Commentationes Mathematicae Universitatis Carolinae

Benito J. González; Emilio R. Negrin

The link between the kernel method and the method of adjoints for the generalized index ${}_2F_1$-transform

Commentationes Mathematicae Universitatis Carolinae, Vol. 37 (1996), No. 4, 691--694

Persistent URL: `http://dml.cz/dmlcz/118878`

Terms of use:

© Charles University in Prague, Faculty of Mathematics and Physics, 1996

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* `http://project.dml.cz`

The link between the kernel method and the method of adjoints for the generalized index ${}_2F_1$−transform

Benito J. González, Emilio R. Negrin

Abstract. In this note we show that the two definitions of generalized index ${}_2F_1$-transform given in the previous works [1] and [2] agree for distributions of compact support.

Keywords: index transform, hypergeometric function, distributions of compact support

Classification: 44A15, 46F12

The generalized index ${}_2F_1$-transform of $f \in \mathcal{E}'(I)$, $I = (0, \infty)$, was defined by the kernel method in [1] as:

$$F(\tau) = \langle f(x), \mathbf{F}(\mu, \alpha, \tau, x)\rangle, \quad \tau \in I, \tag{1}$$

where

$$\mathbf{F}(\mu, \alpha, \tau, x) = {}_2F_1(\mu + \frac{1}{2} + i\tau, \mu + \frac{1}{2} - i\tau; \mu + 1; -x)x^{\alpha}$$

and ${}_2F_1(\mu + \frac{1}{2} + i\tau, \mu + \frac{1}{2} - i\tau; \mu + 1; -x)$ is the Gauss hypergeometric function defined for $|x| < 1$ as the sum of the Gauss series, while for $|x| \geq 1$, is defined as its analytic continuation [4, p. 431]. Here μ and α are complex parameters.

In [2] the generalized index ${}_2F_1$-transform was analyzed by the method of adjoints. It was required to introduce two Fréchet spaces V and W satisfying that the operator $\mathcal{L} : W \longrightarrow V$ given by

$$(\mathcal{L}\psi)(x) = \int_0^\infty \psi(\tau)\mathbf{F}(\mu, \alpha, \tau, x)\, d\tau, \quad x > 0, \tag{2}$$

was continuous whenever $Re\ \mu > -\frac{1}{2}$. To be more precise, the spaces V and W were defined by

$$V = \{\phi \in \mathcal{C}^\infty(I) : \gamma_k(\phi) < \infty, \ \ k \in \mathbb{N} \cup \{0\}\}$$

where

$$\gamma_k(\phi) = \sup_{x \in I} \left| x^{-\alpha}(x+1)^{\mu+\frac{3}{2}} A_x^k \phi(x) \right|,$$

A_x being the differential operator defined by

$$x^{\alpha-\mu}(x+1)^{-\mu}D_x x^{\mu+1}(x+1)^{\mu+1}D_x x^{-\alpha},$$

and

$$W = \{\psi \in \mathcal{C}^\infty(\mathbb{R}), \quad \text{even}: \ \varrho_r(\psi) < \infty, \ \ r \in \mathbb{N}\cup\{0\}\},$$

where

$$\varrho_r(\psi) = \sup_{x\in I}\left| x^{\frac{3}{2}} e^x \left(\mathcal{F}\left(\frac{\left[\left(\mu+\frac{1}{2}\right)^2+\tau^2\right]^r \psi(\tau)}{\tau\Gamma\left(\mu+\frac{1}{2}+i\tau\right)\Gamma\left(\mu+\frac{1}{2}-i\tau\right)}\right)\right)(x)\right|,$$

and $\mathcal{F}$ denotes Fourier transform with respect to the τ-variable.

The generalized index ${}_2F_1$-transform was defined as the adjoint operator of $\mathcal{L}$, namely,

$$\langle \mathcal{L}' f, \psi\rangle = \langle f, \mathcal{L}\psi\rangle, \quad f \in V', \quad \psi \in W. \tag{3}$$

In this note we establish the link between these two definitions. Here, it will be proved that both definitions agree for distributions of compact support. The corresponding result for the Kontorovich-Lebedev transform was established in [3, Proposition 2.4].

Theorem. *Let $f \in \mathcal{E}'(I)$ and α, μ being complex parameters with $Re\ \alpha > 0$, $Re\ \mu > 0$, $\frac{1}{8} < Re\ (\mu-\alpha) < \frac{1}{4}$ and $Re\ (\mu - 2\alpha) < -1$. Then, for any $\psi \in W$, one has*

$$\langle \mathcal{L}' f, \psi\rangle = \langle T_{\langle f(x), \mathbf{F}(\mu,\alpha,\tau,x)\rangle}, \psi(\tau)\rangle, \tag{4}$$

where $T_{\langle f(x), \mathbf{F}(\mu,\alpha,\tau,x)\rangle}$ is the member of W' given by

$$\langle T_{\langle f(x), \mathbf{F}(\mu,\alpha,\tau,x)\rangle}, \psi(\tau)\rangle = \int_0^\infty \langle f(x), \mathbf{F}(\mu,\alpha,\tau,x)\rangle \psi(\tau)\, d\tau. \tag{5}$$

Proof: First, observe that $\mathcal{D}(I) \subset V \subset \mathcal{E}(I)$ and the topology of V is stronger than the one induced on it by $\mathcal{E}(I)$. Thus, $\mathcal{E}'(I)$ is a subspace of V'.

Also, from [1, Theorem 3.2, p. 662] the function $F(\tau)$ given by (1) is an entire function such that

(i) $F(\tau) = O(1)$ as $\tau \to 0^+$;
(ii) there exists a $r \in \mathbb{N}\cup\{0\}$ such that $F(\tau) = O\left(\tau^{2r - Re\ \mu - \frac{1}{2}}\right)$ as $\tau \to \infty$.

From this and having into account that, for any $\psi \in W$ one has

(i) $\psi(\tau) = O(\tau^2)$ as $\tau \to 0^+$;
(ii) for all $p \in \mathbb{N}$, $\psi(\tau) = O\ (\tau^{-p})$, as $\tau \to \infty$;

(cf. [2, Proposition 2.2 and Proposition 2.4]) it follows that the integral of the right hand side of (5) has sense.

Clearly, if $f \in \mathcal{D}(I)$, the result of this Theorem holds. In fact, assume that f has its support contained in the closed interval $[a,b] \subset I$, then, using Fubini theorem, one has

$$\begin{aligned}\langle \mathcal{L}' f, \psi\rangle &= \langle f, \mathcal{L}\psi\rangle \\ &= \int_a^b f(x)(\mathcal{L}\psi)(x)dx = \int_a^b f(x)\int_0^\infty \mathbf{F}(\mu,\alpha,\tau,x)\psi(\tau)\,d\tau\,dx \\ &= \int_0^\infty \psi(\tau)\int_a^b f(x)\mathbf{F}(\mu,\alpha,\tau,x)\,dx\,d\tau = \int_0^\infty F(\tau)\psi(\tau)\,d\tau.\end{aligned}$$

So, the result holds for $f \in \mathcal{D}(I)$.

Now, for $f \in \mathcal{E}'(I)$ and using [5, Theorem 28.2 (i), p. 301], there exists a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ belonging to $\mathcal{D}(I)$ which converges to f in $\mathcal{E}'(I)$ and so, in V'. Furthermore, from [2, Proposition 2.3] it follows that $\mathcal{L}'$ is a continuous mapping from V' into W'. Thus $\left\{\mathcal{L}' f_n\right\}_{n\in\mathbb{N}}$ converges to $\mathcal{L}' f$ in W' as $n \to \infty$.

Since, for all $\psi \in W$,

$$\langle \mathcal{L}' f_n, \psi\rangle = \int_0^\infty F_n(\tau)\psi(\tau)\,d\tau$$

where

$$F_n(\tau) = \langle f_n(x), \mathbf{F}(\mu,\alpha,\tau,x)\rangle,$$

it follows that

$$\langle \mathcal{L}' f, \psi\rangle = \lim_{n\to\infty}\int_0^\infty F_n(\tau)\psi(\tau)\,d\tau. \tag{6}$$

Clearly, from the asymptotic growth of F_n and ψ, the limit in (6) goes into the integral. Also, noting that

$$\begin{aligned}\lim_{n\to\infty} F_n(\tau) &= \lim_{n\to\infty}\langle f_n(x), \mathbf{F}(\mu,\alpha,\tau,x)\rangle \\ &= \langle f(x), \mathbf{F}(\mu,\alpha,\tau,x)\rangle = F(\tau),\end{aligned}$$

the result holds. □

References

[1] Hayek N., González B.J., *The index ${}_2F_1$-transform of generalized functions*, Comment. Math. Univ. Carolinae **34.4** (1993), 657–671.

[2] Hayek N., González B.J., *On the distributional index ${}_2F_1$-transform*, Math. Nachr. **165** (1994), 15–24.

[3] Lisena B., *On the Generalized Kontorovich-Lebedev Transform*, Rend. Math. Appl. (7) **9** (1989), 87–101.

[4] Prudnikov A.P., Brychkov Y.A., Marichev O.I., *Integrals and Series*, vol. 3, Gordon and Breach Science Publishers, New York, 1990.
[5] Treves F., *Topological Vector Spaces, Distributions and Kernels*, Academic Press, New York, 1967.

Departamento de Análisis Matemático, Universidad de La Laguna, 38271 Canary Islands, Spain

E-mail: bgonzalez@ull.es
enegrin@ull.es

(*Received* January 16, 1996)