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Metrizable completely distributive lattices

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Abstract. The purpose of this paper is to study the topological properties of the interval topology on a completely distributive lattice. The main result is that a metrizable completely distributive lattice is an ANR if and only if it contains at most finite completely compact elements.

Keywords: completely distributive lattice, interval topology, AR, ANR

Classification: 54C08, 06B30

0. Introduction

It is a fundamental result in topology that the unit interval $[0, 1]$ is an absolute retract (AR). Since every completely distributive lattice can be embedded as a subcomplete lattice (hence a closed subspace) in some power of $[0, 1]$, it is natural to ask whether a completely distributive lattice is an AR or ANR.

By improving the method of Katětov [6], Liu and Luo [8] have proved the following result.

Let L be a completely distributive lattice with a countable strict join generating set (see [8] for definition or the note after 3.1), X a normal space, $f, g : X \to L$ are lsc, usc in the sense of 1.6 respectively and $g \leq f$, then there is a continuous $h : X \to L$ such that $g \leq h \leq f$. Hence L is an AR by a routine argument like 4.2.

Meanwhile by a very different methods, precisely, making use of the semiuniformity which generates both the order and the topology on a continuous lattice, van Gool [3] has introduced new definitions of lsc and usc functions with values in a continuous lattice, and proved the following result.

Let L be an arcwise connected, metrizable, linked bicontinuous lattice, X a normal space, $f, g : X \to L$ are lsc and usc functions in the sense of van Gool [3] respectively, then there is a continuous $h : X \to L$ such that $g \leq h \leq f$. Therefore by a routine argument L is an AR in this case.

Now questions arise:

(1) What is the relation between the result of Liu and Luo [8] and that of van Gool [3]?

(2) When is a completely distributive lattice an ANR?

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The purpose of this paper is to answer these two questions, precisely we have obtained:

(1) A metrizable completely distributive lattice is an ARN if and only if L contains at most finite completely compact element.

(2) For distributive lattices, the result of Liu and Luo is equivalent to that of van Gool.

1. Completely distributive lattices and semicontinuous mappings

In this section, we recall some basic results about completely distributive lattices, continuous lattices, and some characterizations of semicontinuous mappings.

1.1 Definition. Let a, b be elements in a complete lattice L, we say that a is way below b (wedge below b), in symbols $a \ll b$ ($a \lhd b$), if for every directed (arbitrary) $D \subset L$, $b \leq \bigvee D$ implies there is some $d \in D$ with $a \geq d$. And a complete lattice L is called continuous (completely distributive) if every element of L is the supremum of all the elements which are way below (wedge below) it, i.e. $a = \bigvee\{b \in L \,|\, b \ll a\}$ ($a = \bigvee\{b \in L \,|\, b \lhd a\}$) for all $a \in L$.

Suppose L is a complete lattice, write $Idl(L)$ ($Low(L)$) for the complete lattice of all the ideal (lower sets) in L, and $\sup : Idl(L) \to L$ ($Low(L) \to L$) for the supremum operation, then easily we have

1.2 Theorem. *Let L be a complete lattice, then*

(1) *L is continuous if and only if* $\sup : Idl(L) \to L$ *has a left adjoint* $\Downarrow : L \to Idl(L)$, *or equivalently* $\sup$ *preserves* $\inf$s. *In this case* $\Downarrow a = \{b \in L \,|\, b \ll a\}$ *for* $a \in L$;

(2) *L is completely distributive if and only if* $\sup : Low(L) \to L$ *has a left adjoint* $\beta : L \to Low(L)$. *In this case* $\beta(a) = \{b \in L \,|\, b \lhd a\}$ *for* $a \in L$. □

Note. If L is completely distributive, then for each $a \in L$, $\Downarrow a$ is just the ideal generated by the lower set $\beta(a)$.

1.3 Corollary. (1) *In a continuous lattice L, the way below relation has the interpolation property, that is to say for $a \ll b$ there is some $c \in L$, $a \ll c \ll b$.*

(2) *In a completely distributive lattice L, the wedge below relation has the interpolation property.* □

1.4 Corollary. *Let L be a complete lattice, then*

(1) *L is continuous if and only if the equation*

$$\bigwedge_{i \in I} \bigvee_{j \in J_i} d_{i,j} = \bigvee_{f \in \prod_{i \in I} J_i} \bigwedge_{i \in I} d_{i,f(i)}$$

holds if $I \neq \Phi$ and $D_i = \{d_{i,j} \,|\, j \in J_i\}$ is a directed set for each $i \in I$;

(2) *L is completely distributive if and only if the equation*

$$\bigwedge_{i\in I}\bigvee_{j\in J_i} d_{i,j} = \bigvee_{f\in\prod_{i\in I} J_i}\bigwedge_{i\in I} d_{i,f(i)}$$

holds if $I \neq \Phi$ and $J_i \neq \Phi$ for each $i \in I$. □

Note. In the literature a complete lattice L is called completely distributive if the equations

$$\bigwedge_{i\in I}\bigvee_{j\in J_i} d_{i,j} = \bigvee_{f\in\prod_{i\in I} J_i}\bigwedge_{i\in I} d_{i,f(i)} \tag{1}$$

$$\bigvee_{i\in I}\bigwedge_{j\in J_i} d_{i,j} = \bigwedge_{f\in\prod_{i\in I} J_i}\bigvee_{i\in I} d_{i,f(i)} \tag{2}$$

hold if $I \neq \Phi$ and $J_i \neq \Phi$ for $i \in I$. It is proved in [5] that (1) and (2) are equivalent, hence the definition of complete distributivity in 1.1 is equivalent to that in the literature.

1.5 Corollary. *A distributive complete lattice L is completely distributive if and only if both L and L^{op} are continuous.* □

1.6 Definition. Let L be a complete lattice, then

(1) The lower (upper) topology $\theta_*(L)$ ($\theta^*(L)$) is the topology generated by $\{L\backslash\downarrow a \mid a\in L\}$ ($\{L\backslash\uparrow a \mid a \in L\}$) as a subbase. The interval topology $\theta(L)$ is the coarsest common refinement of both $\theta_*(L)$ and $\theta^*(L)$.

(2) An upper set $U \subset L$ is called Scott open if for every directed set $D \subset L$, $\bigvee D \in U$ implies $u \cap D \neq \Phi$; easily all the Scott open sets form a topology $\sigma(L)$, called the Scott topology. Trivially $\sigma(L)$ is finer than $\theta(L)$.

(3) A mapping f from a topological space to $L : X \to L$ is called continuous (Scott continuous, lsc, usc) if f is continuous with respect to the interval topology $\theta(L)$ ($\sigma(L)$, $\theta_*(L)$, $\theta^*(L)$ respectively).

Trivially $f : X \to L$ is usc if and only if for each $a \in L$, $f_{[a]} = \{x \in X \mid f(x) \geq a\}$ is closed; f is lsc if and only if for each $a \in L$, $f^{[a]} = \{x \in L \mid f(x) \leq a\}$ is closed.

1.7 Lemma ([12], [2]). *Let L be a continuous lattice, X a topological space, write $[X, \sum L]$ for the set of Scott continuous mappings $X \to L$, then $[X, \sum L]$ is closed under pointwise finite* infs *and pointwise arbitrary* sups, *hence a complete lattice.* □

1.8 Theorem ([12]). *Let X be a topological space, L a continuous lattice, $f: X \to L$ a mapping, then*

(1) *f is Scott continuous if and only if for each $x \in X$,*

$$f(x) = \bigvee_{u \in B(x)} \bigwedge_{y \in U} f(y),$$

where $B(x)$ is a neighborhood base of x.

(2) *The biggest Scott continuous mapping $\operatorname{Int} f$ majorized by f satisfies that for each $x \in X$,*

$$\operatorname{Int} f(x) = \bigvee_{u \in B(x)} \bigwedge_{y \in U} f(y),$$

where $B(x)$ is a neighborhood base of x. □

1.9 Theorem ([7]). *Let X be a topological space, L a completely distributive lattice, then $f: X \to L$ is lsc if and only if for each $x \in X$, $f(x) = \bigvee_{U \in B(x)} \bigwedge_{y \in U} f(y)$, where $B(x)$ is a neighborhood base of x; or equivalently f is Scott continuous by the above theorem.* □

1.10 Corollary. *For a completely distributive lattice L, the lower topology coincides with the Scott topology.*

□

1.11 Theorem ([12]). *Let X be a topological space, L a continuous lattice, $f: X \to L$ a mapping, then the biggest Scott continuous mapping $\operatorname{Int} f$ majorized by f satisfies $\operatorname{Int} f = \bigvee_{a \in L} a f^0_{[a]}$, where for each $a \in L$, $a f^0_{[a]}$ is the characteristic mapping on the interior of $f_{[a]} = \{x \in X \mid f(x) \geq a\}$ with value $a \in L$, hence f is Scott continuous if and only if $f = \bigvee_{a \in L} f^0_{[a]}$.* □

1.12 Theorem. *Let X be a topological space, L a completely distributive lattice, $f: X \to L$ a mapping, then*

(1) *the biggest lsc mapping majorized by f is $\operatorname{Int} f = \bigvee_{a \in L} a f^0_{[a]}$;*

(2) *the least usc mapping $\operatorname{cl} f$ majorizing f satisfies $\operatorname{cl} f = \bigvee_{a \in L} a \overline{f_{[a]}}$, where $a \overline{f_{[a]}}$ is the characteristic mapping on the closure of $f_{[a]}$ with value $a \in L$.*

Note. It should be pointed out that (2) is not the dual of (1).

Proof: (1) It follows from the above theorem.

(2) Since a mapping $g: X \to L$ is usc if and only if for each $a \in L$, $g_{[a]} = \{x \in X \mid g(x) \geq a\}$ is closed, hence $\operatorname{cl} f \geq \bigvee_{a \in L} a \overline{f_{[a]}}$ is trivial.

Conversely since $\bigvee_{a \in L} a \overline{f_{[a]}} \geq f$ we need only to prove that $h = \bigvee_{a \in L} a \overline{f_{[a]}}$ is usc, and this follows from the fact that for each $a \in L$, $h_{[a]} = \bigcap_{b \in \beta(a)} \overline{f_{[b]}}$. □

1.13 Corollary. *Let L be a completely distributive lattice, X a topological space, $f : X \to L$ lsc, then for every $a \in L$ and $B \subset \beta(a)$ with $\bigvee B = a$, $f_{[a]} = \bigcap_{b \in B} f^0_{[a]}$.* □

1.14 Proposition. *Let X be a topological space, L a completely distributive lattice, $A \subset L$ is a subset satisfying that for each $a \in L$, the supremum of $A \cap \beta(a)$ is a; $\{F_a \mid a \in A\}$ is a decreasing family of closed sets, i.e. for $b \leq a$ in A, $F_b \geq F_a$, then the mapping $h = \bigvee_{a \in A} aF_a$ is usc, where aF_a is the characteristic mapping on F_a with value a.* □

2. Interval topology on completely distributive lattices

In this section, we study some basic properties of the interval topology on completely distributive lattices.

2.1 Definition. The Lawson topology $\lambda(L)$ on a complete lattice L is defined to be the coarsest common refinement of the Scott topology $\sigma(L)$ and the upper topology $\theta^*(L)$.

By Corollaries 1.5, 1.10 we have

2.2 Proposition. *For a completely distributive lattice L, the following topologies coincide with each other:*

(1) *the interval topology $\theta(L)$;*
(2) *the Lawson topology $\lambda(L)$ on L;*
(3) *the Lawson topology on L^{op};*
(4) *the biscott topology, i.e. the topology generated by $\sigma(L) \cup \sigma(L^{op})$ as a subbase.* □

2.3 Theorem ([2])**.** *Let L be a continuous lattice, then*

(1) *a net $(x_j)_{j \in J}$ converges to $x \in L$ with respect to the Lawson topology if and only if $\bigwedge_{j \in J} \bigvee_{i \geq j} x_i = \bigwedge_{p(j) \geq j} \bigvee_{j \in J} x_{p(j)} = x$, hence a directed (filtered) set D in L converges to $\bigvee D$ ($\bigwedge D$);*
(2) *$(L, \lambda(L))$ is a compact Hausdorff topological semilattice; hence if L is completely distributive, then $(L, \theta(L))$ is a compact Hausdorff topological lattice.* □

2.4 Corollary. *A sublattice of completely distributive lattice is a closed set in the interval topology if and only if it is a subcomplete lattice.* □

2.5 Theorem. *Let L be a completely distributive lattice, then every element $a \in L$ has a neighborhood base consisting of open sublattices in the interval topology.*

Proof: Let a be an element in L. Since L is continuous, a has a neighborhood base $N(a)$ consisting of open subsemilattices which are upper sets in the topology

$\sigma(L) = \theta_*(L)$ by Proposition 3.3 in [2, p. 69]. Similarly by continuity of L^{op}, a has a neighborhood base $N^*(a)$ consisting of open subsemi-join-lattices which are lower sets in the topology $\theta^*(L)$. Then it can be easily verified that $\{U \cap V \mid U \in N(a)$ and $V \in N^*(a)\}$ is a neighborhood base of $a \in L$ in $\theta(L)$, and that $U \cap V$ is sublattice is obvious. $\square$

2.6 Corollary. *Let L be a completely distributive lattice, then every $a \in L$ has a neighborhood base consisting of subcomplete lattices in the interval topology.*

Proof: A direct consequence of the above theorem and the fact that $(L, \theta(L))$ is compact Hausdorff. $\square$

In the remainder of this section, we study the connectedness of the interval topology on a completely distributive lattice.

2.7 Definition. Let L be a complete lattice, $a \in L$, then

(1) a is called compact if $a \ll a$;
(2) a is a coprime if $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$;
(3) a is called completely compact if $a \lhd a$.

Trivially a is completely compact if and only if a is a compact coprime.

2.8 Proposition ([2]). *Every element in a completely distributive lattice L can be represented as union of coprimes in L, hence if $\beta^*(a) = \{b \in \beta(a) \mid b$ is a coprime$\}$, then $\bigvee \beta^*(a) = a$ for every $a \in L$.* $\square$

2.9 Lemma. *If a complete lattice L has a nonzero completely compact element, then $(L, \theta(L))$ is not connected.*

Proof: Let $b = \bigvee\{x \in L \mid a \nleq x\}$, then $L = \uparrow a \cup \downarrow b$ and $\uparrow a \cap \downarrow b = \Phi$, hence $(L, \theta(L))$ is not connected. $\square$

2.10 Lemma ([11]). *A completely distributive lattice L has a nonzero compact element if and only if L has a nonzero completely compact element.*

Proof: Sufficiency is trivial. Now we prove the necessity. Suppose $a \neq 0$ is compact in L, i.e. $a \ll a$. Let $\beta^*(a) = \{x \in \beta(a) \mid x$ is a coprime$\}$, then $\bigvee \beta^*(a) = a$. We claim that every chain in $\beta^*(a)$ has an upper bound. Indeed let $C \subset \beta(a)$ be a chain, then trivially $\bigvee C = c$ is a coprime, and it is left to prove that $c \in \beta(a)$. Suppose $D \subset L$ satisfies $\bigvee D \geq a$, then there are finite elements $\{d_1, \ldots, d_n\} \subset D$ with $\bigvee_{i \leq n} d_i \geq a$.

Since every element in C is a coprime, there exists certain i such that $x \leq d_i$ for all $x \in C$, hence $c \leq d_i$, $c \in \beta(a)$.

Now let $\{c_\gamma \mid \gamma \in \Gamma\}$ be the maximal elements in $\beta^*(a)$, then the supremum of $\bigcup_{\gamma \in \Gamma} \beta^*(c_\gamma)$ is a, hence there exist finite $t_1, \ldots, t_m \in \bigcup_{\gamma \in \Gamma} \beta^*(c_\gamma)$ such that $\bigvee_{i \leq m} t_i = a$. Assume $t_i \in \beta^*(c_{\gamma(i)})$ for each $i \leq m$. Since for each $i \leq m$ $c_{\gamma(i)}$ is a coprime, there exists $t_{i(j)}$ $t_{i(j)} \geq c_{\gamma(i)}$, then by maximality of $c_{\gamma(i)}$ we have $t_{i(j)} = c_{\gamma(i)}$, and $t_{i(j)} \in \beta^*(c_{\gamma(i)})$, hence $c_{\gamma(i)} \lhd c_{\gamma(i)}$, i.e. $c_{\gamma(i)}$ is completely compact. Finally since $\bigvee_{i \leq m} t_i = a$, at least one of the $c_{\gamma(i)}$'s is nonzero. $\square$

2.11 Theorem. *Let L be a completely distributive lattice, then the following statements are equivalent:*

(1) *$(L, \theta(L))$ is connected;*
(2) *L is order sense, i.e. whenever $x < y$ in L, there exists $z \in L$ such that $x < z < y$;*
(3) *L has no nonzero compact element;*
(4) *L has no nonzero completely compact element.*

In this case we say that L is connected.

Proof: The equivalence of (1), (2) and (3) follows from Theorem 5.15 in [2, p. 301], and that of (3) and (4) follows from the above lemma. □

3. Weight for complete lattices

In this section we introduce the concept of weight for a complete lattice and discuss some of its basic properties.

3.1 Definition. A subset B in a complete lattice L is called a base if every element $a \in L$ can be represented as union of subsets of B. And the weight of L $w(L)$ is defined to be $\min\{|B| \mid B \subset L \text{ is a base}\}$.

Note. (1) If X is a topological space, then the weight of the open set lattice $O(X)$ is just the weight of the space $w(X)$.

(2) Trivially the notion of a join generating set defined in [8] is just that of a base in the above definition.

3.2 Proposition. *Suppose B is a subset of a complete lattice, then the following conditions are equivalent:*

(1) *B is a base;*
(2) *whenever $y \not\leq x$, there is a $b \in B$ with $b \not\leq x$ and $b \leq y$.*

If L is continuous, (2) *is equivalent to*

(2′) *whenever $y \not\leq x$, there is a $b \in B$ with $b \not\leq x$ and $b \ll y$.*

If L is completely distributive, (2) *is equivalent to*

(2″) *whenever $y \not\leq x$, there is a $b \in B$ with $b \not\leq x$ and $b \lhd y$.*

□

3.3 Corollary. *If B is a base for a completely distributive lattice L, then for each $a \in L$ the supremum of $B \cap \beta(a)$ is a.* □

3.4 Lemma. *Suppose $w(L) \leq a$, $\{a_s\}_{s \in S} \subset L$ is a subset, then there is a subset $S_0 \subset S$ such that $|S_0| \leq a$ and $\bigvee_{s \in S_0} a_s = \bigvee_{s \in S} a_s$.*

Proof: Suppose $B = \{b_t \mid t \in T\}$ is abase of cardinality $\leq \alpha$ and denote by B_0 the collection of $b \in B$ such that for some $s \subseteq\ b \leq a_s$. To every $b \in B_0$ assign an

$s(b) \in S$ such that $b \leq a_{s(b)}$. In this way a function $s : B_0 \to S$ is defined, now we prove that $S_0 = s(B_0) \subset S$ satisfies the lemma.

Indeed $|S_0| \leq \alpha$ and $\bigvee_{s\in S_0} a_s \leq \bigvee_{s\in S} a_s$ are trivial. Now if $\bigvee_{s\in S} a_s \not\leq \bigvee_{s\in S_0} a_s$ there is some $s_0 \in S$ such that $a_{s_0} \not\leq \bigvee_{s\in S_0} a_s$, hence there is a $b \in B$ with $b \not\leq \bigvee_{s\in S_0} a_s$ and $a_{s_0} \leq b \leq a_{s(b)} \leq \bigvee_{s\in S_0} a_s$, a contradiction. □

3.5 Proposition. *Suppose $w(L) \leq \alpha \geq \omega$, then for every base B of L there exists a subset $B_0 \subset B$ such that $|B_0| \leq \alpha$ and B_0 is a base of L.*

Proof: Take a base $B_1 = \{a_t \mid t \in T\}$ of L such that $|B_1| \leq \alpha$. Let $B = \{b_s \mid s \in S\}$ and for every $t \in T$, let

$$S(t) = \{s \in S \mid b_s \leq a_t\}.$$

Since B is a base we have $\bigvee_{s\in S(t)} b_s = a_t$ and by 3.4 there is a subset $S_0(t) \subset S(t)$ such that $|S_0(t)| \leq \alpha$ and $a_t = \bigvee_{s\in S_0(t)} b_s$.

Let $B_0 = \{b_s \mid s \in S_0(t) \text{ for some } t \in T\}$, then trivially $|B_0| \subset \alpha$, and it is left to prove that B_0 is a base. Suppose $x \not\leq y$ in L, as B_1 is a base there is some $a_t \in B_1$ with $a_t \not\leq y$ and $a_t \leq x$, hence there is some $s \in S_0(t)$ such that $b_s \leq x$ and $b_s \not\leq y$, therefore B_0 is a base. □

A base B of a complete lattice L is called strict if every element $a \in L$ can be represented as union of elements in B which are strictly smaller than a. Trivially L has a strict base if and only if every base of L is strict. Obviously a strict base of a complete lattice L is just a strict join generating set in [8].

3.6 Proposition. *Let L be a completely distributive lattice, then the following conditions are equivalent:*

(1) *$(L, \theta(L))$ is connected;*
(2) *L has no nonzero completely compact element;*
(3) *L has a strict base;*
(4) *every base of L is strict.* □

3.7 Theorem ([2]). *For a continuous lattice L, $w(L) = w(\lambda(L))$, where $w(\lambda(L))$ is the weight of the space $(L, \lambda(L))$, hence $w(L) = w(\theta(L))$ if L is moreover completely distributive.* □

3.8 Theorem. *Let L be a completely distributive lattice, then the following statements are equivalent:*

(1) *$(L, \theta(L))$ is metrizable;*
(2) *$w(L) \leq \omega$, i.e. L has a countable base.* □

4. AR and ANR property of metrizable completely distributive lattices

A subspace $Z \subset X$ is called a retract if there is a continuous mapping $r : X \to Z$ with $ri_z = id_z$. Z is called a neighborhood retract if Z is a retract of some open subspace in X containing Z.

A space Z is called an absolute (neighborhood) retract (AR, ANR in short) if Z is a (neighborhood) retract of every normal space which contains Z as a closed subspace.

It is well known ([4]) that Z is an AR (ANR) if and only if Z is an absolute (neighborhood) extensor, i.e. every continuous $f : A \to Z$ defined on closed subspace A in a normal space X can be extended to X (an open subspace of X).

4.1 Theorem ([8]). *Let L be a connected metrizable completely distributive lattice, X a normal space, $f : X \to L$ is lsc, $g : X \to L$ is usc and $g \leq f$, then there is a continuous $h : X \to L$ with $g \leq h \leq f$.*

Proof: By 3.8. Let $A = \{a_n \,|\, n \in N\}$ be a countable base of L.

(1) For each $n \in N$, we shall define an open set U_n such that

(i) $\bigcup\{g_{[b]} \,|\, b \in A \text{ and } a_n \lhd b\} \subset U_n \subset \overline{U}_n \subset f_{[a_n]}$;

(ii) if $a_n < a_m$ in A, then $U_m \subset U_n$, if moreover $a_n \lhd a_m$, then $\overline{U}_m \subset \overline{U}_n$.

Step 1. We define U_1.

At first $F_1 = \bigcup\{g_{[b]} \,|\, b \in A \text{ and } a_1 \lhd b\}$ is an F_σ set since g is usc; and $G_1 = X \setminus f_{[a_1]}$ is also an F_σ set since $f_{[a_1]} = \bigcap_{b \in \beta(a_1) \cap A} f^0_{[b]}$ is a G_δ set by 1.13, 3.3.

Second F_1 and G_1 are separated, i.e. $\overline{F}_1 \cap G_1 = \overline{G}_1 \cap F_1 = \Phi$.

Indeed since F_1 is contained in the closed set $g_{[a_1]}$ which is contained in $f_{[a_1]}$, hence $\overline{F}_1 \cap G_1 = \Phi$. Next for each $b \in A$ with $a_1 \lhd b$, $g_{[b]} \subset f_{[b]} \subset f^0_{[a_1]}$ by 1.13, hence $g_{[b]} \cap \overline{G}_1 \subset f^0_{[a_1]} \cap \overline{X \setminus f_{[a_1]}} = \Phi$, therefore $F_1 \cap \overline{G}_1 = \Phi$.

Now by normality of X there is an open set U_1 such that $F_1 \subset U_1 \subset \overline{U}_1 \subset f_{[a_1]}$.

Step 2. We define U_n.

Suppose $\{U_i \,|\, i \leq n-1\}$ have been defined satisfying (i) and (ii). Let $D_1 = \{i \leq n-1 \,|\, a_i \lhd a_n\}$, $D_2 = \{i \leq n-1 \,|\, a_n \lhd a_i\}$.

Let $F_n = \bigcup\{g_{[b]} \,|\, b \in A \text{ and } a_n \lhd b\} \cup \bigcup_{i \in D_2} \overline{U}_i$, $G_n = X \setminus (f_{[a_n]} \cap \bigcap_{i \in D_1} U_i)$. Then like in Step 1 it can be verified that F_n and G_n are separated F_σ sets, hence there is an open set V_n such that $F_n \subset V_n \subset \overline{V}_n \subset f_{[a_n]} \cap \bigcup_{i \in D_1} U_i$, and let $U_n = V_n \cap \bigcap\{U_i \,|\, i \leq n-1 \text{ and } a_i < a_n\}$.

(2) Let $h = \bigvee_{n \in N} a_n U_n$, where $a_n U_n$ denotes the characteristic mapping on U_n with value a_n, we claim that h is the desired mapping.

Step 1. h is continuous.

At first that h is lsc is trivial. Next the least usc function $\operatorname{cl} h$ majorizing h satisfies $\operatorname{cl} h = \bigvee_{n \in N} a_n \overline{U}_n$ by Proposition 1.14, since U_n is decreasing. Now we prove

$h = \mathrm{cl}\, h$. Indeed for each $a \in L$, $\mathrm{cl}\, h_{[a]} = \bigcap\{\overline{U}_n \mid a_n \in \beta(a) \cap A\} = \bigcap\{U_n \mid a_n \in \beta(a) \cap A\}$, the last equation holds by the interpolation property of the wedge below relation and that L has no nonzero completely compact element by 2.11. And $h_{[a]} = \bigcap\{U_n \mid a_n \in \beta(a) \cap A\}$ is trivial, hence $h = \mathrm{cl}\, h$.

Step 2. $g \leq h \leq f$.
At first $h \leq f$ is trivial by (i). Next for each $a \in L$, $h_{[a]} = \bigcap\{U_n \mid a_n \in \beta(a) \cap A\} \supset g_{[a]}$ by (i), hence $g \leq h$. □

Remark. The above theorem is proved in [8], and the proof presented here is a simplification of that one in [8] which is rather lengthy.

4.2 Theorem. *Every connected metrizable completely distributive lattice L is an AR.*

Proof: It suffices to prove that every continuous mapping $f : A \to L$ defined on a closed subspace of a normal space X can be extended to X. For each $x \in X$, let

$$f_+(x) = \begin{cases} f(x), & x \in A, \\ 0, & x \notin A; \end{cases}$$

$$f_-(x) = \begin{cases} f(x), & x \in A, \\ 1, & x \notin A; \end{cases}$$

then trivially f_+ is usc and f_- is lsc and $f_+ \leq f_-$, hence there is a continuous $h : X \to L$ such that $f_+ \leq h \leq f_-$, trivially such an h is an extension of f. □

4.3 Corollary. *Every connected metrizable completely distributive lattice is arcwise connected.*

□

Remark. (1) It is proved that there is a unique semiuniformity on a continuous lattice which generates both the Lawson topology and the order, see Theorem 3.6 in van Gool [3].

(2) Making use of the semiuniformity on a uniform ordered space, van Gool [3] has introduced new definitions of lsc and usc mappings But it is not difficult to verify that if L is a continuous lattice, then $f : X \to L$ is lsc in the sense of van Gool if and only if f is Scott continuous by Lemma 4.2 in van Gool [3]. Moreover if L is linked bicontinuous, then $f : X \to L$ is usc in the sense of van Gool if and only if f is dual Scott continuous. Hence if L is a completely distributive lattice, then $f : X \to L$ is lsc (usc) in the sense of van Gool if and only if f is lsc (usc) in the sense of 1.6.

(3) van Gool [3] has proved the following result:

If L as an arcwise connected, metrizable, linked bicontinuous lattice, X a normal space, $f : X \to L$ is Scott continuous, $g : X \to L$ is dual Scott continuous with $g \leq f$, then there is some continuous $h : X \to L$ such that $g \leq h \leq f$.

(4) Since every distributive bicontinuous lattice is completely distributive, every completely distributive lattice is linked bicontinuous and every connected metrizable completely distributive lattice is arcwise connected, hence for distributive lattices Theorem 4.1 is equivalent to the above result of van Gool.

Since an AR is necessarily connected the condition that L is connected in Theorem 4.2 is necessary, the following example showing that L is metrizable is also indispensable. It should be pointed out that the example constructed in [8] for this purpose is not connected.

Example. The extended long line L is constructed from the ordinal space $[0, w_1]$, where w_1 is the least uncountable ordinal, by placing between each ordinal α and it successor $\alpha+1$ a copy of the open interval $(0,1)$, then L is simply a connected completely distributive lattice, but L is not metrizable, since L is not an ANR by Proposition 4.4 in Hu [4, p. 36], then L is not an AR.

Now we prove the main result of this paper, i.e. the ANR property of completely distributive lattices. As shown in the above example, we need only to consider the metrizable case.

4.4 Lemma. *If L is a completely distributive lattice containing at most finite completely compact elements, then $(L, \theta(L))$ has at most finite components, and each of them is a subcomplete lattice.*

Proof: If L contains no nonzero completely compact element, then the conclusion follows from 2.11.

Now suppose $\{a_i \,|\, i \leq n\}$ are the nonzero compact elements in L. For each $i \leq n$, let $b_i = \bigvee\{c \in L \,|\, a_i \not\leq c\}$ then trivially $L = \uparrow a_i \cup \downarrow b_i$ and $\uparrow a_i \cup \downarrow b_i$ and $\downarrow a_i \cap \downarrow b_i = \Phi$. Let A be the collection of subcomplete lattices $\{\uparrow a_i, \downarrow b_i \,|\, i \leq n\}$, and denote by B the family of the nonempty minimal intersections of elements in A, i.e. $K \in B$ if and only if $\Phi \neq K = \bigcap\{M \,|\, M \in A_0$ for some subset $A_0 \subset A\}$ and K is minimal in the sense that $K \cap M = \Phi$ for each $M \notin A_0$.

Trivially B is a collection of pairwise disjoint subcomplete lattices which contain no nonzero completely compact elements as a complete lattice, hence each of them is connected. Therefore B is the collection of components in L, and $|B|$ is finite is obvious. □

4.5 Theorem. *A metrizable completely distributive lattice L is an ANR if and only if L contains at most finite completely compact elements.*

Proof: Sufficiency: By the above lemma, if L contains at most finite completely compact elements, then L can be decomposed into finite components, which are subcomplete lattices. Now suppose $f : A \to L$ is a continuous mapping defined on a closed subspace A in a normal space X, then $\{f^{-1}(K) \,|\, K$ is a component of $L\}$ is a pairwise disjoint closed cover of A.

By normality of X, there exists a collection of pairwise disjoint open sets $\{U(K) \,|\, K$ is a component in $L\}$ such that $f^{-1}(K) \subset U(K)$; again by normality of X there is an open set $V(K)$ with $f^{-1}(K) \subset V(K) \subset \overline{V}(K) \subset U(K)$ for each K.

Now since $\overline{V}(K)$ is normal, we can extend $f \mid f^{-1}(K) : f^{-1}(K) \to K$ to $\overline{V}(K)$ by Theorem 4.2 since K is a connected metrizable completely distributive lattice, trivially the combination of the thus defined extensions is an extension of f to the open set $\bigcup\{V(K) \mid K \text{ is a component in } L\}$.

Necessity: Suppose L has infinitely many completely compact elements $\{a_n \mid n \in N\}$, by compactness of $(L, \theta(L))$, we can assume without loss of generality $\{a_n \mid n \in N\}$ converges to certain $a \in L$.

Let $A = \{\frac{1}{n} \mid n \in N\} \cup \{0\}$, then A is a closed set of the unit interval $[0,1]$. Define $f : A \to L$ as follows: $f(\frac{1}{n}) = a_n$ for each $n \in N$ and $f(0) = a$, then trivially f is continuous.

By Corollary 2.6 $a \in L$ has a neighborhood base consisting of subcomplete lattices $\{U_n \mid n \in N\}$.

Suppose $f : U \to L$ is an extension of f to some open set $U \subset [0,1]$ containing A, then there is some $\varepsilon > 0$ such that $[0,\varepsilon] \subset U$. Without loss of generality we can assume that f maps $[0,\varepsilon]$ into U_n for some $n \in N$. Trivially there exists m, k big enough such that $a_m, a_k \in U_n$ and $\frac{1}{m}, \frac{1}{k} \in [0,\varepsilon]$. Obviously a_m and a_k are completely compact elements in the complete lattice U_n. Since either $a_m \notin \uparrow a_k$ or $a_k \notin \uparrow a_m$, for example, assume $a_m \notin \uparrow a_k$, then a_k is nonzero in U_n, hence $U_n \setminus \uparrow a_k$ and $\uparrow a_k$ are disjoint closed set in U_n. Then a contradiction follows from the fact that $[0,\varepsilon]$ is connected and $f(\frac{1}{m}) \in U_n \setminus \uparrow a_k$, $f(\frac{1}{k}) \in \uparrow a_k$. □

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