lambda\} \in \mathcal{M}$  it follows that  $F \cup \bigcup \{C_\alpha : \alpha < \lambda\} = X$ , a contradiction.  $\square$

By Theorem 5 it follows again that  $|X| \leq 2^{c(X)\chi(X)}$  for every  $T_2$ -space  $X$ . Moreover we have the following

**Corollary 6** ([6]). *If  $X$  is a  $T_3$ -space then  $|X| \leq 2^{c(X)\pi_\chi(X)\psi(X)}$ .*

**Remark 7.** A generalization of the inequality in corollary 6 has also been obtained by Sun in [8]: “ $|X| \leq 2^{c(X)\pi_\chi(X)\psi_c(X)}$  for every Hausdorff space  $X$ ”. Note that even this result is a corollary of Theorem 5. Moreover if  $X$  is the Michael line then  $|X| = 2^{lc(X)\pi_\chi(X)\psi_c(X)} < 2^{c(X)\pi_\chi(X)\psi_c(X)}$ . Observe also that the  $\pi$ -character cannot be omitted in Theorem 5 (see the comment at the end of Remark 3).

Now let us turn our attention to the Arhangel’skii’s inequality: “For  $X \in T_2$ ,  $|X| \leq 2^{L(X)t(X)\psi(X)}$ ”.

**Definitions.** Let  $X$  be a topological space.

(i) ([8]) A subset  $A$  of  $X$  with  $|A| \leq 2^\kappa$  is said to be  $\kappa$ -quasi-dense if for each open cover  $\mathcal{U}$  of  $X$  there exist a  $\mathcal{V} \in [\mathcal{U}]^{\leq \kappa}$  and a  $B \in [A]^{\leq \kappa}$  such that  $(\bigcup \mathcal{V}) \cup \overline{B} = X$ ;  $qL(X)$  is the smallest infinite cardinal  $\kappa$  such that  $X$  has a  $\kappa$ -quasi dense subset.

(ii)  $aqL(X)$  is the smallest infinite cardinal  $\kappa$  such that there is a subset  $S$  of  $X$  with  $|S| \leq 2^\kappa$  such that for every open cover  $\mathcal{U}$  of  $X$  there is a  $\mathcal{V} \in [\mathcal{U}]^{\leq \kappa}$  with  $X = S \cup (\bigcup \mathcal{V})$ .

Clearly  $aqL(X) \leq L(X)$  for every space  $X$ .

**Theorem 8.** *If  $X$  is a Hausdorff space then  $|X| \leq 2^{aqL(X)t(X)\psi_c(X)}$ .*

PROOF: Let  $\lambda = aqL(X)t(X)\psi_c(X)$ ,  $\kappa = 2^\lambda$ , let  $\tau$  be the topology on  $X$  and let  $S$  be an element of  $[X]^{\leq \kappa}$  witnessing that  $aqL(X) \leq \lambda$ . For every  $x \in X$  let  $\mathcal{B}_x$  be a family of open neighbourhoods of  $x$  with  $|\mathcal{B}_x| \leq \lambda$  and  $\bigcap \{\overline{B} : B \in \mathcal{B}_x\} = \{x\}$ , and let  $f : X \rightarrow \mathcal{P}(\tau)$  be the map defined by  $f(x) = \mathcal{B}_x$  for every  $x \in X$ . Let  $A = \kappa \cup \{S, X, \tau, \kappa, f\}$  and take a set  $\mathcal{M} \supset A$  such that  $|\mathcal{M}| = \kappa$  and which reflects enough formulas so that the conditions (i)–(vi) listed in Theorem 1 are satisfied. First observe that  $X \cap \mathcal{M}$  is a closed subset of  $X$ , although this fact follows from a general result which can be found in [4] we give a proof of it for the sake of completeness: let  $x \in \overline{X \cap \mathcal{M}}$ , since  $t(X) \leq \lambda$  there is a  $C \in [X \cap \mathcal{M}]^{\leq \lambda}$  such that  $x \in \overline{C}$ . Since  $C \in \mathcal{M}$  (by (i)), it follows that  $\overline{C} \in \mathcal{M}$  (by (iii)). Now it remains to observe that  $|\overline{C}| \leq \kappa$  (recall that  $t(X)\psi_c(X) \leq \lambda$ ) and hence by (vi)  $x \in \overline{C} \subset X \cap \mathcal{M}$ .

We have done if we show that  $X \subset \mathcal{M}$ . Suppose there is a  $p \in X \setminus \mathcal{M}$ , for every  $y \in X \cap \mathcal{M}$  let  $B_y \in \mathcal{B}_y$  such that  $p \notin B_y$ . Since  $\mathcal{U} = \{B_y : y \in X \cap \mathcal{M}\} \cup \{X \setminus \mathcal{M}\}$  is an open cover of  $X$  and  $aqL(X) \leq \lambda$  there is a  $\mathcal{V} \in [\mathcal{U}]^{\leq \lambda}$  such that  $X = S \cup (\bigcup \mathcal{V})$ . Let  $\mathcal{W} = \{B_y : B_y \in \mathcal{V}\}$ , since  $X \cap \mathcal{M} \subset S \cup (\bigcup \mathcal{W})$  and  $S \cup (\bigcup \mathcal{W}) \in \mathcal{M}$  it follows that  $X = S \cup (\bigcup \mathcal{W})$ , a contradiction ( $p \notin S \cup (\bigcup \mathcal{W})$ ).  $\square$

A consequence of Theorem 8 is the following generalization of the Arhangel'skii's inequality.

**Corollary 9** ([8]). *If  $X$  is a Hausdorff space then  $|X| \leq 2^{qL(X)t(X)\psi_c(X)}$ .*

PROOF: It is enough to note that  $aqL(X) \leq qL(X)t(X)\psi_c(X)$ .  $\square$

**Remark 10.** Let  $\kappa$  be an infinite cardinal number and let  $X$  be the discrete space of cardinality  $2^\kappa$ . This space shows that Theorem 8 can give a better estimation than the one in Corollary 9.

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