Decaying positive solutions of some quasilinear differential equations

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Abstract. The existence of decaying positive solutions in $\mathbb{R}_+$ of the equations (E_λ) and (E^1_λ) displayed below is considered. From the existence of such solutions for the subhomogeneous cases (i.e. $t^{1-p}F(r,tU,t|U'|) \searrow 0$ as $t \nearrow \infty$), a super-sub-solutions method (see § 2.2) enables us to obtain existence theorems for more general cases.

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1. Introduction

Let $F \in C([0,\infty)^3; \mathbb{R}_+)$ and $F_0 \in C([0,\infty)^2; \mathbb{R}_+)$ be such that

$$\text{(f)} \qquad \begin{cases} F(r,T,S) \le f(r)T^\gamma\,(1+S^q); \\ F_0(r,T) \le f(r)\,T^\gamma \\ \text{where} \quad \gamma, q \ge 0; \quad f(r) \simeq r^\theta \quad \text{at} \quad \infty, \quad \theta \in \mathbb{R}. \end{cases}$$

For $a > 1$ and $p \in (1, a+1)$, we investigate the existence of $(u,\lambda) \in C^1([0,\infty)) \times (0,\infty)$ which satisfy for $r \ge 0$ the equations

$$(E_\lambda) \qquad D_a u + \lambda r^a F^u(r) := (r^a|u'|^{p-2}u')' + \lambda r^a F(r,u,|u'|) = 0$$

$$(E^1_\lambda) \qquad \text{and} \quad D_a u + \lambda r^a F_0(r,u) = 0,$$

where u is positive and decaying element of

$$C^1_{ap} := \{u \in C^1([0,\infty)) \quad | \quad r^a|u'|^{p-2}u' \in C^1([0,\infty))\}.$$

For $a = n-1$, $n \in \mathbb{N}$ such u is a radial solution in $\mathbb{R}^n$ of the p-Laplacian equations
$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda F(|x|,u,|\nabla u|) = 0$ and
$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda F_0(|x|,u) = 0$, respectively.
We show that for $\gamma_0 + q_0 < p-1$
(i) such solution U exists for

$$(E^0) \qquad D_a U + r^a f(r) U^{\gamma_0}(1+|U'|^{q_0}) = 0, \quad r \ge 0;$$

(ii) there is $\lambda_0 \equiv \lambda(f,p) > 0$ such that

$$(E^0_{\lambda_0}) \qquad D_a u + \lambda_0 r^a f(r) u^{\gamma_0}(1+|u'|^{q_0}) = 0, \quad r \geq 0$$

has such a solution u_0, say, with $|u_0|_\infty, |u_0'|_\infty \in (0,1]$.
Using u_0 as a supersolution for (E_λ), we extend the result to more general cases where $\gamma \geq \gamma_0$, $q \geq q_0$ and $\lambda \in (0, \lambda_0)$.
We will also consider for $\sigma > 0$ and $\theta, \gamma, q \geq 0$ the equation

$$(F_\sigma) \qquad D_a V + \frac{\sigma r^a}{(1+r)^\theta} V^\gamma \{1 + |V'|^q\} = 0, \quad r \geq 0$$

in the goal to investigate the existence of solutions in C^1_{ap} for (F_σ) where F satisfies

$$(f_\theta) \qquad 0 \leq F(r,T,S) \leq (1+r)^{-\theta} T^\gamma (1 + S^q).$$

It is important to note that the usual condition $F(r,u,0) \not\equiv 0$ found in the literature for the decaying solutions ([7], [8]) is not required here as the use of a sub-super-solutions method enables us to circumvent that condition.
In the sequel the following notations and conventions will be used:
$\mu := 1/(p-1)$; $\quad t_* := \max\{1,t\}$; $\quad \int \phi := \int \phi(s)\,ds$;

$$(1.0) \qquad \begin{cases} w(t) := (1+t)^{-m}, & m = \mu b, \quad b \in (0, a+1-p] \\ \forall R > 0, \quad |u|_R := |u|_{C([0,R])} & \text{and} \quad \psi(t) := w(t)^\gamma f(t). \end{cases}$$

C or c will denote generic positive constants.
The main results are the following:

Theorem 1. *Suppose that $(\gamma_0 + q_0) < p-1$ and that*

$$(1.1) \qquad \int_0^\infty s^{b+p-1}\psi(s) < \infty \quad \text{or} \quad \gamma_0 < (p-1)\Big\{\frac{b+p+\theta}{b}\Big\}.$$

(1) Then (E^0) has a decaying positive solution $U \in C^1_{ap}$ such that at ∞,

$$(1.2) \qquad U(r) \leq C\, r^{-m} \quad (U(r) \simeq r^{-m} \text{ if } b = a+1-p).$$

Moreover $\exists\, \lambda_0 \equiv \lambda(f,p) > 0$ such that $(E^0_{\lambda_0})$ has a similar solution u_0, say, with $|u_0|_\infty, |u_0'|_\infty \in (0,1]$.
(2) For $\lambda \in (0,\lambda_0)$, $\gamma \geq \gamma_0$ and $q \geq q_0$, (E_λ) has a decaying positive solution $u \in C^1_{ap}$ which satisfies (1.2).

Theorem 2. *Suppose that $\theta \in [0,p]$. If*

$$\gamma > \frac{(p-1)\{a+1-\theta\}}{a+1-p}, \tag{1.3}$$

then $\forall\, q \geq 0$, (F_σ) has a decaying positive solution $V \in C^1_{ap}$ and for $\tau > 1$ such that $\gamma = (p-1)[a+1-p+\tau(p-\theta)]/(a+1-p)$, at ∞

$$V(r) \leq C\, r^{-(a+1-p)/\tau(p-1)}, \tag{1.4}$$

provided that σ is small enough e.g.

$$0 < \sigma < \Big\{\max(1, \frac{a+1-p}{\tau(p-1)})\Big\}^{\gamma+1-p} \Big(\frac{a+1-p}{\tau}\Big)^p (p-1)^{1-p}(\tau-1). \tag{1.5}$$

In particular if

$$\gamma \geq \gamma_1 := \{p^2 + (p-1)(a+1-p-\theta)\}/(a+1-p), \tag{1.6}$$

then $\forall\, q \geq 0$ and $0 < 2\sigma < \sigma_1 := (a+1)\{\frac{a+1-p}{p-1}\}^{\gamma_1}$,
(F_σ) has such a solution V with $V(r) \simeq r^{-(a+1-p)/(p-1)}$ at ∞.

Theorem 3. (1) *If $\gamma_0(p-1) < 1$ and* (1.1) *holds, then $\forall\, \lambda > 0$ and $\gamma = \gamma_0$, (E^1_λ) has a decaying positive solution $u_\lambda \in C^1_{ap}$ which satisfies* (1.2).
There is $\lambda_0 \equiv \lambda(f,p) > 0$ such that $(E^1_{\lambda_0})$ has such a solution u with $|u|_\infty, |u'|_\infty \in (0,1]$.
For $\lambda \in (0,\lambda_0)$ and $\gamma \geq \gamma_0$, (E^1_λ) has a decaying solution in C^1_{ap} which satisfies (1.2).
(2) *Let $\theta \in [0,p]$; for $\gamma > (p-1)(a+1-\theta)/(a+1-p)$ and $\tau > 1$ such that*

$$\gamma = (p-1)\,\frac{a+1-p+\tau(p-\theta)}{a+1-p} \tag{1.7}$$

and $0 < \lambda \leq \{\frac{a+1-p}{\tau}\}^p\,(p-1)^{1-p}(\tau-1)$,
(E^1_λ) has a decaying positive solution $u \in C^1_{ap}$ which satisfies (1.4). *In particular if $0 \leq F_0(r,u) \leq u^\gamma/(1+r)^\theta$, $\lambda \leq \{(a+1-p)/\tau\}^p(p-1)^{1-p}(\tau-1)$ and $\gamma \geq \gamma_1$, it has such a solution u such that $u(r) \simeq r^{-(a+1-p)/(p-1)}$ at ∞.*

Remarks 4. (1) In Theorem 1, when $p \geq 2$, θ has to be less than $-p$ and even for this case the existence of solutions for $\gamma > p-1$ is an extension of the known results ([7], [8]).
(2) As concerned (E^1_λ) with F_0 in (f) and $a = n-1$, radial solutions in $C^1([0,\infty)) \cap C^2((0,\infty))$ are known to exist ([3]) for

$\gamma \geq \frac{(p-1)n+p}{n-p}$ if $\theta = 0$; $\gamma > \frac{(p-1)n+p(1+\theta)}{n-p}$ if $\theta \in (-p,0)$;
$p-1 < \gamma < \frac{(p-1)n+p}{n-p}$ if $\theta < -p$;
$\gamma < p-1$ with $\theta < -p$ ([6]).

So, the existence of solutions of (E^1_λ) in C^1_{ap} for $\gamma > \frac{(p-1)(n-\theta)}{n-p}$ and $\theta \in [0,p]$ provided by Theorem 3 seems to be new.

2. Preliminaries

2.1. Properties of some integrals.

Define

$$J(t) := \int_t^\infty \Big(\int_0^r (\frac{s}{r})^a\,\psi(s)\Big)^\mu \quad \text{and} \quad K(t) := J(t)/w(t); \tag{2.1}$$

$$\nu := \begin{cases} 0 & \text{if } \ b = a+1-p \\ a+1-p-b & \text{if } \ b \in (0, a+1-p); \end{cases} \tag{2.2a}$$

$$\Psi_0 := \begin{cases} \frac{1}{m}\big(\int_0^1 s^a\psi\big)^\mu & \text{if } \ b = a+1-p \\ \frac{p-1}{a+1-p}\{\int_0^1 s^a\,\psi(s)\}^\mu & \text{if } \ b < a+1-p; \end{cases} \tag{2.2b}$$

$$\Psi_1 := 2^m\Big\{\int_0^1\Big(\int_0^r \psi\Big)^\mu + \frac{1}{m}\Big(\int_0^\infty s^{b+p-1}\psi\Big)^\mu\Big\}. \tag{2.2c}$$

Lemma 2.1. *If*

$$\int_0^\infty s^{b+p-1}\,\psi(s) < \infty \quad \textit{or} \quad \gamma > (p-1)\frac{(b+p+\theta)}{b}, \tag{2.3}$$

where $b \in (0, a+1-p]$, *then* $\forall\, t \geq 0$

$$\Psi_0 t_*^{-\nu/(p-1)} \leq K(t) \leq \Psi_1; \tag{2.4}$$

$$|J(t)'| \leq \Big(\int_0^\infty (1+s^{b+p-1})\psi\Big)^\mu \quad t_*^{-m-1} := \Psi^1\, t_*^{-m-1}. \tag{2.5}$$

Proof: $J(t) = \int_t^\infty r^{-m-1}\{r^{-a+b+p-1}\int_0^r s^a\psi\}^\mu \leq \int_t^\infty r^{-m-1}(\int_0^\infty s^{b+p-1}\psi)^\mu$ on one hand and
$J(t) \leq \int_0^1(\int_0^r \psi)^\mu + \int_1^\infty(\int_0^\infty s^{b+p-1}\psi)^\mu$ on the other hand; the right hand side of (2.4) then follows from the fact that $(1+t)^m t_*^{-m} \leq 2^m$.
$0 \leq -J(t)' \leq t^{-m-1}(\int_0^\infty s^{b+p-1}\psi)^\mu$ on one hand and
$|J(t)'| \leq (\int_0^\infty \psi)^\mu$ on the other hand; (2.5) is obtained.
$J(t) = \int_t^\infty r^{-a\mu}(\int_0^r s^a\,\psi(s))^\mu \geq (\int_0^1 s^a\,\psi(s))^\mu \int_t^\infty r^{-a\mu}\,dr$ for $t \geq 1$ and for $t < 1$, $J(t) \geq J(1)$. So
$J(t) \geq \Psi_0 t_*^{-(a+1-p)/(p-1)}$ whence $K(t) \geq \Psi_0 t_*^{-\nu/(p-1)}$.
The left hand side of (2.4) is then obtained. □

For $B > A > 0$ define for $C^1 := C^1([0,\infty))$

(2.6) $E := E(A,B) =$

$\{v \in C^1;\ A \leq v \leq B;\ |(wv)'| \leq Bt_*^{-m-1}\}$ if $b = a+1-p$,

$\{v \in C^1;\ 0 \leq v \leq B;\ V \geq A$ in $[0,1];\ |(wv)'| \leq Bt_*^{-m-1}\}$ otherwise.

Define the operator G on E by

$$G\phi(t) := (1+t)^m\int_t^\infty \Big\{r^{-a}\int_0^r s^a\,\psi(s)\phi(s)^\gamma(1+|(w\phi)'|^q)\Big\}^\mu. \tag{2.7}$$

Lemma 2.2. *If (2.3) holds, then $G : E \longrightarrow C^1$ is continuous and GE is equicontinuous in C^1.*

Proof: With $F_1^u := u^\gamma(1 + |(wu)'|^q)$, $\forall\, u, v \in E$,
$\Gamma_1(A) := A^\gamma \leq F_1^u \leq B^\gamma(1+B^q) := \Gamma_2(B)$ and $|F_1^u - F_1^v| \leq C(\gamma, q, A, B)|u-v|_{C^1}$;
Γ standing for $\Gamma_1(A)$ or $\Gamma_2(B)$ according to the sign of $\mu - 1$,

$$(2.8)\quad \Big|\Big(\int_0^r (\frac{s}{r})^a\, \psi(s) F_1^u(s)\Big)^\mu - \Big(\int_0^r (\frac{s}{r})^a\, \psi(s) F_1^v(s)\Big)^\mu\Big| \leq \mu\{\Gamma \int_0^r (\frac{s}{r})^a \psi\}^{\mu-1} \int_0^r (\frac{s}{r})^a\, \psi(s)|F_1^u - F_1^v| \leq C_1(\mu, C, \Gamma)|u-v|_{C^1}\Big\{\int_0^r (\frac{s}{r})^a \psi\Big\}^\mu.$$

From (2.8) simple estimations lead to

$$(2.9)\qquad |(Gu - Gv)'(t)| + |(Gu - Gv)(t)| \leq C\,|u - v|_{C^1}\{|K(t)'| + K(t)\}$$

and the continuity is obtained via Lemma 2.1.

(i) $\forall\, u \in E$,
$|(Gu(t)'| \leq \Gamma^\mu\{(1+t)^m|K(t)'| + m(1+t)^{m-1}K(t)\} \leq C\,(\Gamma, B, \psi)$
by Lemma 2.1 whence GE is equicontinuous in $C([0, \infty))$.

(ii) $\forall\, t > s > 0$ and $u \in E$,
$|(Gu)'(t) - (Gu)'(s)| \leq \Gamma^\mu\{|(1+t)^m t^{-a} - (1+s)^m s^{-a}|(\int_0^s y^a \psi(y))^\mu +$
$+m|(1+t)^{m-1} - (1+s)^{m-1}|K(t) + m(1+s)^{m-1}|K(t) - K(s)|\} := O(t-s)$
and $\{(Gu)' \mid u \in E\}$ is equicontinuous in $C([0, \infty))$. The equicontinuity follows from (i) and (ii).

□

2.2 A super-sub-solutions method.
Consider for $h \in C([0, \infty)^3;\ \mathbb{R}_+)$

$$(\mathrm{H})\qquad H(v) := D_a v + r^a h^v(r) \equiv (r^a|v'|^{p-1}v')' + r^a h(r, v, |v'|) = 0.$$

Definition 2.3. (1) Let $v \in C^1([0, \infty))$ be piecewise C^2. v will be said to be a **supersolution (subsolution)** of (H) if
$H(v) \leq \quad (\geq) \quad 0 \qquad \forall$ a.e. $r \geq 0$.
(2) $w, v \in C^1([0, \infty))$ piecewise C^2 will be said to be **H-compatible** if
$\forall$ a.e. $r \geq 0 \qquad 0 \leq w(r) \leq v(r); \quad v'(r) \leq w'(r) \leq 0; \quad H(v) \leq 0 \leq H(w).$

Lemma 2.4. *Suppose that h^u is non decreasing in u and $|u'|$. Let $w, v \in C^1([0, \infty))$ be H-compatible with $|v|_{C^1} \equiv |v|_{C^1([0,\infty))} < \infty$. Then*
$D_a V + r^a h^v(r) := (r^a|V'|^{p-2}V')' + r^a h^v(r) = 0$ *and* $D_a W + r^a h^w(r) = 0$
have solutions $V, W \in C^1_{ap}$ such that $\forall\, r \geq 0$,

$$(2.10)\qquad w \leq W \leq V \leq v \;\text{ and }\; v' \leq V' \leq W' \leq w' \leq 0.$$

Proof: The existence of solutions of the equations in the lemma is in no doubt in view of the hypotheses on v. We are going to indicate how to construct those which satisfy (2.10). Define the sequences

$$v_n(r) = \begin{cases} v(n) + I_n v(r) & \text{for } r < n \\ v(r) & \text{otherwise} \end{cases}$$

where $I_n v(r) := \int_r^n (\int_0^t (s/t)^a h^v)^\mu$, $\mu := 1/(p-1)$.

$$D_a v_n + r^a h^v(r) = 0 \text{ in } B_n = [0,n); \quad V_n = v \text{ for } r \geq n.$$

w_n are defined from w in the same way.
In B_n, $v_n(r)' = -(\int_0^r (s/r)^a h^v)^\mu \leq -(\int_0^r (s/r)^a h^w)^\mu = w_n(r)'$.
As $v', (v_n)' \leq 0$ in B_n, $\{r^a[|(v_n)'|^{p-1} - |v'|^{p-1}]\}' \leq 0$ whence $v' \leq (v_n)' \leq (w_n)'$ there. Thus $w_n \leq v_n \leq v$ as $v(n) = v_n(n) \geq w(n) = w_n(n)$. Similarly in B_n, $w \leq w_n$ and $(w_n)' \leq w'$. So, $\forall\, n \in \mathbb{N}$ $w \leq w_n \leq v_n \leq v$ and $v' \leq (v_n)' \leq (w_n)' \leq w' \leq 0$.
So, $\forall\, M > 0$ and $B_M := [0, M)$,

$$n > M \Longrightarrow |w_n|_{C^1(\overline{B_M})} \leq |v|_{C^1} \text{ and } |v_n|_{C^1(\overline{B_M})} \leq |v|_{C^1}$$

whence (w_n) and (v_n) have subsequences $(\bar{w_n})$ and $(\bar{v_n})$ say, which converge in $C^1(\overline{B_M})$ to W_M and V_M say, such that for some $w(M) \leq a_M \leq b_M \leq v(M)$, in B_M $W_M(r) = a_M + I_M w(r)$ and $V_M(r) = b_M + I_M v(r)$.
In the same way $(\bar{w_n})_{n>2M}$ and $(\bar{v_n})_{n>2M}$ have subsequences which converge in $C^1(\overline{B_{2M}})$ to W_{2M} and V_{2M} say, and $W_{2M}|_{B_M} = W_M$, $V_{2M}|_{B_M} = V_M$.
W and V are obtained as inductive limit of $(W_{kM})_{k\in\mathbb{N}}$ and $(V_{kM})_{k\in\mathbb{N}}$ ([5]). □

Theorem 2.5. (1) *Suppose that the hypotheses on w and v in the Lemma 2.4 hold. Then* (H) *has a solution $\phi \in C^1_{ap}$ such that $w \leq \phi \leq v$.*

(2) *The existence of such a positive and decreasing supersolution v for* (H) *is sufficient for the existence of a non trivial solution $u \in C^1_{ap}$ of* (H) *such that $0 \leq u \leq v$.*

Proof: (1) Define on $E = \{\phi \in C^1([0,\infty)) \mid w \leq \phi \leq v \text{ and } v' \leq \phi' \leq w'\}$ the operator I by $I\phi(t) := A + \int_t^\infty (\int_0^r (s/r)^a h^\phi(s))^\mu$ where $A := \lim_\infty v(r)$.
(a) Let $\Phi = I\phi$ for $\phi \in E$;
$h^w \leq h^\phi \leq h^v$ whence using the same arguments as in Lemma 2.4,
$IE \subset E$ as $W \leq \Phi \leq V$ and $V' \leq \Phi' \leq W'$, W and V being those in that lemma.
(b) The continuity of $I : E \longrightarrow E$ is easy to verify, following the same steps (with slight modifications) as for Lemma 2.2.
(c) IE is equicontinuous as: (i) $\forall\, \phi \in E$ and $t > s > 0$,

(2.11) $|\Phi'(t) - \Phi'(r)|$

$$\leq \begin{cases} \left\{\frac{t^a - s^a}{t^a}\left(\frac{1}{s}\int_0^s rh^v\right) + \frac{1}{t}\int_s^t rh^v\right\}^\mu & \text{if } \mu \leq 1, \\ \mu\left(\frac{1}{s^a}\int_0^t r^a h^v\right)^{\mu-1}\left\{\frac{t^a - s^a}{t^a}\left(\frac{1}{s}\int_0^s rh^v\right) + \frac{1}{t}\int_s^t rh^v\right\} & \text{if } \mu > 1 \end{cases}$$

and $\{\Phi' \mid \phi \in E\}$ is equicontinuous as a subset of $C([0,\infty))$;
(ii) $|\Phi'(t)| \le |v'|_\infty$ whence IE is equicontinuous as a subset of $C([0,\infty))$.

As E is a closed and convex subset of C^1, the three reasons enable us to apply the Schauder-Tychonoff fixed point theorem to I; I has a fixed point in E which is such a solution.

(2) For $\sigma \ge \mu(2a-p)$ and $z(r) = r^{-\sigma}$ in $D = [1,\infty)$, $D_a z > 0$ in D. Let $\rho > 0$ be such that $z < v/2$ and $v' \le z' \le 0$ for $r > \rho$. Define z_1 and z_2 by

$$z_1(r) = \begin{cases} z(\rho) & \text{for } r \le \rho \\ z(r) & \text{for } r > \rho \end{cases} \quad \text{and} \quad z_2(r) = \begin{cases} 0 & \text{for } r \le \rho \\ |z'(r)| & \text{for } r > \rho. \end{cases}$$

For $h_1^z := h(r, z_1, z_2)$, the function Z constructed from v as W in Lemma 2.4 with h_1^z replacing h^v is such that Z, v are H-compatible and (1) applies. □

Without any extra difficulties, Definition 2.3, Lemma 2.4 and Theorem 2.5 apply to (H) where rather $h \in C([0,\infty)^2; \mathbb{R}_+)$ and $h(r,u)$ non decreasing in $u \ge 0$.

3. Proofs of the main theorems

3.1. Proof of Theorem 1. Let E be that in (2.6). $\forall\, \phi \in E$,
$G\phi(t) = (1+t)^m \int_t^\infty \{\int_0^r (\frac{s}{r})^a\, \psi(s)\phi(s)^{\gamma_0}(1+|(w\phi)'|^{q_0})\}^\mu \le B^{\mu\gamma_0}(1+B^{q_0})^\mu K(t)$
$\le B^{\mu\gamma_0}(1+B^{q_0})^\mu \Psi_1$ by (2.4).
$|(wG\phi)'(t)| \le B^{\mu\gamma_0}(1+B^{q_0})^\mu |J(t)'| \le \Psi^1 B^{\mu\gamma_0}(1+B^{q_0})^\mu$ by (2.5).
For $t \in [0,1]$ if $b < a+1-p$,
$G\phi(t) \ge \int_1^\infty \{\int_0^r (\frac{s}{r})^a\, \psi(s)\phi(s)^{\gamma_0}\}^\mu \ge A^{\mu\gamma_0} J(1) \ge A^{\mu\gamma_0}\frac{1}{m}(\int_0^1 s^a \psi)^\mu := N_2 A^{\mu\gamma_0}$
and for $b = a+1-p$ similar lower bound is obtained $\forall\, t \ge 0$.
$GE \subset E$ if we can find $B > A > 0$ such that

$$\{B^{\gamma_0}(1+B^{q_0})\}^\mu(\Psi^1+\Psi_1) \le B \quad \text{and} \quad N_2 A^{\mu\gamma_0} \ge A. \tag{3.1}$$

Because $\mu(\gamma_0+q_0) < 1$, in $\{(x,y);\quad x>0, y>0\}$ the curve of $y = x$ lies above that of $y = \{x^{\gamma_0}(1+x^{q_0})\}^\mu(\Psi^1+\Psi_1)$ for
$x \ge x_0 \equiv x_0(\Psi^1, \Psi_1, \gamma_0, q_0)$. Also $N_2 A^{\mu\gamma_0} \ge A$ for $A \ge A_0 := A_0(N_2)$ as $\mu\gamma_0 < 1$.
So, with $A_1 := \min\{x_0, A_0\}$,
$\forall\,(A,B) \in (0, A_1] \times [x_0, \infty)$, (3.1) holds and for such A and B, $GE \subset E$.
In that case, as from Lemma 2.2 G is continuous on E and GE equicontinuous in E, G has a fixed point ϕ, say, in E as E is a closed and convex subset of C^1 by Schauder-Tychonoff fixed point theorem. $U(t) := w(t)\phi(t)$ is such a required solution.
For the equation $(E^0_{\lambda_0})$, with $B = 1$,(3.1) reads

$$(2\lambda_0)^\mu(\Psi^1+\Psi_1) \le 1 \quad \text{and} \quad N_2\lambda_0^\mu A^{\mu\gamma_0} \ge A. \tag{3.1a}$$

So, for $\lambda_0 = (1/2)(\Psi^1+\Psi_1)^{-1/\mu}$ and some $A \in (0,1)$, we obtain U_0 as U obtained above.
For $\lambda \in (0, \lambda_0)$, $\gamma \ge \gamma_0$ and $q \ge q_0$ U_0 is a supersolution of (E_λ) and Theorem 2.5 applies.

3.2 Proof of Theorem 2. From Theorem 2.5, it suffices to find a supersolution of the problem in C^1. Define

$$v(r) := (1+r^s)^{-\beta}; \quad s>1; \quad \beta>0, \tag{3.3}$$

then for $a>1$ and $p\in(1,a+1)$

$$D_a v = -r^a \frac{(s\beta)^{p-1} r^{(s-1)(p-1)-1}}{(1+r^s)^{\beta(p-1)+p}} \{(s-1)(p-1)+a+r^s(a+1-p-s\beta(p-1))\}.$$

For $s=p/(p-1)$ and $\beta=(a+1-p)/\tau p,\ \tau>1$,

$$D_a v + r^a \left\{\frac{a+1-p}{\tau(p-1)}\right\}^{p-1} \left\{\frac{a+1+[(a+1-p)(\tau-1)/\tau]r^s}{(1+r^s)^{\beta(p-1)+p}}\right\} = 0. \tag{3.4}$$

This implies that

$$D_a v + \{\frac{a+1-p}{\tau}\}^p (p-1)^{1-p}(\tau-1)\, r^a\,(1+r^s)^{-(p-1)(\beta+1)} \le 0$$

whence $\forall\,\theta\ge 0$

$$\begin{cases} D_a v + D\,\frac{r^a v^\gamma}{(1+r)^\theta} \le 0, \quad r\ge 0 \\ \forall\,\gamma \ge \gamma(\tau,\theta) := (p-1)\frac{a+1-p+\tau(p-\theta)}{a+1-p}\,; \\ D := D(a,p,\tau) = \left(\frac{a+1-p}{\tau}\right)^p (p-1)^{1-p}\,(\tau-1). \end{cases} \tag{3.5}$$

For $v_0 = \max\{1, \frac{a+1-p}{\tau(p-1)}\}$ and $V(r) = v(r)/v_0,\ V(r), |V(r)'| \in [0,1] \quad \forall r\ge 0$ hence

$$\begin{cases} \forall\,\gamma \ge \gamma(\tau,\theta), \sigma\in(0, \quad v_0^{1+\gamma-p}D/2] \ \text{ and } \ q\ge 0 \\ D_a V + \sigma\frac{r^a V^\gamma}{(1+r)^\theta}(1+|V'|^q) \le 0, \quad r\ge 0, \end{cases} \tag{3.6}$$

V is then a supersolution of (F_σ). The proof is completed by the fact that $\forall\,\gamma > (p-1)(a+1-p+\tau(p-\theta)\,)/(a+1-p)$ and $\theta\le p$, there is $\tau>1$ such that $\gamma=\gamma(\tau,\theta)$. For $\tau=1$ in (3.4) and $v_0=(a+1-p)/(p-1)$, (3.6) becomes

$$\begin{cases} D_a V + \frac{\sigma r^a V^\gamma}{(1+r)^\theta}\,(1+|V'|^q) \le 0, \quad r\ge 0 \\ \forall\,q\ge 0, \quad \sigma<\sigma_1 \ \text{ and } \ \gamma\ge\gamma_1. \end{cases} \tag{3.7}$$

The proof is completed by Theorem 2.5.

3.3 Proof of Theorem 3. (1) Adapting the proof of Theorem 1 to (E^1_λ), we see that $GE\subset E$ if for any $\lambda>0$, there are $B>A>0$ such that

$$\lambda^\mu B^{\mu\gamma_0}(\Psi^1+\Psi_1) \le B \quad\text{and}\quad \lambda^\mu A^{\mu\gamma_0} N_2 \ge A;$$

the fact that $\mu\gamma_0<1$ ensures the existence of such A and B.

As $\mu\gamma_0<1$, this part of (1) follows the same process as for Theorem 1. In the same manner, the part (2) of the theorem is obtained by a simple adaptation of the proof of Theorem 2.

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