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On the Noetherian type of topological spaces

S.A. PEREGUDOV

Abstract. The Noetherian type of topological spaces is introduced. Connections between the Noetherian type and other cardinal functions of topological spaces are obtained.

Keywords: Noetherian type, rank weight

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Let X be a topological space and \mathcal{B} be an open family in X . For a set G let us denote by \mathcal{B}_G the family $\{B \in \mathcal{B} : G \subset B\}$.

We define the *Noetherian type* of \mathcal{B} as the cardinal

$$Nt(\mathcal{B}) = \min \{ \alpha : \alpha \text{ is an infinite cardinal} \\ \text{and } |\mathcal{B}_G| < \alpha \text{ for every nonempty open set } G \subset X \}.$$

We define the *lower Noetherian type* of \mathcal{B} as the cardinal

$$lNt(\mathcal{B}) = \sup \{ |\mathcal{B}_G| : G \text{ is a nonempty open subset of } X \}.\omega.$$

The cardinal

$$\min \{ Nt(\mathcal{B}) : \mathcal{B} \text{ is a base of the space } X \}$$

is called the *Noetherian type* of X and is denoted by $Nt(X)$.

The cardinal

$$\min \{ lNt(\mathcal{B}) : \mathcal{B} \text{ is a base of the space } X \}$$

is called the *lower Noetherian type* of X and is denoted by $lNt(X)$.

Considering in the last definitions a π -base in place of a base we obtain definitions of the *Noetherian π -type* and the *lower Noetherian π -type* of X , which are denoted by $N\pi t(X)$ and $lN\pi t(X)$ respectively.

Now let \mathcal{B} be a family of sets. The cardinal

$$rank(\mathcal{B}) = \sup \{ |\mathcal{B}'| : \mathcal{B}' \subset \mathcal{B}, \bigcap \mathcal{B}' \neq \emptyset \text{ and } \mathcal{B}' \text{ is an antichain} \\ \text{(by the set theoretic inclusion)} \}$$

is called the *rank of the family \mathcal{B}* . We define the *rank weight of a topological space X* as the cardinal

$$w_r(X) = \min \{ rank(\mathcal{B}) : \mathcal{B} \text{ is a base of } X \}.\omega.$$

Analogously the *rank π -weight* of X is defined as the cardinal

$$\pi w_r(X) = \min \{ rank(\mathcal{B}) : \mathcal{B} \text{ is a } \pi\text{-base of } X \}.\omega.$$

Lemma 1. *Let X be a topological space. Then*

$$w(X) = lNt(X).\pi w(X) \quad \text{and} \quad lNt(X).\chi(X) = lNt(X).\pi\chi(X).$$

PROOF: Clearly $lNt(X).\pi w(X) \leq w(X)$. Let \mathcal{B} be a base of X such that $|\mathcal{B}| = w(X)$ and $lNt(\mathcal{B}) = lNt(X)$. Let \mathcal{H} be a π -base of X such that $|\mathcal{H}| = \pi w(X)$. For every set $B \in \mathcal{B}$ choose a set $H \in \mathcal{H}$ such that $H \subset B$. We obtain a mapping $f : \mathcal{B} \rightarrow \mathcal{H}$. Since $|f^{-1}.(H)| \leq lNt(X)$ and $\mathcal{B} = \bigcup \{f^{-1}.(H) : H \in \text{rng}(f)\}$ it follows that $|\mathcal{B}| \leq lNt(X).\pi w(X)$. Consequently, $w(X) = lNt(X).\pi w(X)$. Clearly for every point $x \in X$ we have $\chi(x, X) \leq \pi\chi(x, X).lNt(X)$. Consequently, $\chi(X) \leq \pi\chi(X).lNt(X)$ and $\chi(X).lNt(X) \leq \pi\chi(X).lNt(X)$. The inverse inequality is trivial. \square

Lemma 2. *Let X be a topological space. If $Nt(X) < \chi(X)$ then $w_r(X) = \chi(X)$.*

PROOF: Clearly for every base \mathcal{B} of the space X and every point $x \in X$ we have $\text{ord}(x, \mathcal{B}) \leq \chi(x, X).lNt(\mathcal{B})$. Consequently, provided that $Nt(X) < \chi(X)$ we have that $w_r(X) \leq \chi(X)$. Now let \mathcal{B} be an arbitrary base of X and let \mathcal{B}' be a base of X such that $Nt(\mathcal{B}') = Nt(X)$. Take an arbitrary cardinal τ such that $Nt(X) \leq \tau < \chi(X)$. Choose a point $x \in X$ such that $\chi(x, X) \geq \tau^+$. Let $\mathcal{B}'_x = \{B' \in \mathcal{B}' : x \in B'\}$ and for every set $B' \in \mathcal{B}'_x$ choose a set $B'' \subset B'$ such that $B'' \in \mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}$. We obtain a family $\mathcal{B}'' \subset \mathcal{B}$ such that $Nt(\mathcal{B}'') \leq Nt(X)$ and $x \in \bigcap \mathcal{B}''$. Besides it is evident that $|\mathcal{B}''| \geq \tau^+$. Choose a maximal antichain \mathcal{B}''_0 out of \mathcal{B}'' . Consider the family of all sets $B \in \mathcal{B}''$ such that B is contained as a proper subset in some set belonging to \mathcal{B}''_0 . Choose a maximal antichain \mathcal{B}''_1 out of this family. Continuing this process we obtain a $Nt(X)$ -sequence $(\mathcal{B}''_\xi : \xi < Nt(X))$ such that every its element \mathcal{B}''_ξ is a maximal antichain in the family $\{B \in \mathcal{B}'' : B \text{ is contained as a proper subset in some set belonging to } \mathcal{B}''_\gamma \text{ for every } \gamma < \xi\}$. The family $\mathcal{B}''_{Nt(X)} = \bigcup \{\mathcal{B}''_\xi : \xi < Nt(X)\}$ is dense in \mathcal{B}'' . Therefore it is a local base of x in X . Then $|\mathcal{B}''_{Nt(X)}| \geq \tau^+$ and there exists $\xi < Nt(X)$ such that $|\mathcal{B}''_\xi| \geq \tau^+$. Hence, $\text{rank}(\mathcal{B}) \geq \tau^+$. Since it is true for every cardinal τ , such that $Nt(X) \leq \tau < \chi(X)$, we have that $\text{rank}(\mathcal{B}) \geq \chi(X)$. Because \mathcal{B} is an arbitrary base of X it follows that $w_r(X) \geq \chi(X)$. \square

Definitions. A topological space X is called a *Noetherian space* provided that $Nt(X) = \omega$. A topological space X is called a *weakly Noetherian space* provided that $lNt(X) = \omega$.

Corollary 1. *If X is a Noetherian space then $w_r(X) = \chi(X)$.*

Example 1. Let $X = \omega_1 \cup \{\omega_1\}$. Introduce a topology on X as the following. Let every point of ω_1 be isolated. A base of neighborhoods of the point ω_1 is defined as the family $\{(\xi, \omega_1) : \xi < \omega_1\}$ where $(\xi, \omega_1) = \{\eta \leq \omega_1 : \xi < \eta\}$. Evidently X is a regular Lindelöf space for which $lNt(X) = \omega$, $Nt(X) = \omega_1$, $\chi(X) = \omega_1$, and $w_r(X) = \omega$.

Lemma 3. *Let X be a compact Hausdorff space. If $Nt(X)$ is a regular cardinal and $w(X) = Nt(X)$ then $w_r(X) = w(X)$.*

PROOF: Let us assume that $w_r(X) < w(X)$. The space X contains an everywhere dense subset A such that $\pi\chi(x, X) \leq w_r(X)$ for every point $x \in A$ ([2]). Choose a base \mathcal{B} of the space X such that $Nt(\mathcal{B}) = Nt(X)$. Then $ord(x, \mathcal{B}) < Nt(X)$ for every point $x \in A$. By induction in just the same way as in [6], construct a sequence $(S_n : n \in \omega)$ of subsets of the set A such that the following conditions are fulfilled:

- (1) if $n, m \in \omega$ and $n < m$ then $S_n \subset S_m$;
- (2) $|S_n| < w(X)$ for every $n \in \omega$;
- (3) if $n \in \omega$, \mathcal{B}' is a finite subfamily of the family $\{B \in \mathcal{B} : B \cap S_n \neq \emptyset\}$ and $A \setminus \bigcup \mathcal{B}' \neq \emptyset$ then $S_{n+1} \cap (A \setminus \bigcup \mathcal{B}') \neq \emptyset$.

The set $S = \bigcup \{S_n : n \in \omega\}$ by (3) is an everywhere dense subset of X . In addition $|S| < w(X)$ and $ord(x, \mathcal{B}) < w(X)$ for every point $x \in S$. Then $w(X) < w(X)$, a contradiction. It follows that $w_r(X) = w(X)$. □

Theorem 1. *Let X be a compact Hausdorff space. Then*

$$w(X) = lNt(X).\pi w(X) = lNt(X).\pi\chi(X) = lNt(X).t(X) = lNt(X).\chi(X) = lNt(X).w_r(X) = lNt(X).s(X) = lNt(X).hd(X) = lNt(X).hL(X).$$

PROOF: Since X is a compact Hausdorff space it follows from [7] that $\pi\chi(X) \leq t(X)$. By Lemma 1 it implies the first, the third, and the fourth equations. Further, if \mathcal{B} is a base of the space X and $lNt(\mathcal{B}) = lNt(X)$ then $ord(x, \mathcal{B}) \leq lNt(X).\pi\chi(X)$ for every point $x \in X$. By Theorem Mishenko then $|\mathcal{B}| \leq lNt(X).\pi\chi(X)$. It implies the second equation. Now let us assume that $lNt(X).w_r(X) < w(X)$. Then $w_r(X) < w(X)$ and $lNt(X) < w(X)$. By the fourth equation it implies that $w(X) = \chi(X)$ and by Lemma 2 we have $Nt(X) \geq w(X)$. But $w(X) > lNt(X)$, hence $Nt(X) = w(X) = lNt(X)^+$. Then by Lemma 3 we have that $w_r(X) = w(X)$. It is a contradiction, hence $w(X) = lNt(X).w_r(X)$. The other equations are consequences of the inequality $t(X) \leq s(X)$ for compact Hausdorff spaces ([1]). □

Corollary 2. *Let X be a Hausdorff compact weakly Noetherian space. Then $w(X) = \pi w(X) = w_r(X) = \chi(X) = \pi\chi(X) = t(X) = s(X) = hd(X) = hL(X)$.*

Corollary 3. *Let X be a Hausdorff locally compact weakly Noetherian space. Then $w_r(X) = \chi(X)$.*

Corollary 4. *Let X be a compact Hausdorff space. If $Nt(X)$ is a weakly inaccessible cardinal then $w_r(X) = w(X)$.*

Example 2. Let X denote the “two arrows” space. It is known that $w_r(X) = \omega$ ([3]). By Theorem 1 it implies that $lNt(X) = w(X) = 2^\omega$ and because $cf(2^\omega) > \omega$ we get that $Nt(X) = (2^\omega)^+$.

Example 3. Let $X = I^\tau$ where I is the unit segment. Because $lNt(X) = Nt(X) = \omega$ ([5]), it follows from Theorem 1 that $w_r(X) = \tau \cdot \omega$.

Example 4. The space of ordinals $X = \omega_1 \cup \{\omega_1\}$ is not weakly Noetherian because $\chi(X) = \omega_1$ and $w_r(X) = \omega$ ([2]).

Theorem 2. For a compact Hausdorff space X the following conditions are equivalent:

- (1) $w(X) = lNt(X)$;
- (2) $lNt(Y) \leq lNt(X)$ for every subspace Y of X ;
- (3) $lNt(Y) \leq lNt(X)$ for every closed subspace Y of X .

PROOF: Evidently it is sufficient to prove the implication (3) \rightarrow (1). Let the condition (3) be fulfilled and suppose that $w(X) > lNt(X)$. Then by Theorem 1 there exists a discrete set $Y \subset X$ such that $|Y| > lNt(X)$. By the assumption $lNt(clY) \leq lNt(X)$. Choose a base \mathcal{B} of the space clY such that $lNt(\mathcal{B}) = lNt(clY)$. Because $ord(y, \mathcal{B}) \leq lNt(clY)$ for every point y belonging to Y , we have by [6] that $w(clY) \leq lNt(clY)$. Hence $|Y| \leq lNt(X)$. It is a contradiction, consequently, $w(X) = lNt(X)$. □

Theorem 3. If Y is an open or canonical closed subspace of a space X then $lNt(Y) \leq lNt(X)$. If X is a Hausdorff space and \mathcal{F} is a family of compact subspaces of X having in X the character $\leq lNt(X)$, then if $\bigcup \mathcal{F} \subset Y \subset cl \bigcup \mathcal{F}$ then $lNt(Y) \leq lNt(X)$.

PROOF: The first assertion is trivial. To prove the second assertion, take $\mathcal{E}_F = \{A \subset F : A \neq \emptyset, \chi(A, X) \leq lNt(X)\}$ for every $F \in \mathcal{F}$ and put $\mathcal{E} = \bigcup \{\mathcal{E}_F : F \in \mathcal{F}\}$. Let \mathcal{B} be a base of X such that $lNt(\mathcal{B}) = lNt(X)$. If $B \in \mathcal{B}$ and B contains a nonempty open in Y set P , then there exists $E \in \mathcal{E}$ such that $E \subset P$. Because $\chi(E, X) \leq lNt(\mathcal{B})$, it follows that $|\{B \in \mathcal{B} : B \supset E\}|$ is not more than $lNt(\mathcal{B})$. This implies that $lNt(\mathcal{B}|Y) \leq lNt(\mathcal{B})$ and hence $lNt(Y) \leq lNt(X)$. □

Theorem 4. Let $\{Z_\alpha : \alpha \in A\}$ be a family of topological spaces and $\prod \{Z_\alpha : \alpha \in A\}$ is denoted by Z . Then

$$Nt(Z) \leq \sup\{Nt(Z_\alpha) : \alpha \in A\} \text{ and } lNt(Z) \leq \sup\{lNt(Z_\alpha) : \alpha \in A\}.$$

PROOF: To prove this, choose a base \mathcal{B}_α of the space Z_α for every $\alpha \in A$ such that $Nt(\mathcal{B}_\alpha) = Nt(Z_\alpha)$. It is easy to see that a base of Z consisting of sets of the form $B_{\alpha_1} \times \dots \times B_{\alpha_n} \times \prod \{Z_\alpha : \alpha \in A \setminus \{\alpha_1, \dots, \alpha_n\}\}$, where $B_{\alpha_i} \in \mathcal{B}_{\alpha_i}$ for $i = 1, \dots, n$, have the Noetherian type that is not greater than $\sup\{Nt(Z_\alpha) : \alpha \in A\}$. Analogously it can be proved that $lNt(Z) \leq \sup\{lNt(Z_\alpha) : \alpha \in A\}$. □

Remark. The following theorem, which has been proved in [4], is an essential supplement of Theorem 4:

$$\text{if } |A| \geq \sup\{w(Z_\alpha) : \alpha \in A\} \text{ then } Nt(Z) = \omega.$$

Theorem 5. *Let X be a topological space such that $\pi w(X) > N\pi t(X)$ and let κ be a cardinal such that κ^+ is a calibre of X . If $\pi w(X) > \kappa$ then $\pi w_r(X) > \kappa$.*

PROOF: Let \mathcal{H} be a π -base of X . Choose a π -base \mathcal{H}' of X such that $Nt(\mathcal{H}') = N\pi t(X)$. Now choose a cardinal τ such that $N\pi t(X) \cdot \kappa \leq \tau < \pi w(X)$. Take a mapping $f : \mathcal{H}' \rightarrow \mathcal{H}$ such that $f(H') \subset H'$ and put $\mathcal{H}'' = rng(f)$. It is evident that \mathcal{H}'' is a π -base of X . Also it is clear that $|\mathcal{H}''| \geq \pi w(X)$ and $Nt(\mathcal{H}'') = N\pi t(X)$. Consequently $|\mathcal{H}''| \geq \tau^+$. By induction construct a $N\pi t$ -sequence $(\mathcal{H}''_\xi : \xi < N\pi t(X))$ of subsets of \mathcal{H}'' such that the following conditions are fulfilled:

- (1) \mathcal{H}''_0 is a maximal antichain (by inclusion) in \mathcal{H}'' ;
- (2) if $\xi > 0$ then \mathcal{H}''_ξ is a maximal antichain in the family $\{H \in \mathcal{H}'' : \text{for every } \eta < \xi \text{ there exists } H' \in \mathcal{H}''_\eta \text{ such that } H \subset H' \text{ and } H \neq H'\}$.

Put $\mathcal{H}''_{N\pi t(X)} = \bigcup \{\mathcal{H}''_\xi : \xi < N\pi t(X)\}$. It is easy to see that $\mathcal{H}''_{N\pi t(X)}$ is a π -base of X . This implies that $|\mathcal{H}''_{N\pi t(X)}| \geq \tau^+$. Because $N\pi t(X) \leq \tau$, there exists $\xi < N\pi t(X)$ such that $|\mathcal{H}''_\xi| \geq \tau^+$. Since $\kappa \leq \tau$, there exists $\widehat{\mathcal{H}}_\xi \subset \mathcal{H}''_\xi$ such that $|\widehat{\mathcal{H}}_\xi| > \kappa$ and $\bigcap \widehat{\mathcal{H}}_\xi \neq \emptyset$. Because \mathcal{H} is an arbitrary π -base we get that $\pi w_r(X) > \kappa$.

Corollary 5. *Let X be a topological space such that $\pi w(X) > N\pi t(X)$. Then the following assertions are fulfilled:*

- (a) if $\pi w(X) > d(X)$ then $\pi w_r(X) = \pi w(X)$;
- (b) if $\pi w(X) > sh(X)$ then $\pi w_r(X) > sh(X)$.

The following problem was raised by P. Bir'ukov:

Is there a compact Hausdorff space X such that $|X| > 2^{w_r(X)}$?

In view of the above mentioned assertions the space may be encountered only where $Nt(X) = w(X)^+$ or $Nt(X) = w(X)$ is a singular cardinal.

Corollary 6 (MA). *Let X be a compact Hausdorff space. If $Nt(X) \leq 2^\omega$ then $|X| \leq 2^{w_r(X)}$.*

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