Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 38 (1997), No. 2, 205--221

Persistent URL: http://dml.cz/dmlcz/118918

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Comment.Math.Univ.Carolin. 38,2 (1997)205–221

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Relative multiplication and distributive modules

José Escoriza, Blas Torrecillas

Abstract. We study the construction of new multiplication modules relative to a torsion theory τ. As a consequence, τ-finitely generated modules over a Dedekind domain are completely determined. We relate the relative multiplication modules to the distributive ones.

Keywords: torsion theory, semicentered torsion theory, multiplication module, distributive module

Classification: 13A15, 13G13

1. Introduction

Multiplications rings constitute an important class of rings and they have been studied by many authors (cf. [7], [8], [10], [18], [19], and [20]). They are generalizations of Dedekind domains. Two concepts of multiplication module have been given. The first one was due to Singh and Mehdi (cf. [11]) and the second one, the most spread, was introduced by Barnard (cf. [2]). Multiplication modules have been recently considered by many authors, either over a commutative ring ([5], [9], [14] and their references) or over a noncommutative ring (cf. [13], [18], [19] and [20]). Multiplication modules relative to a torsion theory have been defined and studied in [6] as a natural generalization of the absolute case.

The aim of this paper is to study the operations of relative multiplication modules in the commutative case. It is a work which will serve to research into the noncommutative case, which will be exposed in a subsequent paper. Section 2 is devoted to preliminaries and notation. We also include some results on relative multiplication ring and ideals. In [6], it was observed that every Krull domain with the canonical torsion theory is a τ-multiplicaton ring. Now, some examples of τ-multiplication rings which are not Krull domains are given. In Section 3, firstly, some properties for any hereditary torsion theory are found and are applied to find out if a module is or not relative multiplication. Then, operations such as intersection, sum, direct sum, multiplication, etc, between multiplication modules relative to a torsion theory have been studied. Finally, these results are applied to find out what modules over a Dedekind domain are τ-multiplication. In Section 4, relative distributive modules are introduced. Distributive modules have been studied in [1], [2], [4] and [17]. Relative distributive rings have been researched by Năstăsescu (cf. [12]). Some elemental properties of relative distributive modules are shown. It is found the relationship between relative distributive modules and relative multiplication modules in the main theorem. In the case of perfect torsion theories the distributive property of a module is characterized in terms of distributive property for its module of quotients with respect to the torsion theory.

2. Preliminaries and general notation

Throughout this paper, τ is a hereditary torsion theory on a commutative ring R and $M \in R$-*Mod*. The Gabriel filter associated to τ is denoted by $\mathcal{F}$ and the set $Spec(R) - \mathcal{F}$ is denoted by $K(\tau)$. A torsion theory τ is *semicentered* (cf. [3], [16]) if for each $I \notin \mathcal{F}$ there exists a prime ideal P such that $P \notin \mathcal{F}$ and $I \subseteq P$. The ring R *has enough τ-criticals* if for every ideal $I \notin \mathcal{F}$ of R there exists an ideal P such that $I \subseteq P$ and P is maximal with this condition. The set of such ideals is denoted by $Max_{\mathcal{F}}(R)$.

We shall give some easy properties of closure operations that will be useful for future results. If S is a multiplicatively closed subset of R, then $S^{-1}\tau$ is the induced torsion theory by τ on the ring of quotients $S^{-1}R$, whose Gabriel filter is $\{S^{-1}I; I \leq R\}$. If $P \in Spec(R)$, then τ_P is the induced torsion theory in R_P with Gabriel filter $\mathcal{F}_P = \{I_P; I \in \mathcal{F}\}$.

Let M, N be two R-modules. We denote by $(M : N) = \{r \in R; r.N \subseteq M\}$ and by $(M_P : N_P) = \{x \in R_P; x.N_P \subseteq M_P\}$ where P is any prime ideal of R.

R_τ and M_τ represent the ring and the module of quotients with respect to τ respectively (cf. [16]).

The following lemma recollects some useful technical results. They are well-known and the proof is omitted.

Lemma 2.1. *Let S be a multiplicatively closed subset of R. Let τ be a semicentered torsion theory in R-Mod. Let $P \in K(\tau)$. Let M, N be two R-modules. Let $L \leq M$. Then*

1. $S^{-1}Cl^M_\tau(L) = Cl^{S^{-1}M}_{S^{-1}\tau}(S^{-1}L)$;
2. $Cl^{M_P}_{\tau_P}(L_P) = L_P$;
3. *if N is τ-finitely generated, then $(M : N)_P = (M_P : N_P)$ for all $P \in K(\tau)$;*
4. *if M is τ-finitely generated, then, for every $P \in K(\tau)$, $(ann(M))_P = ann(M_P)$.*

Recall that an R-module M is called *τ-multiplication* if for every τ-closed submodule N of M there exists an ideal I of R such that $N = Cl^M_\tau(I.M)$.

The definition of strongly τ-multiplication module is a generalization of Singh and Mehdi's definition (cf. [11]) for multiplication modules.

Definition 2.2. *An R-module M is called strongly τ-multiplication if for all τ-closed submodules $N \subseteq L$, there exists an ideal I of R such that $N = Cl^M_\tau(I.L)$.*

A ring R is called *τ-multiplication* if given A, B τ-closed ideals of R such that $B \subseteq A$, then there exists an ideal I of R verifying $B = Cl^R_\tau(I.A)$.

Obviously, every ring R is τ-multiplication as an R-module and it is strongly τ-multiplication as an R-module if and only if it is a τ-multiplication ring.

Example 2.3. Consider $M = \mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module. Let τ be such that $(p) \in \mathcal{F}$. Since M is τ-simple, it is strongly τ-multiplication, but it is not multiplication according to Singh and Mehdi's definition (cf. [11]).

Let R be an integral domain and let K be its field of quotients. Let τ be a torsion theory in R-*Mod*. A is a *fractional ideal* of R if there exists $d \in R$ such that $d.A \subseteq R$ and it is an R-module.

Definition 2.4. *A fractional ideal A of R is called τ-invertible if there exists a fractional ideal B such that $Cl_\tau^K(A.B) = R$.*

Proposition 2.5. *Every τ-invertible ideal is τ-multiplication.*

PROOF: Let A be a τ-invertible ideal and $B \subseteq A$ another ideal of R such that $Cl_\tau^A(B) = B$. Then, there exists a fractional ideal C such that $Cl_\tau^K(A.C) = R$. Therefore, we have $B = Cl_\tau^A(B.R) = Cl_\tau^K(B.Cl_\tau^K(A.C)) \cap A = Cl_\tau^K(B.C.A) \cap A = Cl_\tau^A((B.C).A)$. Moreover, since $B.C \subseteq A.C$, $B.C$ is an ideal of R. □

If A is an integral ideal and $Cl_\tau^R(A.B) = R$ for some fractional ideal B, then A is a τ-multiplication ideal, i.e., it is τ-multiplication as an R-module. In particular, every ideal belonging to the Gabriel filter is τ-multiplication.

It follows immediately that the product of two τ-invertible ideals is τ-invertible and therefore, it is τ-multiplication.

In [6] it is proved that a Krull domain with the canonical torsion theory is a τ-multiplication ring. The following example is a ring which is not multiplication but is τ-multiplication and is not a Krull domain.

Example 2.6. Let $R = \Pi_{i \in \mathbb{N}} R_i$ where $R_i = \mathbb{Z}_4$. According to [8, Example 3], R is not a multiplication ring. Set $S = \oplus_{i \in \mathbb{N}} R_i$. Obviously, $S^2 = S$ and it is possible to consider the Gabriel filter $\mathcal{F} = \{A \leq R; S \subseteq A\}$. If B is an ideal of R, then $Cl_\tau^R(B) = (B : S)$ clearly. Denote by e_i the element of R which has the i-th coordinate equal to 1 and the others are 0. If $x \in B$ and B is τ-closed, then each component x_i of x has to verify $x_i.e_i \in B$. Let $x = (x_i)_{i \in \mathbb{N}}$ verifying the preceding condition. If $s \in S$, the product $y.s$ can be seen as a finite sum of elements of B and therefore, it belongs to B. This means that $B = \Pi_{i \in \mathbb{N}}(B \cap R_i)$. Thus τ-closed ideals are ideals of the form $B = \Pi_{i \in \mathbb{N}} B_i$ with $B_i \leq \mathbb{Z}_4$. Let A, B be τ-closed ideals of R such that $A \subseteq B$. Since $\mathbb{Z}_4$ is a multiplication ring (it is uniserial), for every $i \in \mathbb{N}$, there exists an ideal C_i of $\mathbb{Z}_4$ such that $A_i = C_i.B_i$. Consequently, $A = Cl_\tau^R(C.B)$, where $C = \Pi_{i \in \mathbb{N}} C_i$.

Remark 2.7. Notice that the ring of quotients with respect to τ is

$$R_\tau = Hom_R(S, R) = \Pi_{i \in \mathbb{N}} Hom_R(R_i, R) = \Pi_{i \in \mathbb{N}} R_i = R.$$

Since R_τ is not a multiplication ring but R is a τ-multiplication ring, it is proved that Proposition 4.14 in [6] is not necessarily true if τ is not perfect.

Example 2.8. If R is a τ-multiplication ring and $\mathcal{F}$ is the corresponding Gabriel filter, then $R\oplus R$ is multiplication with respect to the torsion theory whose Gabriel filter is $\{(I,J); I,J \in \mathcal{F}\}$. Therefore, if D is a Krull domain and τ is the canonical torsion theory, then $D \oplus D$ is a relative multiplication ring and obviously, it is not a Krull domain.

In the relative noetherian case, the following characterization is immediate from [6, Theorem 4.18].

Proposition 2.9. *If R is τ-noetherian and τ is semicentered, then R is a τ-multiplication ring if and only if R_P is a multiplication ring for each $P \in K(\tau)$.*

Some examples of relative multiplication rings appear in [6] and other examples are obtained in forecoming sections.

3. Operations with τ-multiplication modules

If $P \in Spec(R)$, then the set $\{x \in M; c.m = 0 \text{ for some } c \in R - P\}$ is denoted by $T_P(M)$. An R-module M is called *P-torsion* if $M = T_P(M)$.

The starting point is the following result, which appears in [6].

Proposition 3.1. *Let τ be a semicentered torsion theory on R. M is a τ-multiplication module if and only if for all $P \in K(\tau)$, M is P-torsion or $c.M \subseteq Cl^M_\tau(R.m)$ for some $m \in M$ and $c \in R - P$.*

Proposition 3.2. *If R is τ-noetherian (τ-artinian) and M is a τ-multiplication module, then M is τ-noetherian (τ-artinian).*

Proof: Show that M has A.C.C. on τ-closed submodules. In fact, we consider $N_1 \leq N_2 \leq \dots$ with $N_i \leq M$ τ-closed ($i \in I$). Then $N_i = Cl^M_\tau((N_i : M).M)$ by [6, Lemma 3.11]. But $Cl^M_\tau(N_i : M) = (Cl^M_\tau(N_i) : M)$ from [6, Proposition 2.7] and therefore $(N_i : M)$ is a τ-closed ideal for every $i \in I$. Moreover, $(N_1 : M) \leq (N_2 : M) \leq \dots$. By hypothesis, there exists i such that $(N_i : M) = (N_{i+1} : M) = (N_{i+2} : M) = \dots$ and hence

$$N_i = Cl^M_\tau((N_i : M).M) = Cl^M_\tau((N_{i+1} : M).M) = N_{i+1} = \dots$$

and therefore M is τ-noetherian. For the artinian case the proof is analogous.

□

The converse result is false. In fact, consider the ring $\mathbb{Z}$ which is τ-noetherian for any τ. Let $M = \mathbb{Z} \oplus \mathbb{Z}$. M is τ-noetherian but it is not τ-multiplication for any torsion theory τ different from the trivial one (cf. [6, Lemma 3.13]).

Example 3.3. Let $M = \mathbb{Z}[x_1, x_2, \cdots]$ be the $\mathbb{Z}$-module consisting of all polynomials in infinite indeterminates $x_1, x_2, \cdots$ By [16, Corollary VI.6.15], every torsion theory on $\mathbb{Z}$ is semicentered. If τ is different from the trivial one, then M is not τ-noetherian, obviously. By applying Proposition 3.2, M is not τ-multiplication.

Definition 3.4. *Let τ,σ be torsion theories on R with Gabriel filters $\mathcal{F}_\tau$ and $\mathcal{F}_\sigma$ respectively. Then $\tau \wedge \sigma$ is the torsion theory whose Gabriel filter is $\mathcal{F}_\tau \cap \mathcal{F}_\sigma$.*

Proposition3.5. *Let τ and σ be two hereditary torsion theories on R. If $M \in$ Mod-R is τ and σ-multiplication, then M is a $\tau \wedge \sigma$-multiplication module.*

Proof: Let N be $\tau \wedge \sigma$-closed. Then N is τ-closed and σ-closed. By hypothesis and by [6, Lemma 3.11], $N = Cl_\tau^M((N : M).M) = Cl_\sigma^M((N : M).M)$. So, for every $n \in N$ there exist $I_n \in \mathcal{F}_\tau$ and $J_n \in \mathcal{F}_\sigma$ such that $I_n.n \subseteq (N : M).M$ and $J_n.n \subseteq (N : M).M$. Then $I_n \cap J_n \in \mathcal{F}_\tau \cap \mathcal{F}_\sigma$ verifying $(I_n \cap J_n).n \subseteq (N : M).M$. Hence $N = Cl_{\tau\wedge\sigma}^M((N : M).M)$. □

Compare the next result with [14, Lemma 7].

Proposition 3.6. *If M is a τ-multiplication R-module and $M = \sum_{i\in I} M_i$, then $N = Cl_\tau^M(\sum_{i\in I}(N \cap M_i))$ for each N τ-closed submodule of M.*

Proof: Since M is τ-multiplication module we have

$$N = Cl_\tau^M((N : M).M) = Cl_\tau^M((N : M).(\sum_{i\in I} M_i)) \subseteq Cl_\tau^M((N : M)\sum_{i\in I} M_i).$$

Thus $N \subseteq Cl_\tau^M(\sum_{i\in I}(M_i \cap N))$. Therefore $N = Cl_\tau^M(\sum_{i\in I}(N \cap M_i))$. □

Proposition 3.7. *Let τ be a semicentered torsion theory on R and let $M = \sum_{i\in I} Cl_\tau^M(R.m_i)$ for some elements $m_i \in M$ $(i \in I)$. M is a τ-multiplication module if and only if there exists an ideal J_i $(i \in I)$ such that $Cl_\tau^M(R.m_i) = Cl_\tau^M(J_i.M)$ for each $i \in I$.*

Proof: The necessity is clear.

Conversely, suppose the existence of such ideals J_i and let $P \notin \mathcal{F}$. If there exists $i \in I$ such that $J_i \not\subseteq P$, then, by hypothesis, $J_i.M \subseteq Cl_\tau^M(R.m_i)$. Hence, there exists $c \in J_i - P \subseteq R - P$ such that $c.M \subseteq Cl_\tau^M(R.m_i)$. If not, we have $J_i \subseteq P$ for all $i \in I$. So $Cl_\tau^M(R.m_i) = Cl_\tau^M(J_i.M) \subseteq Cl_\tau^M(P.M)$ for all $m_i \in M$ and $M = Cl_\tau^M(P.M)$ by the hypothesis. Therefore, there exists an ideal J_i of R such that $Cl_\tau^M(R.m_i) = Cl_\tau^M(J_i.M) = Cl_\tau^M(J_i.Cl_\tau^M(P.M)) = Cl_\tau^M(P.J_i.M) = Cl_\tau^M(P.Cl_\tau^M(J_i.M)) = Cl_\tau^M(P.m_i)$. Thus $m_i \in Cl_\tau^M(P.m_i)$. So, there exists $H_i \in \mathcal{F}$ such that $H_i.m_i \subseteq P.m_i$ and moreover $H_i \not\subseteq P$. Thus there exists $h - p \in R - P$ such that $(h - p).m_i = 0$ and hence $m_i \in T_P(M)$. Then $R.m_i \subseteq T_P(M)$. Obviously, $Cl_\tau^M(T_P(M)) = T_P(M)$. Thus $Cl_\tau^M(R.m_i) \subseteq T_P(M)$ and therefore $M = T_P(M)$. By Proposition 3.1, M is τ-multiplication. □

It is straightforward from Proposition 3.7 that every τ-cyclic module is a τ-multiplication module.

Example 3.8. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty} = \{\frac{1}{p^i} + \mathbb{Z}; i \in \mathbb{Z}^*\} \subset \mathbb{Q}/\mathbb{Z}$ where p is a prime. If $(p) \in \mathcal{F}$, then $M = Cl_\tau^M(\mathbb{Z}.(\frac{1}{p} + \mathbb{Z}))$ and by Proposition 3.7, M is a τ-multiplication module.

If $(p) \notin \mathcal{F}$, then M is not τ-noetherian. By Proposition 3.2, it is not τ-multiplication.

Proposition 3.9. *Let τ be a semicentered torsion theory on R. If I is a τ-multiplication ideal of R and M is a τ-multiplication R-module, then $I.M$ is a τ-multiplication R-module.*

Proof: Consider $P \in K(\tau)$. It is clear that if $I = T_P(I)$ or $M = T_P(M)$, then $T_P(I.M) = I.M$. If $I \neq T_P(I)$ and $M \neq T_P(M)$, then there exist $c, d \in R - P$ such that $c.I \subseteq Cl_\tau^R(R.a)$ and $d.M \subseteq Cl_\tau^M(R.m)$ for some $a \in R$ and $m \in M$. Therefore $c.d.I.M \subseteq Cl_\tau^R(R.a).Cl_\tau^M(R.m) \subseteq Cl_\tau^M(R.a.m)$. Hence by Proposition 3.1, $I.M$ is a τ-multiplication module. □

The next result answers the question of when the sum of τ-multiplication modules is τ-multiplication. It is the analogous one to [14, Theorem 2].

Theorem 3.10. *Let τ be a semicentered torsion theory on R. Let M_i $(i \in I)$ be a family of τ-multiplication τ-closed submodules of an R-module M such that $M = \sum_{i\in I} M_i$. Let $A = \sum_{i\in I}(M_i : M)$. Then the following conditions are equivalent:*

1. *M is a τ-multiplication module;*
2. *$M_i = Cl_\tau^M((M_i : M).M)$ for all $i \in I$;*
3. *$ann(m) + A \in \mathcal{F}$ for all $m \in M$;*
4. *for every $P \in K(\tau)$ either $M = T_P(M)$ or there exist $z \in \cup_{i\in I} M_i$ and $c \in R - P$ such that $c.M \subseteq Cl_\tau^M(R.z)$.*

Proof: $1 \Rightarrow 2$ is clear. Now suppose 2 holds. Suppose that $m \in M$ and $ann(m) + A \notin \mathcal{F}$. Since τ is semicentered, there exists $P \in K(\tau)$ such that $ann(m) + A \subseteq P$. So, $(M_i : M) \subseteq P$ for all $i \in I$. Hence $(M_i : M).M \subseteq P.M$ and we have $M_i = Cl_\tau^M((M_i : M).M) \subseteq Cl_\tau^M(P.M)$. Thus $M = Cl_\tau^M(P.M)$. As $m \in M$, then $m = x_1 + x_2 + \cdots + x_n$ with $x_i \in M_i$ for $i \in \{1, 2, \ldots, n\}$. Since M_i is a τ-multiplication module, we have $Cl_\tau^M(R.x_i) = Cl_\tau^M(B_i.M_i)$ for some ideal B_i of R. Then, by the same argument as in the proof of Proposition 3.7, $Cl_\tau^M(R.x_i) = Cl_\tau^M(P.x_i)$. Therefore there exists $K_i \in \mathcal{F}$ such that $K_i.x_i \subseteq P.x_i$ for each $i \in \{1, 2, \ldots, n\}$. Hence there exists $c_i \in R - P$ such that $c_i.x_i = 0$ for each $i \in \{1, 2, \ldots, n\}$. Therefore there exists $c \in R - P$ such that $c.m = 0$. But then, $c \in ann(m)$ which contradicts $ann(m) \subseteq P$. Thus 3 is satisfied.

$3 \Rightarrow 4$. Let $P \in K(\tau)$ and suppose that $T_P(M) \neq M$. Then there exists $m \in M$ such that $ann(m) \subseteq P$. By condition 3, $A \nsubseteq P$. Therefore, there exists $i \in I$ such that $(M_i : M) \nsubseteq P$. Hence there exists $c \in R - P$ such that $c.M \subseteq M_i$. Moreover, $M_i \neq T_P(M_i)$ because if not, then $c.M \subseteq T_P(M_i)$ and M would be P-torsion. By Proposition 3.1, there exist $c' \in R - P$ and $y \in M_i$ such that $c'.M_i \subseteq Cl_\tau^M(R.y)$. Therefore $c.c'.M \subseteq c'.M_i \subseteq Cl_\tau^M(R.y)$ and $c.c' \in R - P$.

$4 \Rightarrow 1$ by Proposition 3.1. □

Remark 3.11. The result is still true if $M = Cl_\tau^M(\sum_{i\in I} M_i)$.

Corollary 3.12. *Let τ be a semicentered torsion theory on R. Let M_i $(i \in I)$ be a family of τ-multiplication τ-closed submodules of an R-module M. If $\sum_{i\in I}(M_i : M) \in \mathcal{F}$, then M is a τ-multiplication module.*

Proof: We have $M = Cl_\tau^M((\sum_{i\in I}(M_i : M)).M) \subseteq Cl_\tau^M(\sum_{i\in I}(M_i : M).M) \subseteq Cl_\tau^M(\sum_{i\in I} M_i)$. Since $ann(m) + \sum_{i\in I}(M_i : M) \in \mathcal{F}$, it suffices to apply Theorem 3.10 and Remark 3.11. □

In these conditions we denote by $A = \sum_{i\in I}(M_i : M)$.

Corollary 3.13. *Let τ be a semicentered torsion theory on R. Let M_i $(i \in I)$ be a family of τ-closed τ-multiplication finitely generated submodules of M. If $M = \sum_{i\in I} M_i$, then M is a τ-multiplication module if and only if $ann(M_i) + A \in \mathcal{F}$.*

Proof: Suppose that M is τ-multiplication and $M_i = \langle x_1, \dots, x_n\rangle$ (n depending on i). From Theorem 3.21, $ann(x_j) + A \in \mathcal{F}$ $(1 \le j \le n)$. Hence

$$[ann(x_1) \cap \cdots \cap ann(x_n)] + A \supseteq \Pi_{j=1}^n (ann(x_j) + A) \in \mathcal{F}.$$

Therefore $ann(M_i) + A = [\cap_{j=1}^n ann(x_j)] + A \in \mathcal{F}$ for all $i \in I$.

Now, suppose that $ann(M_i) + A \in \mathcal{F}$ for all $i \in I$. Let $m \in M$. Since $M = \sum_{i\in I} M_i$, $m = m_1 + \cdots + m_r$ with $m_j \in M_j$ $(1 \le j \le r)$. Since $ann(m_j) \supseteq ann(M_j)$, $ann(m_i) + A \in \mathcal{F}$ for $1 \le j \le n$. Moreover, $ann(m) + A = [\cap_{j=1}^n ann(m_j)] + A \supseteq \Pi_{j=1}^n (ann(m_j) + A) \in \mathcal{F}$. Therefore $ann(m) + A \in \mathcal{F}$ for all $m \in M$. By Theorem 3.10, M is a τ-multiplication module. □

Example 3.14. Let $M = \oplus_{n=1}^\infty C_{p^n}$ where C_{p^n} is the cyclic group of order p^n and p a prime integer. M is a $\mathbb{Z}$-module. Every C_{p^i} is cyclic and therefore it is τ-multiplication.

If $(p) \notin \mathcal{F}$, then every C_{p^i} is τ-closed. Moreover, $(C_{p^i} : M) = 0$. Thus $A = 0$. It holds $ann(C_{p^i}) + A = (p^i) \notin \mathcal{F}$ for all $i \ge 0$. By Corollary 3.13, M is not τ-multiplication.

Suppose that $(p) \in \mathcal{F}$. Let $x = x_{i_1} \oplus \cdots \oplus x_{i_n} \in M$ where each $x_{i_j} \in C_{p^j}$. We have $(p^{i_n}).x = 0$. Hence M is τ-torsion. Therefore M is τ-multiplication in this case.

Corollary 3.15. *Let τ be a semicentered torsion theory on R. Let $M = \sum_{i\in I} M_i$, M_i being a τ-closed τ-multiplication finitely generated submodule of M for all $i \in I$. M is τ-finitely generated if and only if there exists a finite subset $J \subseteq I$ such that $\sum_{i\in J}(M_i : M) \in \mathcal{F}$.*

Proof: Since M is τ-finitely generated, there exists a finitely generated submodule F of M such that $M = Cl_\tau^M(F)$. Therefore there exists a finite subset J of I such that $M = Cl_\tau^M(\sum_{i\in J} M_i)$. By Theorem 3.10 and Remark 4.12, $ann(m) + \sum_{i\in J}(M_i : M) \in \mathcal{F}$ for all $m \in M$, in particular for all $m \in F$. As F is finitely generated, it holds that $ann(F) + \sum_{i\in J}(M_i : M) \in \mathcal{F}$. However, $ann(F) \subseteq (M_i : M)$ for all $i \in J$. Hence $\sum_{i\in J}(M_i : M) \in \mathcal{F}$.

Conversely, suppose that $\sum_{i\in J}(M_i : M) \in \mathcal{F}$ for some finite subset J of I. By Theorem 3.10, M is τ-multiplication. Moreover, $M = Cl_\tau^M((\sum_{i\in J}(M_i : M)).M) = Cl_\tau^M(\sum_{i\in J}(M_i : M).M) = Cl_\tau^M(\sum_{i\in J} M_i)$. Therefore M is τ-finitely generated. □

Corollary 3.16. *Let τ be a semicentered torsion theory on R. Let $K, L_1, \dots, L_n$ be τ-closed submodules of M. If $K, K+L_i$ $(1 \leq i \leq n)$, $L_1 \cap \dots \cap L_n$ are τ-multiplication modules, then $K + (L_1 \cap \dots \cap L_n)$ is a τ-multiplication module.*

Proof: Let $P \in K(\tau)$. Call $L = L_1 \cap \dots \cap L_n$. Clearly, L is τ-closed. Suppose that $T_P(K+L) \neq K+L$. Then $T_P(K+L_i) \neq K+L_i$ for each $1 \leq i \leq n$. Consider $A = (K : (K+L_i)) + (L_i : (K+L_i))$. By applying Theorem 3.10 to $K+L_i$, we obtain $A \nsubseteq P$ as there exists $m \in K+L_i$ such that $ann(m) + A \subseteq P$. However, $A = (K : L_i) + (L_i : K)$. Since $(K : L_i) \subseteq (K : L)$, we deduce that $(K : L) + (L_i : K) \nsubseteq P$ for $1 \leq i \leq n$. Hence $(K : L) + (L : K) = (K : L) + [(L_1 : K) \cap \dots \cap (L_n : K)] \nsubseteq P$. Therefore there exists $c' \in R - P$ such that $c' = a_1 + a_2$ with $a_1 \in (K : L)$ and $a_2 \in (L : K)$. Thus there exists $c \in R - P$ (a_1 or a_2) such that $c \in (K : L)$ or $c \in (L : K)$. Hence $c.L \subseteq K$ or $c.K \subseteq L$ and therefore $c.(K+L) \subseteq K$ or $c.(K+L) \subseteq L$. By [6, Corollary 4.24], $K+L$ is a τ-multiplication module. □

Corollary 3.17. *Let τ be a semicentered torsion theory on R. If K, L are τ-closed submodules of an R-module M such that $(K : L) + (L : K) \in \mathcal{F}$, then $K+L$ is a τ-multiplication module.*

Lemma 3.18. *Let τ be a semicentered torsion theory on R. Let N_1 and N_2 be τ-closed submodules of an R-module M. If N_1, N_2 and N_1+N_2 are τ-multiplication, then $N_1 \cap N_2$ is a τ-multiplication module.*

Proof: Let $P \in K(\tau)$. If $T_P(N_1 \cap N_2) \neq N_1 \cap N_2$, then it is clear that $T_P(N_1) \neq N_1$, $T_P(N_2) \neq N_2$ and $T_P(N_1+N_2) \neq N_1+N_2$. By Theorem 3.10, there exist $x \in N_1$, $y \in N_2$, $z \in N_1 \cup N_2$, $c_1, c_2, c \in N$ such that $c_1.N_1 \subseteq Cl_\tau^{N_1}(R.x)$, $c_2.N_2 \subseteq Cl_\tau^{N_2}(R.y)$ and $c.(N_1+N_2) \subseteq Cl_\tau^{N_1+N_2}(R.z)$.

Suppose $z \in N_1$ (similarly if $z \in N_2$). Then $Cl_\tau^{N_1}(R.z) \subseteq Cl_\tau^{N_1+N_2}(N_1) = N_1$. Moreover, $c.y \in N_2$ because $y \in N_2$, and $c.y \in N_1$ because $c.y \in c.(N_1+N_2) \subseteq N_1$. Therefore we have $c_2.c.(N_1 \cap N_2) \subseteq c.Cl_\tau^{N_2}(R.y) \subseteq Cl_\tau^{N_2}(R.c.y)$.

On the other hand, it is obvious that $c_2.c.(N_1 \cap N_2) \subseteq N_1$. So, there exists $c_2.c \in R - P$ with $c_2.c.(N_1 \cap N_2) \subseteq Cl_\tau^{N_2}(R.c.y) \cap N_1 = Cl_\tau^{N_1 \cap N_2}(R.c.y)$ and by Proposition 3.1, $N_1 \cap N_2$ is a τ-multiplication module. □

Theorem 3.19. *Let τ be a semicentered torsion theory on R. Let $N_1, \dots, N_k$ be τ-closed submodules of an R-module M such that $N_i + N_j$ is a τ-multiplication module for all i, j, such that $1 \leq i < j \leq k$. Then*

1. *$N_1 + \dots + N_k$ is a τ-multiplication module;*
2. *$N_1, \dots, N_k$ are τ-multiplication modules if and only if $N_1 \cap \dots \cap N_k$ is a τ-multiplication module.*

Proof: To prove the first part, it suffices to follow the proof of [14, Theorem 8] with slight modifications. Proposition 3.1 and Theorem 3.10 are needed.

For the second part, we use induction on k. Suppose $N_1, \dots, N_k$ are τ-multiplication modules. Consider the τ-multiplication module $X = N_2 \cap \dots \cap N_k$.

By Corollary 3.16, $N_1 + X$ is a τ-multiplication module and by Lemma 3.18, $N_1 \cap X$ is a τ-multiplication module.

Let $P \in K(\tau)$. If $T_P(N_1 + N_i) = N_1 + N_i$, then $T_P(N_1) = N_1$. Suppose that $N_1 \cap \cdots \cap N_k$ is a τ-multiplication module. Let $P \in K(\tau)$. Suppose $T_P(N_1 + N_i) \neq N_1 + N_i$ for all $i \in \{2, 3, \dots n\}$. By Theorem 3.10, there exist $u_i \in N_1 \cup N_i$ and $c_i \in R - P$ such that $c_i.(N_i + N_1) \subseteq Cl_\tau^{N_1+N_i}(R.u_i)$. If for some i, $u_i \in N_1$, then $c_i.N_1 \subseteq Cl_\tau^{N_1}(R.u_i)$ and by Proposition 3.1, N_1 is τ-multiplication. If $u_i \in N_i$ for every $2 \leq i \leq k$, then we have $c_2 \dots c_k.N_1 \subseteq N_1 \cap \cdots \cap N_k$ as $c_i.N_1 \subseteq c_i.(N_1 + N_i) \subseteq N_i$ $(2 \leq i \leq k)$. By [6, Corollary 4.24], N_1 is a τ-multiplication module. □

Corollary 3.20. *Let τ a semicentered torsion theory on R. Let K_i $(1 \leq i \leq n)$ be a family of τ-closed submodules of an R-module M which are τ-multiplication modules and such that $K_i + K_j$ is τ-multiplication for $1 \leq i < j \leq n$. Then $(K_1 \cap \cdots \cap K_m) + (K_{m+1} \cap \cdots \cap K_n)$ is a τ-multiplication module for every positive integer $m < n$.*

Proof: Consider $L = K_{m+1} \cap \cdots \cap K_n$. By Theorem 3.19, L is a τ-multiplication module. By Corollary 3.16, $K_i + L$ is a τ-multiplication module $(1 \leq i \leq n)$ and by Corollary 3.16 again, $L + (K_1 \cap \cdots \cap K_n)$ is a τ-multiplication module. □

Denote $\hat{M}_i = \oplus_{j \neq i} M_j$. Compare the next result with [5, Theorem 2.2].

Theorem 3.21. *Let τ be a semicentered torsion theory on R. Let M an R-module such that $M = \oplus_{i \in I} M_i$ where $M_i's$ are τ-closed submodules of M. Then M is a τ-multiplication module if and only if the two following conditions are satisfied:*

1. *M_i is a τ-multiplication module for each $i \in I$;*
2. *for each $i \in I$ there exists an ideal A_i of R, such that $M_i = Cl_\tau^M(A_i.M_i)$ and $A_i.\hat{M}_i = 0$.*

Proof: Suppose that M is a τ-multiplication module. Then $M_i \cong M/\hat{M}_i$ and therefore it is a quotient of a τ-multiplication module. Thus M_i is a τ-multiplication module.

On the other hand, since M_i is τ-closed, there exists an ideal A_i of R such that $M_i = Cl_\tau^M(A_i.M) = Cl_\tau^{M_i}(A_i.M)$. So $A_i.M \subseteq M_i$. But $A_i.M = (\oplus_{j \in I} A_i.M_j) = \oplus_{j \in I}(A_i.M_j) \subseteq M_i$. Therefore $A_i.M_j = 0$ for all $j \neq i$ and hence $A_i.\hat{M}_j = 0$. Moreover, $A_i.M = A_i.M_i$ and $M_i = Cl_\tau^M(A_i.M_i)$.

Suppose that $P \in K(\tau)$. If $M_i = T_P(M_i)$ for all $i \in I$, then for each $m \in M_i$ there exists $c \in R - P$ such that $c.m = 0$. Hence, for each $x \in M$ there exists $c \in R - P$ such that $c.x = 0$. Thus $T_P(M) = M$.

Suppose that there exists $j \in I$ such that $M_j \neq T_P(M_j)$. Then by Proposition 3.1, there exist $c \in R - P$ and $m \in M_j$ such that $c.M_j \subseteq Cl_\tau^M(R.m)$. By condition 2, there exists an ideal $A_j \leq R$ such that $Cl_\tau^M(A_j.M_j) = M_j$ and $A_j.\hat{M}_j = 0$. We have $c.A_j.M_j \subseteq c.Cl_\tau^M(A_j.M_j) = M_j.c \subseteq Cl_\tau^M(R.m)$. If

$A_j \subseteq P$, then $M_j = Cl_\tau^M(A_j.M_j) \subseteq Cl_\tau^{M_j}(P.M_j)$ and hence $M_j = Cl_\tau^{M_j}(P.M_j)$. Therefore $M_j = T_P(M_j)$, a contradiction. Thus there exists $d \in (R-P) \cap A_j$ such that $c.d.M \subseteq c.d.(\oplus_{j\in I} M_j) \subseteq c.d.M_j \subseteq Cl_\tau^M(R.m)$ and by Proposition 3.1, M is a τ-multiplication module. □

Corollary 3.22. *Let τ be a semicentered torsion theory on R. Let M_i $(i \in I)$ be a family of finitely generated τ-closed modules such that $M = \oplus_{i\in I} M_i$. Then, M is a τ-multiplication module if and only if M_i is a τ-multiplication module and $ann(M_i) + ann(\hat{M}_i) \in \mathcal{F}$ for each $i \in I$.*

Proof: Suppose that M is τ-multiplication module. By Theorem 3.21, the first condition, M_i is a τ-multiplication module for each $i \in I$, is true. Suppose that there exists $i \in I$ such that $ann(M_i) + ann(\hat{M}_i) \notin \mathcal{F}$. Since τ is semicentered, there exists $P \in K(\tau)$ such that $ann(M_i) + ann(\hat{M}_i) \subseteq P$. From Theorem 3.21, there exists $A_i \leq R$ verifying $M_i = Cl_\tau^M(A_i.M_i)$ and $A_i.\hat{M}_i = 0$. Therefore $A_i \subseteq ann(\hat{M}_i)$. Thus $A_i \subseteq P$. Hence $M_i = Cl_\tau^M(P.M_i)$. Since M_i is finitely generated, there exists $c \in R - P$ such that $c.M_i = 0$, a contradiction because $c \in ann(M_i) \cap (R-P) = \emptyset$.

Conversely, it suffices to apply Corollary 3.13. □

Corollary 3.23. *Let τ be semicentered. Let $M = M_1 \oplus \cdots \oplus M_n$ where M_i is a τ-closed τ-multiplication finitely generated module for $1 \leq i \leq n$. Then M is τ-multiplication if and only if $ann(M_i) + ann(M_j) \in \mathcal{F}$ for all $1 \leq i \neq j \leq n$.*

Proof: Suppose that M is τ-multiplication. Clearly $ann(\hat{M}_i) \subseteq ann(M_j)$, if $j \neq i$. By Corollary 3.22, $ann(M_i) + ann(\hat{M}_i) \in \mathcal{F}$ for $1 \leq i \leq n$. Thus $ann(M_i) + ann(M_j)$ contains an element of the Gabriel filter and therefore it belongs to the filter.

Suppose that the second part of the equivalence is true. Since $ann(\hat{M}_i) = \cap_{j\neq i} ann(M_j)$,

$$ann(M_i)+ann(\hat{M}_i) = ann(M_i)+\cap_{j\neq i} ann(M_j) \supseteq \Pi_{j\neq i}[ann(M_i)+ann(M_j)] \in \mathcal{F}.$$

□

We can apply this corollary to find all finitely generated τ-multiplication modules over Dedekind domains.

Corollary 3.24. *Finitely generated τ-multiplication modules over a Dedekind domain are just modules of the form $M = Cl_\tau^M(N)$ where N is isomorphic to an ideal of R and τ-cyclic modules.*

Proof: Let R be a Dedekind domain. A finitely generated R-module M is of the form

$$M \cong I_1 \oplus \cdots \oplus I_r \oplus R/\alpha_1 \oplus \cdots \oplus R/\alpha_n,$$

where $\alpha_i \subseteq \alpha_{i+1}$ $(1 \leq i \leq n)$, I_j $(1 \leq j \leq n)$ is an ideal of R and every R/α_i is a cyclic R-module.

Since R is a commutative noetherian ring, by [16, Corollary VI.6.15], every torsion theory is semicentered.

If τ is trivial, then every module is τ-torsion. Thus every module is τ-multiplication. Suppose that τ is not trivial. Consider three cases.

Case A: $n = 0$.

If $r = 1$, then M is a projective ideal. By [15, Theorem 1], it is multiplication.

If $r \geq 2$, then I_j is τ-closed ($1 \leq j \leq r$) and τ-multiplication. Moreover, $ann(I_1) + ann(I_2) = 0 \notin \mathcal{F}$. By Corollary 3.23, M is not τ-multiplication.

Case B: $\alpha_1 \in \mathcal{F}$.

Assume $r = 0$. Then M is τ-torsion and hence it is τ-multiplication.

Suppose that $r \geq 1$. For each $x \in M$ we have $x \in Cl^M_\tau(I_1 \oplus \cdots \oplus I_r \oplus 0 \oplus \cdots \oplus 0)$. Let $N = I_1 \oplus \cdots \oplus I_r$. By [6, Theorem 3.7], M is τ-multiplication if and only if N is τ-multiplication.

If $r = 1$, then N is a projective ideal and therefore it is τ-multiplication. Consequently, $M = Cl^M_\tau(N)$ where N is isomorphic to an ideal of R.

If $r > 1$, then $ann(I_1) + ann(I_2) = 0 \notin \mathcal{F}$. By Corollary 3.23, N is not a τ-multiplication module.

Case C: $\alpha_1 \notin \mathcal{F}$.

Suppose that $n = 1$.

If $r = 0$, then $M = R/\alpha_1$ is a cyclic module and therefore it is a multiplication module.

Now, assume $r \geq 1$. We have $ann(I_1) + ann(R/\alpha_1) = \alpha_1 \notin \mathcal{F}$. By Corollary 3.23, M is not τ-multiplication.

Suppose $n \geq 2$.

If there exists α_k ($2 \leq k \leq n$) such that $\alpha_k \in \mathcal{F}$, then $\alpha_k.x \subseteq Cl^M_\tau(I_1 \oplus \cdots \oplus I_r \oplus R/\alpha_1 \oplus \cdots \oplus R/\alpha_{k-1} \oplus 0 \oplus \cdots 0)$. Let $N = I_1 \oplus \cdots \oplus I_r \oplus R/\alpha_1 \oplus \cdots \oplus R/\alpha_{k-1} \oplus 0 \oplus \cdots 0$. By [6, Theorem 3.7], M is τ-multiplication if and only if N is τ-multiplication.

If $r \geq 2$, then $ann(I_1) + ann(I_2) = 0 \notin \mathcal{F}$.

If $r = 1$, then $ann(I_1) + ann(R/\alpha_1) = \alpha_1 \notin \mathcal{F}$.

If $r = 0$ and $k - 1 = 1$, then N is cyclic and therefore it is multiplication. In this case M is τ-cyclic.

If $r = 0$ and $k - 1 \geq 2$, then $ann(R/\alpha_1) + ann(R/\alpha_2) = \alpha_2 \notin \mathcal{F}$. By Corollary 3.23, M is not τ-multiplication.

If none of α_j's belongs to the Gabriel filter, the situation is absolutely similar to the preceding one. $\square$

Immediately it follows the next corollary.

Corollary 3.25. *Finitely generated τ-multiplication modules over a P.I.D. are just τ-cyclic modules.*

Proposition 3.26. *Let R, S be rings such that $R \subseteq S$. If X, Y are τ-multiplication R-modules inside S, then $X.Y$ is a τ-multiplication R-module.*

Proof: Let $N = Cl_\tau^{X.Y}(N)$. Since X and Y are τ-multiplication, $Cl_\tau^X(N \cap X) = Cl_\tau^X(I.X)$ and $Cl_\tau^Y(N \cap Y) = Cl_\tau^Y(J.Y)$ for some $I, J \leq R$. By applying the properties of the closure operation which appear in [6], we have $Cl_\tau^{X.Y}(N) = Cl_\tau^{X.Y}(N \cap X.Y) = Cl_\tau^{X.Y}((N \cap X).(N \cap Y)) = Cl_\tau^{X.Y}(Cl_\tau^X(N \cap X).Cl_\tau^Y(N \cap Y)) = Cl_\tau^{X.Y}(Cl_\tau^X(I.X).Cl_\tau^Y(J.Y)) = Cl_\tau^{X.=Y}(I.J.X.Y)$. □

4. τ-distributive modules

An R-module M is called *distributive* if it has distributive property of the sum with respect to the intersection or distributive property of the intersection with respect to the sum, for the lattice of submodules.

Definition 4.1. *A module M is called τ-distributive if the lattice of τ-closed submodules, denoted by $C_\tau(M)$, is a distributive lattice.*

The case $M = R$ has been considered in [12]. Obviously, every distributive module is a τ-distributive module for any τ. It is also immediate that every ring is τ-distributive if and only if it is τ-distributive as an R-module. If $\mathcal{F} = \{R\}$ and τ the corresponding torsion theory, then τ-distributive modules are just the distributive modules. If τ is perfect, then the R-module M is τ-distributive if and only if the R_τ-module M_τ is distributive. This is due to the isomorphism of lattices which appears in [16, Proposition 3.7].

Recall that a τ-torsionfree module M is called *τ-uniserial* if its only τ-closed submodules are M and a chain (finite or infinite) of the form

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n \subset \ldots$$

Example 4.2. Every τ-uniserial module is τ-distributive.

Theorem 4.3. *Let τ be a semicentered torsion theory over R-Mod. The following sentences are equivalent:*

1. *M is a τ-distributive R-module;*
2. *if N, L, K are submodules of M, then $Cl_\tau^M((N+L) \cap (N+K)) = Cl_\tau^M(N + (L \cap K))$;*
3. *if N, L, K are submodules of M, then $Cl_\tau^M(N \cap (L+N)) = Cl_\tau^M((L+N) \cap (L+K))$;*
4. *M_P is distributive as an R_P-module for all $P \in K(\tau)$;*
5. *$(Rm : Rn) + (Rn : Rm) \in \mathcal{F}$ for all $m, n \in M$;*
6. *$Cl_\tau^M(R(m+n)) = Cl_\tau^M((Rm \cap R(m+n)) + (Rn \cap R(m+n)))$ for all $m, n \in M$;*
7. *$Cl_\tau^M(Rm + Rn) = Cl_\tau^M(R(m+n) + (Rm \cap Rn))$ for all $m, n \in M$;*
8. *$Cl_\tau^R((K+L) : N) = Cl_\tau^R((K : N) + (L : N))$ for all $K, L, N \leq M$, N being τ-finitely generated;*

9. $Cl^R_\tau(K:(L\cap N)) = Cl^R_\tau((K:L)+(K:N))$ *for all* $K,L,N \le M$, L, N *being* τ*-finitely generated;*
10. $Hom_{R_P}((N/(N\cap L))_P,(L/(N\cap L))_P) = 0$ *for all* $L,N \le M$ *and for all* $P \in K(\tau)$.

Proof: $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4$ it is similar to [12, Theorem 7.3].

$1 \Rightarrow 5$. Suppose that 1 is true. Let $m,n \in M$. By hypothesis, $Cl^R_\tau((Rm:Rn)+(Rn:Rm)) = R$. Let $P \in K(\tau)$. Then $Cl^{R_P}_{\tau_P}((R_P\frac{m}{1}:R_P\frac{n}{1})+(R_P\frac{n}{1}:R_P\frac{m}{1})) = R_P$. By Lemma 2.1, $(R_P\frac{m}{1}:R_P\frac{n}{1})+(R_P\frac{n}{1}:R_P\frac{m}{1}) = R_P$. Hence $(R_P\frac{m}{s}:R_P\frac{n}{t})+(R_P\frac{n}{t}:R_P\frac{m}{s}) = R_P$ for all $\frac{m}{s},\frac{n}{t} \in M_P$. By [17, Theorem 1.6], M_P is a distributive R_P-module for all $P \in K(\tau)$. By 4, M is τ-distributive.

$5 \Rightarrow 1$. Conversely, suppose that M is τ-distributive and $(Rn:Rm)+(Rm:Rn) \notin \mathcal{F}$. Since τ is semicentered, there exists $P \in K(\tau)$ such that $(Rn:Rm)+(Rm:Rn) \subseteq P$. Thus $[(Rm:Rn):(Rn:Rm)]_P \subseteq P_P \subset R_P$. Thus M_P is distributive as an R_P-module by Lemma 4.3 and by [17, Theorem 1.6], $[(Rm:Rn)+(Rn:Rm)]_P = R_P$, a contradiction.

$6 \Rightarrow 5$. By using [9, Lemma 3.1], we have

$$\begin{aligned} Cl^M_\tau(R(m+n)) &= Cl^M_\tau((Rm:R(m+n))(m+n)+(Rn:R(m+n))(m+n)) \\ &= Cl^M_\tau((Rm:R(m+n))+(Rn:R(m+n))(m+n)). \end{aligned}$$

Since $ann(m+n) \subseteq (Rm:Rn)+(Rn:Rm)$, $(Rm:Rn)+(Rn:Rm) \in \mathcal{F}$.

$1 \Rightarrow 6$ is trivial.

$7 \Rightarrow 5$. By applying [9, Lemma 3.1], we have $Cl^M_\tau(Rm) = Cl^M_\tau(Rm \cap (Rm+Rn)) = Cl^M_\tau(((Rm:Rn)+(Rn:Rm))m)$.
Since $ann(m) \subseteq (Rn:Rm)$, 2 follows.

$1 \Rightarrow 7$. We have $Rm \subseteq Rn+R(m+n)$. Since M is τ-distributive, $Cl^M_\tau(Rm) = Cl^M_\tau((Rn\cap Rm)+(R(m+n)\cap Rm))$. Analogously, $Cl^M_\tau(Rn) = Cl^M_\tau((Rm \cap Rn)+(R(m+n)\cap Rn))$. Easily, it can be checked that $Cl^M_\tau(Rm+Rn) = Cl^M_\tau((Rm\cap Rn)+R(m+n))$.

$1 \Rightarrow 8$. Let $P \in K(\tau)$. By 4, M_P is distributive as an R_P-module. By [1, Theorem 1.9], $((K_P+L_P):N_P) = ((K_P:N_P)+(L_P:N_P))$. Since N_P is finitely generated, we have $((K+L):N)_P = ((K:N)+(L:N))_P$ for all $P \in K(\tau)$. Since τ is semicentered, 2 follows.

$8 \Rightarrow 1$. We shall prove that M_P is distributive as an R_P-module for each $P \in K(\tau)$. Let $K_P, L_P, N_P \le M_P$, N_P being finitely generated. Since $N_P = \langle \frac{x_1}{1},\dots,\frac{x_r}{1}\rangle$, there exists $N' = \langle x_1,\dots,x_n\rangle \le N$ such that $(Cl^M_\tau(N'))_P = N'_P = N_P$ and obviously, $Cl^M_\tau(N')$ is τ-finitely generated. From the hypothesis, by using localization, we obtain $((K+L):N')_P = ((K:N)+(L:N))_P$. Hence $(K_P+L_P):N_P = (K_P:N_P)+(L_P:N_P)$. By [1, Theorem 1.9], M_P is distributive.

$8 \Leftrightarrow 9$. From [1, Theorem 1.9], it suffices to use localization.

$1 \Leftrightarrow 10$ is straightforward by applying [17, Proposition 1.1]. □

Corollary 4.4. *Let τ be a semicentered torsion theory on R. Let M be a τ-distributive R-module. Let L, N be submodules of M. Then*

1. *if N is finitely generated and $N \cap L = 0$, then $Hom_R(N, L)$ is τ-torsion;*
2. *if M/L is finitely generated and $N + L = M$, then $Hom_R(M/L, M/N)$ is τ-torsion.*

Proof: We shall prove 1. Let $P \in K(\tau)$. By Theorem 4.3.10, $Hom_{R_P}(N_P, L_P) = 0$. Since N is finitely generated, the canonical morphism $(Hom_R(N, L))_P \to Hom_{R_P}(N_P, L_P)$ is injective. Thus $(Hom_R(N, L))_P = 0$ for all $P \in K(\tau)$. Hence 1 follows.

Now, prove 2. Let $P \in K(\tau)$. By Theorem 4.3.10, we have

$$Hom_{R_P}((N/(N \cap L))_P, (L/(N \cap L))_P) = 0 \cong Hom_{R_P}(((N + L)/L)_P, ((N + L)/N)_P).$$

By hypothesis this module is $Hom_{R_P}((M/L)_P, (M/N)_P)$. Since M/L is finitely generated, $(Hom_R(M/L, M/N))_P = 0$ for all $P \in K(\tau)$. Therefore, the R-module $Hom_R(M/L, M/N)$ is τ-torsion. □

Example 4.5. Let τ be a semicentered torsion theory. By Theorem 4.3.4, every Krull domain R is a τ-distributive ring, i.e., it is τ-distributive as an R-module.

Proposition 4.6. *If τ is semicentered, then every submodule and every quotient of a τ-distributive module is a τ-distributive module.*

Proposition 4.7. *Let τ be a semicentered torsion theory over R-Mod. If $M = Cl_\tau^M(N)$, then M is τ-distributive if and only if N is τ-distributive.*

Proof: Suppose that M is τ-distributive. By Theorem 4.3.4, M_P is a distributive R_P-module for all $P \in K(\tau)$. Then $M_P = (Cl_\tau^M(N))_P = Cl_{\tau_P}^{M_P}(N_P)$ by Lemma 2.1. Moreover, $M_P = N_P$. Thus N_P is distributive as an R_P-module. By Theorem 4.3.4 again, N is τ-distributive. The converse can be proved in the same way. □

Definition 4.8. *A module is called τ-Bezout if every τ-finitely generated submodule is τ-cyclic.*

Proposition 4.9. *τ-distributive modules over a P.I.D. are just τ-Bezout modules.*

Proof: Straightforward from Corollary 3.25. □

The following results give different ways to obtain new relative distributive modules from relative distributive modules .

Proposition 4.10. *Let τ be a semicentered torsion theory on R. Let M, N be two τ-distributive R-modules. Then*

1. *$M \otimes_R N$ is a τ-distributive R-module;*
2. *if M is finitely generated, then $Hom_R(M, N)$ is a τ-distributive R-module.*

PROOF: By Theorem 4.3.4, 1 is trivial.

Since M, N are τ-distributive modules, M_P, N_P are distributive R_P-modules for all $P \in K(\tau)$. By [1, Lemma 4.1], $Hom_{R_P}(M_P, N_P)$ is distributive as an R_P-module. Since M is a finitely generated R-module, the canonical morphism $[Hom_R(M, N)]_P \to Hom_{R_P}(M_P, N_P)$ is injective. By Proposition 4.6, $[Hom_R(M, N)]_P$ is distributive for all $P \in K(\tau)$. By Theorem 4.3.4, $Hom_R(M, N)$ is a τ-distributive R-module. $\square$

Remark 4.11. If M is τ-finitely generated and N is τ-torsion free the same result is obtained. It suffices to realize that with the above hypothesis if $M = Cl_\tau^M(F)$, then two maps belonging to $[Hom_R(M, N)]_P$ which are equal over F are equal over M.

For an R-module M, set $\tau - Supp(M) = \{P \in K(\tau); M_P \neq 0\}$.

Proposition 4.12. *Let M_i $(i \in I)$ be a family of τ-distributives modules. Then $\oplus_{i\in I} M_i$ is τ-distributive if and only if $\tau - Supp(M_i) \cap \tau - Supp(M_j) = \emptyset$ for all $i, j \in I$ $i \neq j$.*

PROOF: Suppose that $\oplus_{i\in I} M_i$ is τ-distributive and for some $i \neq j$, there exists $P \in K(\tau)$ such that $(M_i)_P \neq 0 \neq (M_j)_P$. The R_P-module $(\oplus_{i\in I}(M_i))_P \cong \oplus_{i\in I}(M_i)_P$ is distributive. By [1, Proposition 1.8], $Supp(M_i) \cap Supp(M_j) = \emptyset$. However $P \in Supp(M_i) \cap Supp(M_j) = \emptyset$, a contradiction.

Conversely, let $P \in K(\tau)$. If there exists $Q.R_P \in Supp((M_i)_P) \cap Supp((M_j)_P)$, then, since $(M_i)_{P_{Q.R_P}} \cong M_Q$, $(M_i)_Q \neq 0 \neq (M_j)_Q$ for $i \neq j$. If $Q \in \mathcal{F}$, then $P \in \mathcal{F}$ as $Q \subseteq P$. Therefore, $Q \in K(\tau)$, a contradiction. $\square$

The following theorem establishes a relationship between τ-distributive modules and τ-multiplication modules. It is a generalization of [2, Proposition 7].

Theorem 4.13. *Let τ be a semicentered torsion theory on R-Mod. Then M is τ-distributive if and only if every τ-finitely generated submodule of M is τ-multiplication.*

PROOF: Suppose that M is τ-distributive. Let $N = Cl_\tau^N(F) \leq M$, F being finitely generated. By Theorem 4.3.4, M_P is distributive for all $P \in K(\tau)$. Since F_P is finitely generated for all $P \in K(\tau)$ as an R_P-module, by [2, Proposition 7], the R_P-module F_P is multiplication for all $P \in K(\tau)$. By [6, Theorem 4.18], F is a τ-multiplication R-module. By [6, Theorem 3.7], N is τ-multiplication as an R-module.

Conversely, suppose that every τ-finitely generated submodule of M is τ-multiplication. Let $P \in K(\tau)$. We shall prove that M_P is distributive as an R_P-module. Let $N_P = \langle \frac{x_1}{1}, \ldots, \frac{x_r}{1} \rangle$ with $x_i \in N$ $(1 \leq i \leq r)$. Let $L_P \leq N_P$.

Consider $K = \langle x_1, \ldots, x_r \rangle \leq N$. Obviously, $K_P = N_P$. Since $Cl_\tau^N(K) \leq M$ is τ-finitely generated, it is τ-multiplication. Hence $Cl_\tau^N(L \cap N) = Cl_\tau^N(I.Cl_\tau^N(K))$ for some ideal I of R. By localization, $L_P = I_P.N_P$. Thus N_P is multiplication as an R_P-module. $\square$

Corollary 4.14. *Let τ be a semicentered torsion theory. Every τ-noetherian τ-distributive module is a strongly τ-multiplication module.*

Corollary 4.15. *If τ is semicentered, then every τ-uniserial τ-noetherian module M is strongly τ-multiplication.*

Corollary 4.16. *Let τ be a semicentered torsion theory. Every τ-noetherian τ-distributive ring is a τ-multiplication ring.*

In particular, if τ is the canonical torsion theory, then every Krull domain is a τ-multiplication ring, by Corollary 4.16.

The following example shows that the τ-distributive modules class is strictly wider than the distributive modules class.

Example 4.17. Let $R = K[x, y]$, K being a field. Let τ be the canonical torsion theory. Since R is a Krull domain, it is a τ-multiplication ring. Thus every submodule of K is τ-multiplication. By Theorem 4.13, R is a τ-distributive R-module.

R is a integral domain which is not a Dedekind domain. By [10, Proposition 9.13], there exists some ideal (which must be finitely generated as R is noetherian) which is not multiplication as an R-module. By [2, Proposition 7], R is not distributive as an R-module.

Acknowledgments. This work will form part of the first author's doctoral thesis. The authors wish to thank Professor Patrick F. Smith (University of Glasgow) for his suggestions and ideas.

This work was started when the first author was an Erasmus student (P.I.C. No. 93-B-1033/11) in the University of Glasgow.

The second author has been partially supported by the grant PB91-706 from DGICYT. We should also thank Professor C. Năstăsescu for his comments about some parts of the work.

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(*Received* June 10, 1996)